SUMMA DE ARITHMETICA GEOMETRIA PROPORTIONI ET PROPORTIONALITA
F. 197 R. AND 198 V.

LUCA PACIUOLO, DE BORGO SAN SEPOLCRO (PACIOLI)

INTRODUCTION

Here we present the relevant text[1] of the *Summa* of Pacioli[1] associated with the division of stakes and a translation of it into English. A translation into German appears in the anthology of Ivo Schneider [2] which has been consulted. Footnotes have been placed at the end of each problem.

Two images of the text are readily available. The better is part of the Archimedes Project at ECHO (the European Cultural Heritage Online)[2] and the other through the University of Seville.[3]

TEXT AND ENGLISH TRANSLATION

PROBLEM 1

Una brigata gioca a palla a 60 el giuoco e 10 p̲ caccia.

E fãno posta duc̃ 10. Acade p̲ certi acidẽti che non possano fornire, e̲ l'una p̲te a 50 e l'altra 20. Se dimanda che tocca p̲ parte de la posta. In q̃sto caso o trovato diverse opinioni si i un lato cõmo in l'altro; e tutte mi parõ frasche certi loro argumenti.

A company plays a ballgame to 60 and each goal is 10.

They stake 10 ducats in all. It happens by certain accidents they are not able to finish; and one party has 50 and the other 20. One asks what portion of the stake is due to each party. For this problem I have found different opinions, going in the one

Date: Venice, 1494.

English paraphrase by Richard J. Pulskamp, Department of Mathematics and Computer Science, Xavier University, OH. Document created July 18, 2009.

[1]An examination of the printed text reveals the difficulty that exists in both reading and reproducing the original. One can certainly choose to display the text as accurately as possible. However, the language and the spelling of it have evolved. Early printers made use of several devices to pack words into a line which can cause problems for the reader. Consequently, one may choose to totally rewrite the text so as to bring it "up to date." This will include inserting letters which are missing due to the use of contractions and abbreviations, modernizing spelling, introducing better punctuation and breaking words that have been joined. In the passages reproduced here a compromise has been to make but minor changes in spelling (for example, "uero" becomes "vero"), to insert an apostrophe when two words have been contracted (for example, "laltro" becomes "l'altro"), and to break the text into sentences corresponding to the translation at the right. Note that the symbol ˜ is an indicator that one or more letters have been suppressed. For example, we have "pãni" instead of "panni" or "q̃sto" instead of "questo," but we have chosen to replace ch̃ by che. A symbol which we render as p̲ is used typically as an abbreviation for "per."

[2]`http://echo.mpiwg-berlin.mpg.de/content`, images 409 and 410.

[3]`http://fondosdigitales.us.es/`, images 413 and 414.

Ma la veritae questa ch'io diro e la retta via.

Dico che poi sequire in 3 modi.

Prima die cõsiderare quante caccie al piu fra l'una e l'altra p̲te si possino fare; che siran 11 cioe quando sonno a lª 50 p̲ uno. Ora vedi quelli da 50 che parte hano de tutte queste cacce che hanno li $\frac{5}{11}$ e quelli da 20 hanno li $\frac{2}{11}$. Donca di che l'una p̲te deve tirar p̲ $\frac{5}{11}$ e l'altra parte p̲ $\frac{2}{11}$. Sũmati fãno $\frac{7}{11}$. Poi di $\frac{7}{11}$ guadagna 10; che tocca a $\frac{5}{11}$ e che a $\frac{2}{11}$? Che a quel da 50 virra $7\frac{1}{7}$ e a 20.$2\frac{6}{9}$. Fatta.

Un'altro mõ sie simile. Cioe in tutto possan fare 110. Vedi che parte sia 50 de questo che harai ut supra $\frac{5}{11}$ e cosi 20 sira $\frac{2}{11}$. E sequi ut supra.

El terço brevissimo sia, che sũmi ĩsiemi quello che hano fra tutte doi le parti: cioe 50 e 20 fa 70. E questo e partitor e di 70 guadagna 10. Che tocca a 50 e che a 20? etc.

E cosi farai d̃ una corsa a pede o a cavallo vedẽdo quãti miglia a fatto per uno etc. E similiter quando giocano a la morra a 10 o 5 deta, che l'una parte hara 9 e l'altra 7. etc.

O vero quando giocano a l'arco a tanti colpi chi prima giõgni habia el pregio e cetera.

E guarda di sopra in quello de la palla, che tu non dicesse poi che l'una parte a li $\frac{5}{11}$ di cio che possa no fare i tutto due tirare li $\frac{5}{11}$ d'la posta; che nõ veria, perche avãª d̃ i postª nõ serebboº ne d̃ lũo ne d̃ l'altro p̲ cha 50 toccaria $4\frac{6}{11}$ e a 20. $2\frac{9}{11}$ che sõ $7\frac{4}{11}$ e li $2\frac{7}{11}$ verrieno a eẽr di q̃ilo che tene li pãni iõ etc.

direction to the other; all appear to me incoherent in their arguments. But the truth is what I will say, together with the right way.

I say then pursue these in 3 ways.

First consider how many goals at most are able to be made between the one and the other party; this will be 11, that is, when they are at 50 p(oints) each. Now you see what this part with 50 has of all of these goals; this gives $\frac{5}{11}$; and 20 gives $\frac{2}{11}$. Therefore from this the one p(ar)ty must take $\frac{5}{11}$ p(arts) and the other party $\frac{2}{11}$ p(arts). The sum makes $\frac{7}{11}$. Then $\frac{7}{11}$ is worth 10. What is due with $\frac{5}{11}$ and what with $\frac{2}{11}$? Thus to the one with 50 will come $7\frac{1}{7}$ and $2\frac{6}{7}$ to 20. Finished.

Another way is similar: That is, in all they are able to make 110. See what part of these is 50; you will find, as above, $\frac{5}{11}$ and thus 20 will be $\frac{2}{11}$. And do further as above.

The third is a very short way, that you add what both parties have together: that is, 50 and 20, makes 70. And this is the divisor, by which 70 is worth 10. What is due with 50 and what with 20? etc.

And so you will do with a race on foot or on horseback, if you know how many miles each has made etc. And similarly with Morra[4] with 10 or with 5 fingers, if the one party has 9 and the other 7.

Indeed also, when they shoot bows; who obtains the prize when he first attains so many wins. etc.

And pay attention in the above to that of the ballgame, that you don't say then, that the party with the $\frac{5}{11}$ against those able to be made in all may take only $\frac{5}{11}$ of the stake; that would not be correct, because what remains of the rest of the stake will go to neither the one nor the other. There will come $4\frac{6}{11}$ to the one with 50 points and $2\frac{9}{11}$ to the one with 20 points,[5] which makes $7\frac{4}{11}$, and the remainder $2\frac{7}{11}$ will be for the cloak room attendant.

Ne ãche dire cõmo alcũi che si sõdão a la morra cõmo se doi fãno a 5 d̃ta che lũ habia 4 l'altro 3 e dicano torniamo ĩ drieto 1. si che lũ hara 2. l'altro 3. che nõ e el dovere: p̲ che colui butta el $\frac{1}{3}$ de cio che a e colui butta el $\frac{1}{4}$ siche non buttano a un modo.

E cosi dicono alcuni che si buti 20 da ciascuna p̲te l'uno hara nulla l'altro 30 e poi dicano che quel da 30 torra la $\frac{1}{2}$ de la posta, che 5, per che a la $\frac{1}{2}$ del gioco e li altri 5 dividerãno ĩ meçço fra loro: si che quel da 50 haria $7\frac{1}{2}$ e l'altro $2\frac{1}{2}$ che non screbbe giusto p̲ la ragiõ gia ditta inãçe etc.

Also do not say as some, who cite the game of Morra and say, if in one game with 5 fingers one has 4, the other 3, "we deduct 1," so that then the one has 2 and the other 3;[6] then this is not fair, because the one resigns $\frac{1}{3}$ and the other $\frac{1}{4}$ of his claim, so that they do not resign to the same extent.

So say some,[7] each should resign 20 points; then the one will have none, the other 30 points; and then they say, that the one with 30 points will receive $\frac{1}{2}$ of the stake, which is 5, because they have the half of the points for the win of the complete game, and of the remaining 5 each will take the half; so that the one with 50 points had $7\frac{1}{2}$ and the other $2\frac{1}{2}$, which would not be just according to the aforesaid rationale.

PROBLEM 2

Tre fãno a balestrare; chi prima fa 6 colpi meglio quello tiri. E fãno posta fra tutti duc̃ 10. Quãdo el primo ha 4 colpi el 2º 3 colpi el 3º 2 colpi no vogliã far piu e da cordo vogliano p̲ tire la posta. Dimando q̃ti ne toca p̲ uno?

Fa cosi: Prĩa vedi quãti colpi possano fare al piu fratutti 3 loro. Troverai che non ne possõ far piu che 16; per che esser po, che tutti 3 ognuno habia 5 colpi e uno Poi sene fara p̲ averne 6 che tiri la posta: Donca fãno 16 al piu di quali 16 el pº ha 4, che e el $\frac{1}{4}$; donca deve havere il $\frac{1}{4}$ de la posta: cioe de 10 ₰[8] che son $2\frac{1}{2}$. El 2º a 3 colpi che sõ li $\frac{3}{16}$ de cio che possã fare. Dõca

Three compete at crossbow; who first makes 6 best goals wins. They stake altogether 10 ducats. When the first has 4 goals, the second 3, and the third 2, they intend to continue no longer and agree to share the stake. I ask, to how much is each entitled?

Do thus: First of all look how many best goals at most all three together are able to make. You will find that they are able to make no more than 16; because it could be that all three each have 5 goals and one next will be made in order to have 6 which wins the stake. Thus they make at most 16; of these 16 the first has 4, which makes $\frac{1}{4}$; thus he has $\frac{1}{4}$ of the stake: that is of

[4]Morra is an ancient game of luck played yet today. Both players simultaneously stretch however many fingers on one hand on a table as they wish and shout simultaneously a number between 2 and 10. One finger or the closed fist are regarded as the same, namely 1. Whoever has guessed the correct sum of fingers, wins one point. If both have declared the same number or both are in error, no one receives a point.

[5]He must mean $1\frac{9}{11}$, because $\frac{2}{11}$ of 10 ducats results in $1\frac{9}{11}$ ducats. The subsequent text corrects the error.

[6]Actually the sequence must be: "…that one has 3, the other 2".

[7]Pacioli now refers back to the original problem.

hara li $\frac{3}{16}$ de la posta che sõ $1\frac{7}{8}$. El 3º ha 2 che son li $\frac{2}{16}$. Donca li tocca li $\frac{2}{16}$ cioe $\frac{1}{8}$ de la posta che sõ $1\frac{1}{4}$. Quali sũma tutti i siemi: cioe $2\frac{1}{2}.1\frac{7}{8}.1\frac{1}{4}$. Fãno $5\frac{5}{8}$ e questo cava de 10 cioe de tutta la posta. Resta $4\frac{3}{8}$.

10 Denari, which is $2\frac{1}{2}$. The second has 3 goals, which amounts to $\frac{3}{16}$ of what he could make. Therefore he will have $\frac{3}{16}$ of the stake, which is $1\frac{7}{8}$ ducats. The third has two, which is $\frac{2}{16}$. Therefore to him is $\frac{2}{16}$, that is $\frac{1}{8}$ of the stake, which is $1\frac{1}{4}$. All these you add up: that is $2\frac{1}{2}, 1\frac{7}{8}, 1\frac{1}{4}$. They make $5\frac{5}{8}$, and you subtract this from 10, that is from the whole stake. There remains $4\frac{3}{8}$.

E questo mo se deve dividere cõmo cõpagnia e dire l'uno a 4 l'altro 3 l'altro 2 colpi o vero $\frac{1}{4}, \frac{3}{16}, \frac{1}{8}$ e ãno a partire $4\frac{3}{8}$. Che tocca p̱ uno. O̱pa! Harai che al pº toccara $1\frac{17}{18}$, al 2º $1\frac{11}{24}$ al 3º $\frac{35}{36}$. Fatta.

One must now divide this as in a trading-company[9] and say: one has 4 and the other 3 and the other 2 goals, indeed $\frac{1}{4}, \frac{3}{16}$ and $\frac{1}{8}$, and they have to divide $4\frac{3}{8}$. How much for each? Work! You will find, that the first will amount to $1\frac{17}{18}$, the second to $1\frac{11}{24}$ and the third to $\frac{35}{36}$. Finished.

La pua: giongni insiemi q̃llo che li tocca pª cõ quello che li tocco. Poi convensar 10. E pero dirai, ch'el primo in tutto nebbe $2\frac{1}{2}$ e $1\frac{17}{18}$ che son $4\frac{4}{9}$ al secõdo $1\frac{7}{8}$ e $1\frac{11}{24}$, che sõ $3\frac{1}{3}$; el 3º. $1\frac{1}{4}$ e $\frac{35}{36}$ che son $2\frac{2}{9}$; che sta bene.

The proof: Join together, what they are due first with what is due to them later. It must make 10. Consequently you will say, that the first received altogether of it $2\frac{1}{2}$ and $1\frac{17}{18}$, which is $4\frac{4}{9}$; the second $1\frac{7}{8}$ and $1\frac{11}{24}$, which is $3\frac{1}{3}$; the third $1\frac{1}{4}$ and $\frac{35}{36}$, which is $2\frac{2}{9}$; which is well.

Ora questo medesimo re dara a un tratto per via d̃ compagnia cõmo qui in quelli de la palla diciamo. Siche qui la fai in doi volte: li la fai i una. Pero che a dire: 3 fan compagnia; lũ mette 4, el 2º, 3. el 3º, 2, e hano a partire 10. Che tocca p̱ uno. Opera! Harai cõmo e detto etc.

Now it will give this same thing immediately according to the way of a trading company, as we say in that of the ball-game. What you make there in two steps, here you make in one. There is however to say: 3 establish a trading company; the one puts 4 shares, the second 3 and the third 2, and they have to divide 10. What is due to each? Work! You find it, as it has been said already etc.

Pacioli's solution

Pacioli belabors his solution. In the end, each player receives a share of the stake proportional to his number of points with respect to the total number of points achieved. This holds regardless of the number of players. Thus, suppose, for example, there are two players. Player A has a points and Player B b. Player A receives $\frac{a}{a+b}$ of the stake and Player B $\frac{b}{a+b}$.

[8] Schneider [2] notes here that Pacioli uses an abbreviation for the Denarius which we render as $\mathcal{D}$, although previously the use of 10 ducats was stipulated. We accept this interpretation.

[9] In the late-medieval Italian cities there was already the form of business, the so-called *Compagnia*, by which the profits or losses that these companies made were shared proportionally according to the shares of the parties.

The number of points required to win as well as the maximal number of points that may be achieved among the players are completely irrelevant.

References

[1] Fra Luca Pacioli. *Summa de Arithmetica, Geometria, Proportioni et Proportionalita*. Venice, 1494.
[2] Ivo Schneider. *Die Entwicklung der Wahrscheinlichkeitstheorie von den Anfängen bis 1933*. Wissenschaftliche Buchgesellschaft, Darmstadt, 1988.