The Poisson Distribution

Extracted from
Researches sur la probabilité des jugements en matière criminelle et en matière civile *

Simon Denis Poisson

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§73. I return actually to the case where the chances p and q of the two events E and F are constants, I am going to consider the probability that in a number μ or $m+n$ trials, E will arrive at least m times and F at most n times. This probability will be the sum of the first m terms of the development of $(p+q)^\mu$, ordered according to the increasing powers of q; so that by designating it by P, one will have (n$^{|}$*textrmo* 15)

$$P = p^\mu + \mu p^{\mu-1} q + \frac{\mu.\mu-1}{1.2} p^{\mu-2} q^2 + \cdots + \frac{\mu.\mu-1\ldots\mu-n+1}{1.2.3\ldots n} p^{\mu-n} q^n : \qquad (8)$$

but under this form, it will be difficult to transform it into an integral to which one is able to next apply the method of n$^{|}$*textrmo* 67, when m and n will be very great numbers. We seek therefore first another expression of P which agrees better to this object.

One is able also to say that the composite event of which the concern consists in this that F will not arrive more than n times in the μ trials. By considering it in this manner, I will name it G. It will be able to take place in the $n+1$ following cases:

1°. If the first m trials bring forth each the event E; because then, there will no longer remain but $\mu - m$ or n trials that will not be able to bring forth F more than n times. The probability of this first case will be p^m.

2°. If the first $m+1$ trials bring forth m times E and one time F, without that F occupy the last rank, a necessary condition in order that this second case not return to the first. It is evident that the $n-1$ following trials being able to bring forth F only $n-1$ times more, this event will not arrive more than n times in the totality of trials. The probability of the arrival of E m times and of F one time, which would occupy a determined rank, being $p^m q$, and this rank being able to be the first m, it follows that the probability of the second case favorable to G, will be $mp^m q$.

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3°. If the first $m+2$ trials bring forth E m times and F twice, without that F occupies the second rank, this which is necessary and sufficient in order that this third case neither return to the first, nor to the second. The probability of the arrival of E m times and of F twice, in determined ranks, would be $p^m q^n$; by taking two-by-two the first $m+1$ ranks in order to place F, one has $\frac{1}{2}m(m+1)$ different combinations; the probability of the third case favorable to G will have therefore $\frac{1}{2}m(m+1)p^m q^n$ for value.

By continuing thus, one will arrive finally to the $n+1^{\text{st}}$ case, in which the μ trials will bring forth E m times and F n times, without that F occupy the last rank, so that this case return to none of the preceding; and its probability will be

$$\frac{m.m+1.m+2\ldots m+n-1}{1.2.3\ldots n}p^m q^n.$$

These $n+1$ cases being distinct from one another, and presenting all different ways in which the event G is able to take place, its complete probability will be the sum of their respective probabilities (n° 10);

$$P = p^m\left[1+mq+\frac{m.m+1}{1.2}q^2+\frac{m.m+1.m+2}{1.2.3}q^3+\cdots\right.$$
$$\left.+\frac{m.m+1.m+2\ldots m+n-1}{1.2.3\ldots n}q^n\right]; \quad (9)$$

an expression which must coincide with formula (8), but which has the advantage to be able to be transformed easily into definite integrals, of which the numerical values will be able to be calculated by the method of n° 67, with so much more approximation as m and n will be greater numbers.

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§81. In the preceding calculation, we have excluded (n° 78) the case where one of the two chances p and q is very small, which remains to us, consequently, to consider in particular.

I suppose that q is a very small fraction, or that it is the event F which has a very feeble probability. In a very great number μ of trials, the ratio $\frac{n}{\mu}$ of the number of times that F will arrive to this number number μ will be also a very small fraction; by putting $\mu-n$ in the place of m in formula (9), making

$$q\mu = w, \qquad q = \frac{w}{\mu},$$

and neglecting next the fraction $\frac{n}{\mu}$, the quantity contained between the parentheses, in that formula, will become

$$1+w+\frac{w^2}{1.2}+\frac{w^3}{1.2.3}+\cdots+\frac{w^n}{1.2.3\ldots n}.$$

At the same time, one will have

$$p = 1-\frac{w}{\mu}, \quad p^m = \left(1-\frac{w}{\mu}\right)^{\mu}\left(1-\frac{w}{\mu}\right)^{-n};$$

one will be able to replace by the exponential e^{-w}, the first factor of this value of p^m, and to reduce the second to unity; consequently, according to equation (9), we will have, very nearly,

$$P = \left(1 + w + \frac{w^2}{1.2} + \frac{w^3}{1.2.3} + \cdots + \frac{w^n}{1.2.3\ldots n}\right) e^{-w},$$

for the probability that an event of which the chance at each trial is the very small fraction $\frac{w}{\mu}$, will not arrive more than n times in a very great number μ of trials.

In the case of $n = 0$, this value of P is reduced to e^{-w}; there is therefore this probability e^{-w} that the event of which there is concern will not arrive one single time in the number μ of trials, and consequently, the probability $1 - e^{-w}$ that it will arrive at least one time, as one has already seen in n° 8. Since n will no longer be a very small number, the value of P will differ very little from unity, as one sees it, by observing that the preceding expression of P is able to be written under the form

$$P = 1 - \frac{w^{n+1} e^{-w}}{1.2.3\ldots n+1} \left(1 + \frac{w}{n+2} + \frac{w^2}{n+2.n+3} + \text{etc.}\right).$$

If one has, for example, $w = 1$, and if one supposes $n = 10$, the difference $1 - P$ will be very nearly one hundred-millionth, so that it is near certain that an event of which the very feeble chance is $\frac{1}{\mu}$ at each trial, will not arrive more than 10 times, in the number μ of trials.