ESSAI
sur la maniére de trouver le terme général des séries recurrentes.*

JEAN TREMBLEY

Mémoires de l'Académie royale des sciences et belles-lettres... Berlin, 1797 pp. 84-105†

One knows that the general term of a recurrent series of which the generating fraction is $\frac{a'}{1-ax}$ is $a'a^nx^n$. One knows also that in order to find the general term of any recurrent series, it is necessary to equate to zero the denominator of the generating fraction, to seek the roots of this equation, to raise each of these roots to the power n, to multiply each of them by an appropriate constant & to add all these products. This method has all the inconveniences attached to the seeking of the roots of the equations, & besides the form of these roots can be such that their elevation to the power n is very painful. I am going to expose here a method which leads to the same end, without that there be need to know separately the roots of which we just spoke & which give immediately the general term by means of series regular & easy to form.

§ 1. Let the fraction be $\frac{a'+b'x}{1-ax+bx^2}$, which is the general generating fraction of all the recurrent series of the second order, that is to say of all those where any term is formed by means of the two preceding. Let m & p be the roots of the equation

$$a - bx + cx^2 = 0,$$

I make according to the ordinary method of the decomposition of the fractions,

$$\frac{a'+b'x}{1-ax+bx^2} = \frac{A}{1-ms} + \frac{B}{1-px}$$

whence I deduce

$$A = \frac{b'+a'm}{m-p}, \quad B = -\frac{(b'+a'p)}{m-p}.$$

One has also $m+p=a$, $mp=b$. The general term of these sorts of series being as we just said it $= Am^n + Bp^n$, n being any number, in order to have the first term of the series, I make $n=0$, this which gives this term

$$= A + B = \frac{a'(m-p)}{m-p} = a'.$$

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. December 22, 2009

†Read to the Academy 23 November 1797.

In order to have the second term, I make $n = 1$, & I have this term

$$= Am + Bp = \frac{a'(m^2 - p^2) + b'(m - p)}{m - p} = a'(m + p) + b' = a'a + b'.$$

In order to have the third term, I make $n = 2$, & I have this term

$$\begin{aligned} &= Am^2 + Bp^2 = \frac{a'(m^3 - p^3) + b'(m^2 - p^2)}{m - p} \\ &= a'(m^2 + mp + p^2) + b'(m + p) = a'((m + p)^2 - mp) + b'(m + p) \\ &= a'(a^2 - b) + b'a. \end{aligned}$$

In order to have the fourth term, I make $n = 3$, & I have this term

$$\begin{aligned} &= Am^3 + Bp^3 = \frac{a'(m^4 - p^4) + b'(m^3 - p^3)}{m - p} \\ &= a'(m^3 + m^2p + mp^2 + p^3) + b'(m^2 + mp + p^2) \\ &= a'((m + p)^3 - 2(m + p)mp) + b'((m + p)^2 - mp)) \\ &= a'(a^3 - 2ab) + b'(a^2 - b). \end{aligned}$$

One will find likewise the fifth term

$$= Am^4 + bp^4 = a'(a^4 - 3a^2b + b^2) + b'(a^3 - 2ab),$$

the sixth term

$$= Am^5 + Bp^5 = a'(a^5 - 4a^3b + 3ab^2) + b'(a^4 - 3a^2b + b^2),$$

the seventh term

$$= Am^6 + Bp^6 = a'(a^6 - 5a^4b + 6a^2b^2 - b^3) + b'(a^5 - 4a^3b + 3ab^2),$$

the eighth

$$= Am^7 + Bp^7 = a'(a^7 - 6a^5b + 10a^3b^2 - 4ab^3) + b'(a^6 - 5a^4b + 6a^2b^2 - b^3),$$

the ninth

$$\begin{aligned} &= Am^8 + Bp^8 \\ &= a'(a^8 - 7a^6b + 15a^4b^2 - 10a^2b^3 + b^4) + b'(a^7 - 6a^5b + 10a^3b^2 - 4ab^3), \end{aligned}$$

The law of these terms being now manifest, one sees that one will have in general the $(n + 1)^{\text{st}}$ term

$$\begin{aligned} &= Am^n + Bp^n \\ &= a'(a^n - (n - 1)a^{n-2}b + \frac{(n - 3)(n - 2)}{1.2}a^{n-4}b^2 \\ &\quad - \frac{(n - 5)(n - 4)(n - 3)}{1.2.3}a^{n-6}b^3 + \frac{(n - 7)(n - 6)(n - 5)(n - 4)}{1.2.3.4}a^{n-8}b^4 \text{ \&c.})x^n \\ &\quad + b'(a^{n-1} - (n - 2)a^{n-3}b + \frac{(n - 4)(n - 3)}{1.2}a^{n-5}b^2 \\ &\quad - \frac{(n - 6)(n - 5)(n - 4)}{1.2.3}a^{n-7}b^3 + \frac{(n - 8)(n - 7)(n - 6)(n - 5)}{1.2.3.4}a^{n-9}b^4 \text{ \&c.})x^n. \end{aligned}$$

One continues these series until one arrives to some terms $= 0$. One sees first that the second series which is multiplied by b' is the same as the first which is multiplied by a', by changing only n into $n-1$, thus all that which I will say on the first of these series will apply to the second with this single observation. The law of the same terms of the first series, setting aside their coefficients, is quite simple since the exponent of a diminishes by two units at each term, while the one of b increases by one unit. As for the coefficients, they are so easy to form that there is no need to write them in the general formula, a single rule suffices to form all of them. This rule is that if one has the term $a^\mu b^\nu$, the coefficient of this term will be equal to the number of permutations of $\mu + \nu$, letters of which the μ are the same & the ν others are also the same, this which gives by the theory of permutations

$$\frac{1.2.3\ldots\mu+\nu}{1.2.3\ldots\mu.1.2.3\ldots\nu} = \frac{\mu+1.\mu+2\ldots\mu+\nu}{1.2.3\ldots\nu} = \frac{\nu+1.\nu+2\ldots\mu+\nu}{1.2.3\ldots\mu}.$$

Thus, for example, the coefficient of the 4th term $a^{n-6}b^3$ is

$$= \frac{1.2\ldots n-3}{1.2\ldots n-6.1.2.3} = \frac{n-5.n-4.n-3}{1.2.3}.$$

The coefficient of the 5th term $a^{n-8}b^4$ is

$$= \frac{1.2\ldots n-4}{1.2\ldots n-8.1.2.3.4} = \frac{n-7.n-6.n-5.n-4}{1.2.3.4}$$

& thus in sequence, this which agrees with the terms of the series reported above, & which I had first deduced from that consideration that the coefficients of the first term were the figurate numbers of the first order, that the coefficients of the second were the figurate numbers of the second, & that in general the coefficients of the n^{th} term were the figurate numbers of the n^{th} order. We can therefore in the general formula set aside the coefficients that one will form always easily for each term according to the preceding rule, & one will have by making

$$A^{(n)} = a^n - a^{n-2}b + a^{n-4}b^2 - a^{n-6}b^3 + a^{n-8}b^4,$$

the $(n+1)^{\text{th}}$ term of the recurrent series of the second order

$$= (a'A^{(n)} + b'A^{(n-1)})x^n.$$

This last form of which one was able to happen in this case here, will be useful to us in the more composed cases.

§ 2. In order to give an example of our formula, I take in the introduction to the calculus of the infinite of Mr. Euler T. 1 p. 179 the recurrent series

$$1 + 3z + 4z^2 + 7z^3 + 11z^4 + 18z^5 + 29z^6 + 47z^7 \text{ \&c.} = \frac{1+2z}{1-z-2z}.$$

By comparing the terms of this generating fraction with those of our general fraction, one has $a' = 1$, $b' = 2$, $a = 1$, $b = -1$, in order to have the 7th term we will make $n+1 = 7$, this which gives $n = 6$, & we will have

$$A^{(6)} = a^6 - a^4b + a^2b^2 - b^3,$$

& by forming the coefficients,

$$A^{(6)} = a^6 - 5a^4b + 6a^2b^2 - b^3 = 1 + 5 + 6 + 1 = 13,$$
$$A^{(5)} = a^5 - 4a^3b + 3ab^2 = 1 + 4 + 3 = 8.$$

The 7th term will be therefore

$$= 13a' + 8b' = 13 + 16 = 29z^6$$

as this is effectively. In order to have the 8th term, we will make $n = 7$, & we will have

$$A^{(7)} = a^7 - 6a^5b + 10a^3b^2 - 4ab^3 = 1 + 6 + 10 + 4 = 21,$$

it will be therefore

$$= 21a' + 26b' = 47z^7,$$

this which is true. The advantage of this method is that it is the same, whether the roots are equals or they are not. If one seeks the terms by the method of the general term that Mr. Euler finds, one will find the method longer at least in all the parts which consist in finding the roots of the scale of relation, & if one compares the general form which results from the method of Mr. Euler (p. 180) & which is by taking our denominations

$$\left(\frac{a'(\sqrt{aa+4b}+a)+2b'}{2\sqrt{aa+4b}}\right)\left(\frac{a+\sqrt{aa+4b}}{2}\right)^n z^n$$
$$+\left(\frac{a'(\sqrt{aa+4b}-a)-2b'}{2\sqrt{aa+4b}}\right)\left(\frac{a-\sqrt{aa+4b}}{2}\right)^n z^n$$

one will find it more complicated & less convenient.

§ 3. Let the fraction be $\frac{a'+b'x+c'x^2}{1-ax+bx^2-cx^3}$ which is the generating fraction of all the recurrent series of third order, that is to say of those where any term is formed by means of the three preceding. Let m, p, q be the roots of the equation $1 - ax + bx^2 - cx^3 = 0$, I make according to the ordinary method

$$\frac{a' + b'x + c'x^2}{1 - ax + bx^2 - cx^3} = \frac{A}{1 - mx} + \frac{B}{1 - px} + \frac{C}{1 - qx},$$

whence I deduce

$$A = \frac{c' + b'm + a'm^2}{(m-p)(m-q)}, \quad B = -\frac{(c' + b'p + a'p^2)}{(m-p)(p-q)}, \quad C = \frac{c' + b'q + a'q^2}{(m-q)(p-q)}.$$

One has also $m + p + q = a$, $mp + mq + pq = b$, $mpq = c$. The general term of these series is $Am^n + Bp^n + Cq^n$. In order to have the first term of the series I make $n = 0$, this which gives me this term $= A + B + C =$

$$\frac{c'p + b'mp + a'm^2p + c'q + b'pq + a'p^2q + c'm + b'mq + a'mq^2}{(m-p)(m-q)(p-q)}$$
$$- \frac{(c'p + b'pq + a'pq^2 + c'q + b'mq + a'm^2q + c'm + b'mp + a'mp^2)}{(m-p)(m-q)(p-q)}$$
$$= \frac{a'(m^2p - m^2q - mp^2 + p^2q + mq^2 - pq^2)}{(m-p)(m-q)(p-q)} = a'.$$

In order to have the second term I make $n = 1$, this which gives me this term

$$= \frac{a'(m^3p - m^3q - mp^3 + p^3q + mq^3 - pq^3) + b'(m^2p - m^2q - mp^2 + p^2q + mq^2 - pq^2)}{(m-p)(m-q)(p-q)}$$
$$= a'(m+p+q) + b' = a'a + b'.$$

In order to find the third term, I make $n = 2$, this which gives this term=

$$\frac{\begin{gathered}a'(m^4p - m^4q - mp^4 + p^4q + mq^4 - pq^4) + b'(m^3p - m^3q - mp^3 + p^3q + mq^3 - pq^3)\\ +c'(m^2p - m^2q - mp^2 + p^2q + mq^2 - pq^2)\end{gathered}}{(m-p)(m-q)(p-q)}$$

$$\begin{aligned}&= a'(m^2 + mp + mq + p^2 + pq + q^2) + b'(m+p+q) + c'\\ &= a'((m+p+q)^2 - (mp + mq + pq)) + b'(m+p+q) + c'\\ &= a'(a^2 - b) + b'a + c'.\end{aligned}$$

In order to have the fourth term I make $n = 3$, this which gives me this term

$$\frac{\begin{gathered}a'(m^5p - m^5q - mp^5 + p^5q + mq^5 - pq^5) + b'(m^4p - m^4q - mp^4 + p^4q + mq^4 - pq^4)\\ +c'(m^3p - m^3q - mp^3 + p^3q + mq^3 - pq^3)\end{gathered}}{(m-p)(m-q)(p-q)}$$

$$\begin{aligned}=&a'(m^3 + m^2p + mp^2 + p^3 + m^2q + mpq + p^2q + mq^2 + pq^2 + q^3)\\ &+ b'(m^2 + mp + mq + p^2 + pq + q^2) + c'(m+p+q)\\ =&a'((m+p+q)^3 - 2(m+p+q)(mp + mq + pq) + mpq)\\ &+ b'((m+p+q)^2 - (mp + mq + pq)) + c'(m+p+q)\\ &= a'(a^3 - 2ab + c) + b'(a^2 - b) + c'a.\end{aligned}$$

One will find likewise for the 5th, 6th, 7th, 8th terms,

$$\begin{aligned}&Am^4 + Bp^4 + Cq^4 = a'(a^4 - 3a^2b + b^2 + 2ca)\\ &+ b'(a^3 - 2ab + c) + c'(a^2 - b),\\ &Am^5 + Bp^5 + Cq^5 = a'(a^5 - 4a^3b + 3ab^2 + c(3a^2 - 2b))\\ &+ b'(a^4 - 3a^2b + b^2 + 2ca) + c'(a^3 - 2ab + c),\\ &Am^6 + Bp^6 + Cq^6 = a'(a^6 - 5a^4b + 6a^2b^2 - b^3 + c(4a^3 - 6ab + c))\\ &+ b'(a^5 - 4a^3b + 3ab^2 + c(3a^2 - 2b)) + c'(a^4 - 3a^2b + b^2 + 2ca),\\ &Am^7 + Bp^7 + Cq^7 = a'(a^7 - 6a^5b + 10a^3b^2 - 4ab^3\\ &\qquad + c(5a^4 - 12a^2b + 3b^2 + 3ca)) + b'(a^6 - 5a^4b + 6a^2b^2 - b^3\\ &\qquad + c(4a^3 - 6ab + c)) + c'(a^5 - 4a^3b + 3ab^2 + c(3a^2 - 2b)),\\ &Am^8 + Bp^8 + Cq^8 = a'(a^8 - 7a^6b + 15a^4b^2 - 10a^2b^3 + b^4\\ &\qquad + c(6a^5 - 20a^3b + 12ab^2 + c(6a^2 - 3b)) + b'(a^7 - 6a^5b\\ &\qquad + 10a^3b^2 - 4ab^3 + c(5a^4 - 12a^2b + 3b^2 + 3ca))\\ &\qquad + c'(a^6 - 5a^4b + 6a^2b^2 - b^3 + c(4a^3 - 6ab + c)) \text{ \&c.}\end{aligned}$$

One can now discover the law of these terms, & one will have in general the $(n{+}1)^{\text{st}}$ term $= Am^n + Bp^n + Cq^n =$

$$
\begin{aligned}
& a'(a^n - (n-1)a^{n-2}b + \frac{(n-3)(n-2)}{1.2}a^{n-4}b^2 \\
& \quad - \frac{(n-5)(n-4)(n-3)}{1.2.3}a^{n-6}b^3 + \frac{(n-7)(n-6)(n-5)(n-4)}{1.2.3.4}a^{n-8}b^4 + \text{\&c.}) \\
& + c((n-2)a^{n-3} - (n-4)(n-3)a^{n-5}b \\
& \quad + \frac{(n-6)(n-5)(n-4)}{1.2}a^{n-7}b^2 - \frac{(n-8)(n-7)(n-6)(n-5)}{1.2.3}a^{n-9}b^3 \text{\&c.}) \\
& + c^2\left(\frac{(n-5)(n-4)}{1.2}a^{n-6} - \frac{(n-7)(n-6)(n-5)}{1.2}a^{n-8}b \right. \\
& \quad \left. + \frac{(n-9)(n-8)(n-7)(n-6)}{1.2 \quad 1.2}a^{n-10}b^2 - \frac{(n-11)(n-10)(n-9)(n-8)(n-7)}{1.2 \quad 1.2.3}a^{n-12}b^3 \text{\&c.}\right) \\
& + c^3\left(\frac{(n-8)(n-7)(n-6)}{1.2.3}a^{n-9} - \frac{(n-10)(n-9)(n-8)(n-7)}{1.2.3}a^{n-11}b \right. \\
& \quad \left. + \frac{(n-12)(n-11)(n-10)(n-9)(n-8)}{1.2.3 \quad 1.2}a^{n-13}b^2 \text{\&c.}\right) + c^4(\text{\&c.} + \text{\&c.}) \\
& + b'(\text{the terms which must multiply } b' \text{ are the preceding by changing } n \text{ to } n-1 \\
& + c'(\text{the terms which must multiply } c' \text{ are the preceding by changing } n \text{ into } n-2).
\end{aligned}
$$

One continues the terms of these series until one arrives to zero. The law of the coefficients of these terms is the same as we have found for the series of the second order, as one can see it immediately. We take for example, the term $a^{n-13}b^2c^3$, the number of permutations will be

$$= \frac{1.2.3\ldots n-8}{1.2.3\ldots n-13.1.2 \quad 1.2.3} = \frac{n-12.n-11.n-10.n-9.n-8}{1.2.3 \quad 1.2}$$

as one sees it in our formula; thus one can set aside some coefficients which one will find easily for each term, & this done one will find without pain the law of the same terms, as one is going to see. We take the value of $A^{(n)}$ enunciated in the preceding problem, & we will have in general for the $(n{+}1)^{\text{st}}$ term of the series of the third order,

$$
\begin{aligned}
& a'(A^{(n)} + cA^{(n-3)} + c^2A^{(n-6)} + c^3A^{(n-9)} + c^4A^{(n-12)} + \text{ \&c.}) \\
& \quad + b'(A^{(n-1)} + cA^{(n-4)} + c^2A^{(n-7)} + c^3A^{(n-10)} + c^4A^{(n-13)} + \text{ \&c.}) \\
& \quad + c'(A^{(n-2)} + cA^{(n-5)} + c^2A^{(n-8)} + c^3A^{(n-11)} + c^4A^{(n-14)} + \text{ \&c.})
\end{aligned}
$$

One will continue the values of $A^{(n)}$ until the exponents become negative. We make now

$$B^{(n)} = A^{(n)} + A^{(n-3)}c + A^{(n-6)}c^2 + A^{(n-9)}c^3 + A^{(n-12)}c^4 + \text{ \&c.}$$

one will have for the $(n{+}1)^{\text{st}}$ term of the recurrent series of the third order the following formula,

$$(a'B^{(n)} + b'B^{(n-1)} + c'B^{(n-2)})x^n.$$

There will be therefore only to form the terms $A^{(n)}$, $A^{(n-1)}$, $A^{(n-2)}$ &c. & next after these the terms $B^{(n)}$, $B^{(n-1)}$, $B^{(n-2)}$ &c. & to form in measure the coefficient of each term according to the rule which we have given above.

§ 4. In order to give an example of this formula, I take in the work cited of Mr. Euler p. 180 the recurrent series

$$1 + z + 2z^2 + 2z^3 + 3z^4 + 3z^5 + 4z^6 + 4z^7 \text{ \&c.} = \frac{1}{1 - z - zz + z^3}.$$

By comparing the generating fraction with ours, I find

$$a' = 1,\ b' = 0,\ c' = 0,\ a = 1,\ b = -1,\ c = -1;$$

in order to find the seventh term I make $n + 1 = 7$, therefore $n = 6$, & our formula will be reduced to

$$a'B^{(6)} = A^{(6)} + A^{(3)}c + c^2;$$

now

$$A^{(6)} = a^6 - a^4b + a^2b^2 - b^3, \quad A^{(3)} = a^3 - ab,$$

therefore

$$B^{(6)} = a^6 - a^4b + a^2b^2 - b^3 + a^3c - abc + c^2,$$

& by forming the coefficients,

$$\begin{aligned} B^{(6)} =& a^6 - 5a^4b + 6a^2b^2 - b^3 + 4a^3c - 6abc + c^2 \\ =& 1 + 5 + 6 + 1 - 4 - 6 + 1 = 4, \end{aligned}$$

& it is effectively the seventh term.

§ 5. In order to give a second example, I take the recurrent series

$$1 + 2x + 3x^2 + 3x^3 + 7x^4 + 5x^5 + 15x^6 + 9x^7 + 31x^8 + 17x^9 + 63x^{10} \text{ \&c.}$$
$$= \frac{1 + 3x + 3x^2}{1 + x - 2x^2 - 2x^3}.$$

In comparing the generating fraction with ours, I obtain

$$a' = 1,\ b' = 3,\ c' = 3,\ a = -1,\ b = -2,\ c = 2.$$

In order to find the tenth term, I make $n + 1 = 10$, therefore $n = 9$ & the formula is reduced to

$$a'B^{(9)} + b'B^{(8)} + c'B^{(7)} = B^{(9)} + 3B^{(8)} + 3B^{(7)}.$$

Now I find

$$\begin{aligned} B^{(9)} &= A^{(9)} + A^{(6)}c + A^{(3)}c^2 + c^3, \\ B^{(8)} &= A^{(8)} + A^{(5)}c + A^{(2)}c^2, \\ B^{(7)} &= A^{(7)} + A^{(4)}c + A^{(1)}c^2; \\ A^{(9)} &= a^9 - a^7b + a^5b^2 - a^3b^3 + ab^4, \\ A^{(6)} &= a^6 - a^4b + a^2b^2 - b^3, \\ A^{(3)} &= a^3 - ab; \end{aligned}$$

therefore

$$\begin{aligned}B^{(9)} =& a^9 - a^7b + a^5b^2 - a^3b^3 + ab^4 + a^6c - a^4bc + a^2b^2c - b^3c + a^3c^2 - abc^2 + c^3\\ &=\text{(by forming the coefficients)}\\ &\ a^9 - 8a^7b + 21a^5b^2 - 20a^3b^3 + 5ab^4 + 7a^6c - 30a^4bc + 30a^2b^2c - 4b^3c\\ &\ + 10a^3c^2 - 12abc^2 + c^3\\ &= -1 - 16 - 84 - 160 - 80 + 14 + 120 + 240 + 64 - 40 - 96 + 8\\ &= -31.\end{aligned}$$

We will have likewise

$$\begin{aligned}A^{(8)} &= a^8 - a^6b + a^4b^2 - a^2b^3 + b^4,\\ A^{(5)} &= a^5 - a^3b + ab^2,\\ A^{(2)} &= a^2 - b,\end{aligned}$$

therefore by forming the coefficients immediately,

$$\begin{aligned}&B^{(8)} = a^8 - 7a^6b + 15a^4b^2 - 10a^2b^3 + b^4 + 6a^3c - 20a^3bc + 12ab^2c + 6a^2c^2 - 3bc^2\\ &= 1 + 14 + 60 + 80 + 16 - 12 - 80 - 96 + 24 + 24 = 31.\end{aligned}$$

We will have finally

$$\begin{aligned}A^{(7)} &= a^7 - a^5b + a^3b^2 - ab^3,\\ A^{(4)} &= a^4 - a^2b + b^2,\\ A^{(1)} &= a,\end{aligned}$$

therefore

$$\begin{aligned}B^{(7)} =& a^7 - 6a^5b + 10a^3b^2 - 4ab^3 + 5a^4c - 12a^2bc + 3b^2c + 3ac^2\\ &= -1 - 12 - 40 - 32 + 10 + 48 + 24 - 12 = -15.\end{aligned}$$

The tenth term will be therefore

$$= -31 + 3.31 - 3.15 = 93 - 76 = 17,$$

as this is really.

§ 6. Let be $\frac{a'+b'x+c'x^2+d'x^3}{1-ax+bx^2-cx^3+dx^4}$ which is the generating fraction of all the recurrent series of the fourth order, that is to say of all those where any term is formed by means of the preceding four, one finds by a process analogous to the one which I just used & which I suppress because of its length, that by leaving to $A^{(n)}$ & $B^{(n)}$ the same values as above, & making

$$C^{(n)} = B^{(n)} - B^{(n-4)}d + B^{(n-8)}d^2 - B^{(n-12)}d^3 + B^{(n-16)}d^4 \text{ \&c.}$$

one will have in general the $(n+1)^{\text{st}}$ term of a recurrent series of the fourth order

$$= (a'C^{(n)} + b'C^{(n-1)} + c'C^{(n-2)} + d'C^{(n-3)})x^n.$$

In order to give an example, let the series

$$\begin{aligned}&1 + x + x^2 + 2x^3 + 4x^4 + 6x^5 + 7x^6 + 7x^7 + 7x^8 + 8x^9 + \text{ \&c.}\\ &= \frac{1 - 2x + 2x^2}{1 - 3x + 4x^2 - 3x^3 + x^4},\end{aligned}$$

we will have by comparing this generating fraction with our general fraction

$$a' = 1,\ b' = -2,\ c' = 2,\ d' = 0,\ a = 3,\ b = 4,\ c = 3,\ d = 1.$$

In order to have the tenth term, we make $n + 1 = 10$, this which gives $n = 9$, & this term will be

$$= C^{(9)} - 2C^{(8)} + 2C^{(7)}.$$

Now we have

$$\begin{aligned}C^{(9)} &= B^{(9)} - B^{(5)}d + B^{(1)}d^2,\\ C^{(8)} &= B^{(8)} - B^{(4)}d + d^2,\\ C^{(7)} &= B^{(7)} - B^{(3)}d.\end{aligned}$$

Now one will find as in the preceding problem,

$$\begin{aligned}B^{(9)} =&a^9 - 8a^7b + 21a^5b^2 - 20a^3b^3 + 5ab^4 + 7a^6c - 30a^4bc\\ &+ 30a^2b^2c - 4b^3c + 10a^3c^2 - 12abc^2 + c^3;\\ &B^{(5)} = a^5 - a^3b + ab^2 + a^2c - bc,\end{aligned}$$

therefore (by forming the coefficients)

$$\begin{aligned}&B^{(5)}d = 6a^5d - 20a^3bd + 12ab^2d + 12a^2cd - 6bcd,\\ &B^{(1)} = a,\end{aligned}$$

therefore $B^{(1)}d^2 = 3ad^2$, therefore

$$\begin{aligned}C^{(9)} =&a^9 - 8a^7b + 21a^5b^2 - 20a^3b^3 + 5ab^4 + 7a^6c - 30a^4bc\\ &+ 30a^2b^2c - 4b^3c + 10a^3c^2 - 12abc^2 + c^3 - 6a^5d\\ &+ 20a^3bd - 12ab^2d - 12a^2cd + 6bcd + 3ad^2\\ =&19683 - 69984 + 81684 - 34560 + 3840 + 15309 - 29160\\ &+ 12960 - 768 + 2430 - 1296 + 27 - 1458 + 2160 - 576\\ &- 324 + 72 + 9 = 12.\end{aligned}$$

One will have also

$$\begin{aligned}B^{(8)} =&a^8 - 7a^6b + 15a^4b^2 - 10a^2b^3 + b^4 + 6a^5c - 20a^3bc\\ &+ 12ab^2c + ba^2c^2 - 3bc^2,\\ &B^{(4)} = a^4 - a^2b + b^2 + ac,\end{aligned}$$

therefore

$$B^{(4)}d = 5a^4d - 12a^2bd + 3b^2d + 6acd,$$

therefore

$$\begin{aligned} C^{(8)} =& a^8 - 7a^6b + 15a^4b^2 - 10a^2b^3 + b^4 + 6a^5c - 20a^3bc + 12ab^2c \\ &+ 6a^2c^2 - 3bc^2 - 5a^4d + 12a^2bd - 3b^2d - 6acd + d^2 \\ =& 6561 - 20412 + 19440 - 5760 + 256 + 4374 - 6480 + 1728 \\ &+ 486 - 108 - 405 + 432 - 48 - 54 + 1 = 11. \end{aligned}$$

One has finally

$$B^{(7)} = a^7 - 6a^5b + 10a^3b^2 - 4ab^3 + 5a^4c - 12a^2bc + 3b^2c + 3ac^2,$$
$$B^{(3)} = a^3 - ab + c,$$

therefore

$$B^{(3)}d = 4a^3d - 6abd + 2cd,$$

therefore

$$\begin{aligned} C^{(7)} =& a^7 - 6a^5b + 10a^3b^2 - 4ab^3 + 5a^4c - 12a^2bc + 3b^2c \\ &+ 3ac^2 - 4a^3d + 6abd - 2cd \\ =& 2187 - 5852 + 4320 - 768 + 1215 - 1296 + 144 + 81 - 108 + 72 - 6 \\ =& 9, \end{aligned}$$

therefore

$$C^{(9)} - 2C^{(8)} + 2C^{(7)} = 12 - 22 + 18 = 8,$$

& it is effectively the tenth term.

§ 7. Let there be the fraction $\frac{a'+b'x+c'x^2+d'x^3+e'x^4}{1-ax+bx^2-cx^3+dx^4-ex^5}$ which is the generating fraction of all the recurrent series of the fifth order, one will find that by leaving to $A^{(n)}$, $B^{(n)}$, $C^{(n)}$ their values found above, & making

$$D^{(n)} = C^{(n)} + C^{(n-5)}e + C^{(n-10)}e^2 + C^{(n-15)}e^3\text{\&c.}$$

one will have in general the $(n+1)^{\text{st}}$ term of a recurrent series of the fifth order

$$= (a'D^{(n)} + b'D^{(n-1)} + c'D^{(n-2)} + d'D^{(n-3)} + e'D^{(n-4)}x^n.$$

In order to give an example I take in the work cited of Euler p. 188 the series

$$1 + 2z + 3z^2 + 3z^3 + 4z^4 + 5z^5 + 6z^6 + 6z^7 + 7z^8 + 8z^9 + \text{\&c.} = \frac{1+z+z^2}{1-z-z^4+z^5}.$$

We will have by comparing this generating fraction with our general fraction,

$$a' = 1,\ b' = 1,\ c' = 1,\ d' = 0,\ e' = 0;\ a = 1,\ b = 0,\ c = 0,\ d = -1,\ e = -1.$$

In order to have the tenth term we will make $n+1=10$, this which gives $n=9$, & this term will be $= D^{(9)} + D^{(8)} + D^{(7)}$. Now we have

$$\begin{aligned} D^{(9)} &= C^{(9)} + C^{(4)}e, \\ D^{(8)} &= C^{(8)} + C^{(3)}e, \\ D^{(7)} &= C^{(7)} + C^{(2)}e. \end{aligned}$$

Now

$$\begin{aligned} C^{(9)} &= B^{(9)} - B^{(5)}d + B^{(1)}d^2, \\ C^{(4)} &= B^{(4)} - d, \\ B^{(9)} &= A^{(9)} = a^9, \\ B^{(4)} &= A^4 = a^4, \end{aligned}$$

therefore

$$\begin{aligned} C^{(9)} &= a^9 - a^5 d + ad^2, \\ C^{(4)}e &= a^4 e - de, \end{aligned}$$

&

$$D^{(9)} = a^9 - 6a^5 d + 3ad^2 + 5a^4 e - 2de = 1 + 6 + 3 - 5 - 2 = 3.$$

One has likewise

$$\begin{aligned} C^{(8)} &= B^{(8)} - B^{(4)}d + d^2 = a^8 - a^4 d + d^2, \\ C^{(3)} &= B^{(3)} = a^3, \\ C^{3)}e &= a^3 e, \end{aligned}$$

therefore

$$D^{(8)} = a^8 - 5a^4 d + d^2 + 4a^3 e = 1 + 5 + 1 - 4 = 3.$$

One has finally

$$\begin{aligned} C^{(7)} &= B^{(7)} - B^{(3)}d = a^7 - a^3 d, \\ C^{(2)} &= B^{(2)} = a^2, \end{aligned}$$

therefore

$$D^{(7)} = a^7 - 4a^3 d + 3a^2 e = 1 + 4 - 3 = 2.$$

The tenth term will be therefore

$$= D^{(9)} + D^{(8)} + D^{(7)} = 3 + 3 + 2 = 8,$$

as this is effectively.

§ 8. Let the fraction be $\frac{a'+b'x+c'x^2+d'x^3+e'x^4+f'x^5}{1-ax+bx^2-cx^3+dx^4-ex^5+fx^6}$ which is the generating fraction of all the recurrent series of the sixth order, one will find by conserving the values of $A^{(n)}$, $B^{(n)}$, $C^{(n)}$, $D^{(n)}$ & making

$$E^{(n)} = D^{(n)} - D^{(n-6)}f + D^{(n-12)}f^2 - D^{(n-18)}f^3 + D^{(n-24)}f^4 + \text{ \&c.}$$

the $(n+1)^{\text{st}}$ term of a recurrent series of the sixth order

$$= (a'E^{(n)} + b'E^{(n-1)} + c'E^{(n-2)} + d'E^{(n-3)} + e'E^{(n-4)} + f'E^{(n-5)}x^n.$$

In order to give an example, we take in the work cited of Mr. Euler p. 186 the series

$$1+z+2z^2+3z^3+4z^4+5z^5+7z^6+8z^7+10z^8+12z^9+\text{\&c.} = \frac{1}{1-z-z^2+z^4+z^5-z^6},$$

we will have by comparing this generating fraction with our general fraction,

$$a' = 1,\ b' = 0,\ c' = 0,\ d' = 0,\ e' = 0,\ f' = 0;\ a = 1,\ b = -1,\ c = 0,\ d = 1,\ e = -1,\ f = -1.$$

In order to have the tenth term we will make $n = 9$, & this term will be $= E^{(9)}$. But $E^{(9)} = D^{(9)} - D^{(3)}f$, $D^{(9)} = C^{(9)} + C^{(4)}e$, $D^{(3)} = C^{(3)}$, $C^{(9)} = B^{(9)} - B^{(5)}d + B^{(1)}d^2$, $C^{(4)} = B^{(4)} - d$, $C^{(3)} = B^{(3)}$, $B^{(9)} = A^{(9)}$, $B^{(5)} = A^{(5)}$, $B^{(1)} = A^{(1)}$, $B^{(4)} = A^{(4)}$, $B^{(3)} = A^{(3)}$, therefore

$$\begin{aligned} C^{(9)} =& a^9 - a^7b + a^5b^2 - a^3b^3 + ab^4 - a^5d + a^3bd - ab^2d + ad^2 \\ &= \text{(by forming the coefficients)} \\ &\quad a^9 - 8a^7b + 21a^5b^2 - 20a^3b^3 + 5ab^4 - 6a^5d + 20a^3bd - 12ab^2d + 3ad^2. \end{aligned}$$

One will have likewise

$$C^{(4)}e = 5a^4e - 12a^2be + 3b^2e - 2de.$$

One will have finally

$$D^{(3)}f = 4a^3f - 6abf.$$

Therefore reuniting these values, one will have

$$\begin{aligned} E^{(9)} =& a^9 - 8a^7b + 21a^5b^2 - 20a^3b^3 + 5ab^4 - 6a^5d + 20a^3bd - 12ab^2d \\ &+ 3ad^2 + 5a^4e - 12a^2be + 3b^2e - 2de - 4a^3f + 6abf \\ =& 1 + 8 + 21 + 20 + 5 - 6 - 20 - 12 + 3 - 5 - 12 - 3 + 2 + 4 + 6 \\ =& 12, \end{aligned}$$

as this must be.

§ 9. In order to have the general term of the series of the seventh order of which the generating fraction is $\frac{a'+b'x+c'x^2+d'x^3+e'x^4+f'x^5+g'x^6}{1-ax+bx^2-cx^3+dx^4-ex^5+fx^6-gx^7}$, it is necessary by conserving all the preceding denominations to make

$$F^{(n)} = E^{(n)} - E^{(n-7)}g + E^{(n-14)}g^2 - E^{(n-21)}g^3 + E^{(n-28)}g^4 + \text{ \&c.}$$

& one will have in general the $(n+1)^{\text{st}}$ term of a recurrent series of the seventh order,

$$= (a'F^{(n)} + b'F^{(n-1)} + c'F^{(n-2)} + d'F^{(n-3)} + e'F^{(n-4)} + f'F^{(n-5)} + g'F^{(n-6)}x^n.$$

§ 10. The law of these progressions is actually manifest, & it is evident by continuing to form some quantities following the same law, one will have the general term of any series. Let therefore the generating fraction be

$$\frac{a' + b'x + c'x^2 + d'x^3 \cdots + px^{m-1}}{1 - ax + bx^2 - cx^3 + dx^4 \cdots \pm px^m}$$

(the $+$ sign has place if m is even, & the $-$ sign if m is odd) one will form the quantities represented by the following tableau,

$$A^{(n)} = a^n - a^{n-2}b + a^{n-4}b^2 - a^{n-6}b^3 + a^{n-8}b^4 + \&\text{c.}$$
$$B^{(n)} = A^{(n)} + A^{(n-3)}c + A^{(n-6)}c^2 + A^{(n-9)}c^3 + A^{(n-12)}c^4 + \&\text{c.}$$
$$C^{(n)} = B^{(n)} - B^{(n-4)}d + B^{(n-8)}d^2 - B^{(n-12)}d^3 + B^{(n-16)}d^4 + \&\text{c.}$$
$$D^{(n)} = C^{(n)} + C^{(n-5)}e + C^{(n-10)}e^2 + C^{(n-15)}e^3 + C^{(n-20)}e^4 + \&\text{c.}$$
$$E^{(n)} = D^{(n)} - D^{(n-6)}f + D^{(n-12)}f^2 - D^{(n-18)}f^3 + D^{(n-24)}f^4 + \&\text{c.}$$
$$\cdots$$
$$M^{(n)} = K^{(n)} \pm K^{(n-m)}p + K^{(n-2m)}p^2 \pm K^{(n-3m)}p^3 + K^{(n-4m)}p^4 \pm \&\text{c.}$$

I call here $K^{(n)}$ the quantities corresponding to the recurrent series of order $m-1$. If m is odd all the terms are positive., if it is even they are alternately positive & negative. It is this which I have marked by the ambiguous signs of the value of $M^{(n)}$ which is the quantity corresponding to the recurrent series of order m. Now one will have in general the $(n+1)^{\text{st}}$ term of the recurrent series of order m

$$= (a'M^{(n)} + b'M^{(n-1)} + c'M^{(n-2)} + d'M^{(n-3)} \cdots + p'M^{(n-m+1)}x^n.$$

For example, if one makes $m = 7$, one will have

$$p' = g',\ p = g,\ K^{(n)} = E^{(n)},\ M^{(n)} = F^{(n)},$$

one will have therefore

$$F^{(n)} = E^{(n)} + E^{(n-7)}g + E^{(n-14)}g^2 + E^{(n-21)}g^3 + E^{(n-28)}g^4 + \&\text{c.}$$

& the $(n+1)^{\text{st}}$ term will be in general

$$(a'F^{(n)} + b'F^{(n-1)} + c'F^{(n-2)} \cdots + g'F^{(n-6)})x^n,$$

this which is the formula which we have found above. This general formula alone will serve us therefore to find all the particular formulas, by descending from m to 1.

§ 11. In order to give an example of the usage of these series, I will apply them to a problem of the theory of probabilities that Mr. de la Place has treated quite differently in a beautiful memoir on this material inserted into T. 7 of the Memoirs[1] presented to the Academy. Here is this problem.

[1] "Recherches, sur l'integration des équations differentielles aux différences finies, & sur leur usage dans la théorie des hasards." *Savants étranges*, 1773 (1776) p. 37-162. This refers to Problem XII.

If one imagines a solid composed of a number n faces perfectly equal, designated by the numbers $1,\ 2,\ 3,\ \ldots n$, one demands the probability that in a number x coups one will bring forth these n faces in sequence in the order $1,\ 2,\ 3,\ \ldots n$.

§ 12. In order to begin with the simplest case, I will suppose first $n = 2$ or that it has only two faces. The probability that the numbers 1, 2 follow themselves in this order, in supposing only two coups, is $= \frac{1}{4}$, & the probability to the contrary $= \frac{3}{4}$. In order to have the probability that the same thing will arrive in three coups, it is necessary to add to the probability found the probability that in the last two coups, one will bring forth 1, 2 in this order, whatever had been the first coup, a probability which is evidently $= \frac{1}{4}$. Thus for three coups the probability will be $= \frac{1}{2}$, & the probability to the contrary $= \frac{1}{2}$. For four coups, it is necessary to add to that which one just found, the probability that in the last two coups one will bring forth 1, 2 in this order, multiplied by the probability that one will not bring them forth in the first two coups; this probability is $= \frac{1}{4} \cdot \frac{3}{4} = \frac{3}{16}$, the total probability will be therefore $= \frac{1}{2} + \frac{3}{16} = \frac{11}{16}$ & the probability of the contrary will be $= \frac{5}{16}$. For five coups, it is necessary to add to this that one just found, the probability that in the last two coups one will bring forth 1, 2 in this order, multiplied by the probability that one will not bring them forth in the first three coups; this probability is $= \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$, the total probability will be therefore $= \frac{11}{16} + \frac{1}{8} = \frac{13}{16}$ & the probability of the contrary will be $= \frac{3}{16} = \frac{6}{32}$. For six coups, it is necessary to add to that which one just found the probability that in the last two coups one will bring forth in this order 1, 2 multiplied by the probability that one will not bring them forth in the first four coups; this probability is $= \frac{1}{4} \cdot \frac{5}{16} = \frac{5}{64}$, the total probability will be therefore $= \frac{13}{16} + \frac{5}{64} = \frac{57}{64}$ & the probability of the contrary is $= \frac{7}{64}$. For seven coups it is necessary to add to that which one just found $\frac{1}{4} \cdot \frac{3}{16} = \frac{3}{64}$, the total probability will be therefore $= \frac{57}{64} + \frac{3}{64} = \frac{60}{64}$ & the probability of the contrary will be $= \frac{4}{64} = \frac{8}{128}$. One will find likewise for eight coups the total probability $= \frac{247}{256}$, & the probability of the contrary will be $= \frac{9}{256}$. For nine coups the total probability will be $= \frac{251}{256}$, & the probability of the contrary $= \frac{5}{256} = \frac{10}{512}$. We form now the tableau of the probabilities contrary to the event demanded for 2, 3, 4, 5, 6, 7, 8, 9, coups & one will have

Coups	2	3	4	5	6	7	8	9
Contrary probability	$\frac{3}{4}$	$\frac{4}{8}$	$\frac{5}{16}$	$\frac{6}{32}$	$\frac{7}{64}$	$\frac{8}{128}$	$\frac{9}{256}$	$\frac{10}{512}$

The analogy is evident, the denominators follow a geometric progression of which the exponent is 2, & the numerators an arithmetic progression of which the exponent is 1, or else these numerators form a recurrent series of the second order, where each term is equal to two times the one which precedes it, less the one which precedes the preceding. The generating fraction of the sequence of the numerators is therefore $= \frac{1}{1-2x+x^2}$.

§ 13. We suppose now $n = 3$, the probability that the numbers 1, 2, 3 follow themselves in this order by supposing only three coups is $= \frac{1}{3^3} = \frac{1}{27}$ & the probability of the contrary is $= \frac{26}{27}$. In order to have the probability that the same thing will arrive in four coups it is necessary to add to the probability found the probability that in the last three coups one will bring forth 1, 2, 3 in this order, whatever be the first coup; this probability is $= \frac{1}{27}$, the total probability will be therefore $= \frac{2}{27}$ & the probability of the contrary$= \frac{25}{27}$. For five coups, it is necessary to add to that which one just found the probability that in the last three coups one will bring forth 1, 2, 3 in this order, whatever be the first two coups, now there are 3^2 possible cases for these first two coups, because by admitting the combinations where two numbers of the same kind are found together, each number can be joined to each of the three numbers; this probability is therefore $= \frac{3^2}{3^5} = \frac{1}{27}$, the total probability will be therefore $\frac{3}{27}$, & that of the contrary $= \frac{24}{27}$. For six coups, it is necessary to add to that which one just found the probability that in the last three coups one will bring forth 1, 2, 3 in this order, multiplied by the probability that one will not bring them forth in the first three coups; this probability is $= \frac{1}{3^3} \cdot \frac{26}{3^3} = \frac{26}{3^6}$, the total probability will be therefore $= \frac{3}{3^3} + \frac{26}{3^6} = \frac{107}{3^6}$ & the probability of the contrary will be $\frac{622}{3^6}$. For seven coups it is necessary to add to that which one just found the probability that in the last three coups one will bring forth 1, 2, 3 in this order, multiplied by the probability that one will not bring them forth in the first four coups; this probability is $= \frac{1}{3^3} \cdot \frac{25}{3^3} = \frac{25}{3^6}$, the total probability will be therefore $= \frac{107}{3^6} + \frac{25}{3^6} = \frac{132}{3^6}$ & the probability of the contrary will be $= \frac{597}{3^6} = \frac{1791}{3^7}$. For eight coups it is necessary to add to that which one just found, the probability that in the last three coups one will bring forth 1, 2, 3 in this order, multiplied by the probability that one will not bring them forth in the first five coups; this probability is $= \frac{1}{3^3} \cdot \frac{24}{3^3} = \frac{24}{3^6}$, the total probability will be therefore $= \frac{132}{3^6} + \frac{24}{3^6} = \frac{156}{3^6}$ & the probability of the contrary will be $\frac{573}{3^6} = \frac{5157}{3^8}$. For nine coups, it is necessary to add to that which one just found $\frac{1}{3^3} \cdot \frac{622}{3^9} = \frac{622}{3^9}$, the total probability will be therefore $= \frac{156}{3^6} + \frac{622}{3^9} = \frac{4834}{3^9}$ & the probability of the contrary will be $= \frac{14849}{3^9}$. For ten coups, it is necessary to add to that which one just found $\frac{1}{3^3} \cdot \frac{597}{3^9} = \frac{597}{3^9}$, the total probability will be therefore $= \frac{4834}{3^9} + \frac{597}{3^9} = \frac{5431}{3^9}$ & the probability of the contrary will be $= \frac{14252}{3^9} = \frac{42756}{3^{10}}$. For eleven coups, it is necessary to add to that which one just found $\frac{1}{3^3} \cdot \frac{573}{3^6} = \frac{573}{3^9}$, the total probability will be therefore $= \frac{5431}{3^9} + \frac{573}{3^9} = \frac{6004}{3^9}$ & the probability of the contrary will be $= \frac{13679}{3^9} = \frac{123111}{3^{11}}$. We form now the tableau of the probabilities contrary to the event demanded for 3, 4, 5, 6, 7, 8, 9, 10, 11 coups, we will have

Coups	3	4	5	6	7	8	9	10	11
Contrary probability	$\frac{26}{3^3}$	$\frac{75}{3^4}$	$\frac{216}{3^5}$	$\frac{622}{3^6}$	$\frac{1791}{3^7}$	$\frac{5157}{3^8}$	$\frac{14849}{3^9}$	$\frac{42756}{3^{10}}$	$\frac{123111}{3^{11}}$

The denominators follow a geometric progression of which the exponent is 3; as for the numerators, each is equal to three times the one which precedes it by one place less the one which precedes it by three places. The generating fraction of the sequence of the numerators will be therefore $= \frac{1}{1-3x+x^3}$.

§ 14. We make next $n = 4$, the probability that the numbers 1, 2, 3, 4 follow themselves in this order, by supposing only four coups is $= \frac{1}{4^4} = \frac{1}{256}$ & the probability of the contrary is $= \frac{255}{256}$. In order to have the probability that the same thing will arrive in five coups, it is necessary to add to the probability found, the probability that in the last three coups, one will bring forth in this order 1, 2, 3, 4, whatever be the first coup, this which gives evidently $\frac{4}{4^5} = \frac{1}{4^4}$. The total probability will be therefore $= \frac{2}{4^4} = \frac{2}{256}$ & the probability of the contrary $= \frac{256}{4^4}$. For six coups it is necessary to add to that which one just found, the probability that in the last three coups one will bring forth in this order 1, 2, 3, 4 whatever are the first two coups, now there are 4^2 possible cases for these first two coups, because one can join to each number, the four numbers, this which will give $\frac{4^2}{4^6} = \frac{1}{4^4}$, the total probability will be therefore $= \frac{3}{4^4}$ & the probability of the contrary $= \frac{253}{4^4}$. For seven coups, it is necessary to add to that which one just found the probability that in the last four coups one will bring forth 1, 2, 3, 4 in this order whatever are the first three coups, now there are 4^3 possible cases for these first three coups, because one can combine with each number all the 4^2 combinations of two numbers, this which gives $\frac{4^3}{4^7} = \frac{1}{4^4}$. The total probability will be therefore $= \frac{4}{4^4}$ & the probability of the contrary will be $= \frac{252}{4^4}$. For eight coups it is necessary to add to that which one just found the probability that in the last four coups one will bring forth 1, 2, 3, 4 in this order, multiplied by the probability that one will not bring them forth; in the first four coups, this probability is $\frac{1}{4^4} \cdot \frac{255}{4^4} = \frac{255}{4^8}$, the total probability will be therefore $= \frac{4}{4^4} + \frac{255}{4^8} = \frac{1279}{4^8}$ & the probability of the contrary will be $= \frac{64257}{4^8}$. For nine coups it is necessary to add to that which one just found the probability that in the last four coups one will bring forth 1, 2, 3, 4 in this order, multiplied by the probability that one will not bring them forth in the first five coups; this probability is $\frac{1}{4^4} \cdot \frac{254}{4^4} = \frac{254}{4^8}$, the total probability will be therefore $= \frac{1279}{4^8} + \frac{254}{4^8} = \frac{1533}{4^8}$ & the probability of the contrary will be $= \frac{64003}{4^8} = \frac{256012}{4^9}$. For ten coups it is necessary to add to that which one just found the probability that in the last four coups one will bring forth 1, 2, 3, 4 &c. in this order, multiplied by the probability that one will not bring them forth in the first six coups; this probability is $\frac{1}{4^4} \cdot \frac{253}{4^4} = \frac{253}{4^8}$, the total probability will be therefore $= \frac{1533}{4^8} + \frac{253}{4^8} = \frac{1786}{4^8}$ & the probability of the contrary will be $= \frac{63750}{4^8} = \frac{1020000}{4^{10}}$. For eleven coups, it is necessary to add to that which one just found $\frac{1}{4^4} \cdot \frac{252}{4^4} = \frac{252}{4^8}$; the total probability is therefore $= \frac{1786}{4^8} + \frac{252}{4^8} = \frac{2038}{4^8}$ & the probability of the contrary is $= \frac{63498}{4^8} = \frac{4063872}{4^{11}}$. We form now the tableau of the probabilities contrary to the event demanded, for 4, 5, 6, 7, 8, 9, 10, 11 coups and one will have

Coups	4	5	6	7	8	9	10	11
Contrary probability	$\frac{255}{4^4}$	$\frac{1016}{4^5}$	$\frac{4048}{4^6}$	$\frac{16128}{4^7}$	$\frac{64257}{4^8}$	$\frac{256012}{4^9}$	$\frac{1020000}{4^{10}}$	$\frac{4063872}{4^{11}}$

The denominators form a geometric progression of which the exponent is 4; as for the numerators, each is equal to four times the one which precedes it by one place less the one which precedes it by four places. The generating fraction of the sequence of the numerators is therefore $= \frac{1}{1-4x+x^4}$.

§ 15. If we make $n = 5$, we will find by the same process that the denominators furnish a geometric progression of which the exponent is 5, & the first term 5 & that each numerator is equal to five times the one which precedes it by one place less the one which precedes it by five places. The generating fraction of the sequence of the numerators is therefore $= \frac{1}{1-5x+x^5}$.

§ 16. One will find similarly by making $n = 6$ that the denominators form a geometric progression of which the exponent is 6 & the first term 6, & that each numerator is equal to six times the one which precedes it by one place, less the one which precedes it by six places. The generating fraction of the sequence of the numerators will be therefore $= \frac{1}{1-6x+x^6}$.

§ 17. The analogy is now evident & one sees that by leaving to n its indeterminate value, the denominators will form a geometric progression of which the exponent will be n, & the first term n, & that each numerator will be equal to n times the term which precedes it by one place, less the one which precedes it by n places. The generating fraction of the sequence of the numerators will be therefore $= \frac{1}{1-nx+x^n}$.

§ 18. Thus for a number of coups x, the denominator will be in general n^x, & in order to have the general term of the numerators, it is necessary to compare the fraction

$\frac{1}{1-nx+x^n}$ with the general fraction of § 10. We will have $a' = 1$, $b' = c' =$&c.$= p' = 0$, $a = n$, $b = c =$&c.$= 0$, $p = \pm 1$, $m = n$, $n = x$. We will have therefore $A^{(x)} = B^{(x)} = C^{(x)} =$&c.$= K^{(x)} = n^x$, & the general term

$$\begin{aligned} M^{(x)} =& K^{(x)} \pm K^{(x-n)}p + K^{(x-2n)}p^2 \pm K^{(x-3n)}p^3 + K^{(x-4n)}p^4 \text{ \&c.} \\ =& n^x \pm n^{x-n}p + n^{x-2n}p^2 \pm n^{x-3n}p^3 + n^{x-4n}p^4 \text{ \&c.} \end{aligned}$$

We form now the coefficients; the one of the first term is evidently $= 1$, in order to have the one of the second, we take the combinations of $x - n + 1$ things, of which $x - n$ are the same, this which gives $x - n + 1$; in order to have the one of the third we take the combinations of $x - 2n + 2$ things, of which $x - 2n$ are the same & the two others the same, this which gives

$$\frac{(x-2n+2)(x-2n+1)}{1.2};$$

in order to have the one of the fourth term, we take the combinations of $x - 3n + 3$ things, of which $x - 3n$ are the same, & the three others the same, this which gives

$$\frac{(x-3n+3)(x-3n+2)(x-3n+1)}{1.2.3},$$

& thus in sequence. As for the signs it is necessary to observe that if n is even, one has $p = 1$, & that the signs must alternate in the general formula; if on the contrary n is odd, one has $p = -1$, & the general formula must have the $+$ sign in all the terms. Therefore in all the cases the signs must alternate, by supposing simply $p = 1$. The general term of the numerators will be therefore

$$\begin{aligned} n^x -& (x-n+1)n^{x-n} + \frac{(x-2n+2)(x-2n+1)}{1.2}n^{x-2n} \\ &- \frac{(x-3n+3)(x-3n+2)(x-3n+1)}{1.2.3}n^{x-3n} \\ &+ \frac{(x-4n+4)(x-4n+3)(x-4n+2)(x-4n+1)}{1.2.3.4}n^{x-4n} + \text{\&c.} \end{aligned}$$

One will have therefore finally the probability contrary to the event demanded $=$

$$\frac{(n^x - (x-n+1)n^{x-n} + \frac{(x-2n+2)(x-2n+1)}{1.2}n^{x-2n} - \frac{(x-3n+3)(x-3n+2)(x-3n+1)}{1.2.3}n^{x-3n} \text{ \&c.})}{n^x}$$

The sought probability will be therefore

$$\frac{(x-n+1)n^{x-n} - \frac{(x-2n+2)(x-2n+1)}{1.2}n^{x-2n} + \frac{(x-3n+3)(x-3n+2)(x-3n+1)}{1.2.3}n^{x-3n} \text{ \&c.}}{n^x}$$

§ 19. Let for example $n = 2$, the contrary probability will be

$$\frac{2^x - (x-1)2^{x-2} + \frac{(x-2)(x-3)}{1.2}2^{x-4} - \frac{(x-3)(x-4)(x-5)}{1.2.3}2^{x-6} + \frac{(x-4)(x-5)(x-6)(x-7)}{1.2.3.4}2^{x-8} + \text{\&c.}}{2^x}$$

$= \frac{x+1}{2^x}$, therefore the probability demanded is $= 1 - \frac{x+1}{2^x}$ as Mr. de la Place finds it in the memoir cited. This result emanated immediately from that which we have said § 12. that the sequence of the numerators was that of the natural numbers, but I have taken the form of recurrent series of the second degree, because this form was the base of the analogy which must serve to resolve the general problem.

§ 20. One would have been able to resolve the problem in another way by leaving the terms which express the contrary probability separate, without confounding them by addition. For example for the case of $n = 2$, one would have had the following table:

Coups	Probabilities
2	$1 - \frac{1}{2^2}$
3	$1 - \frac{2}{2^2}$
4	$1 - \frac{3}{2^2} + \frac{1}{2^4}$
5	$1 - \frac{4}{2^2} + \frac{3}{2^4}$
6	$1 - \frac{5}{2^2} + \frac{6}{2^4} - \frac{1}{2^6}$
7	$1 - \frac{6}{2^2} + \frac{10}{2^4} - \frac{4}{2^6}$
8	$1 - \frac{7}{2^2} + \frac{15}{2^4} - \frac{10}{2^6} + \frac{1}{2^8}$
9	$1 - \frac{8}{2^2} + \frac{21}{2^4} - \frac{20}{2^6} + \frac{1}{2^8} +$ &c.

One sees that the numerators of the second vertical column are the natural numbers, those of the third the triangular numbers, those of the fourth the pyramidal numbers &c., & this consideration gives the same general term as above. One will find the same laws for the case of $3,\ 4,\ 5 \ldots n$ faces, whence one will draw the same general result by the other other method, this which confirms the goodness of the solution.