Extrait d'un mémoire sur la probabilité des erreurs des tribunaux*

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In the session of the Academy of 12 June 1834, Mr. Ostrogradsky has read a memoir on the probability of the errors of tribunals, where he has considered the case of unequal veracity of the judges, which case is the one of all the tribunals.

By supposing that the limits of the veracity of each judge are known, the author gives the analytic formulas, relative to the different cases which are able to present themselves, for the probability of the error of a tribunal composed of a given number of judges. He supposes first that one knows nominally the judges who decide affirmatively a litigious question, and consequently that one knows also those who vote against; he examines also the case, where there is only one part of judges of one same opinion, and one part of those of contrary opinion, who are known nominally; next he indicates the means in order to determine the limits of the veracity of the judges according to experience, and he terminates his memoir by the consideration of the case of equal veracity, a case that Condorcet and Laplace have already treated.

Mr. Ostrogradsky, under the hypothesis that the veracities of the judges are found all comprehended between the same limits, finds that the probability of the error to fear depends only on the majority, that is on the difference between the numbers of judges of opposes opinions. Laplace and Condorcet have thought that a similar result would be contrary to the indication of the simple natural ratio; but Mr. Ostrogradsky does not surrender himself to the authority of these celebrated geometers, he claims that the result of his analysis has nothing which is able to offend good sense, and after having cited the passage where Laplace speaks of the extreme difference between the probability of the error of a judgment rendered by unanimity, by a tribunal of twelve judges, and the probability of the error of judgment rendered by the majority of 12 votes by a tribunal of two hundred twelve judges,[1] Mr. Ostrogradsky says:

> "In order to have less to discuss, we compare a single judge pronouncing himself affirmatively on a question, in a tribunal A of three judges, of

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[1] *Théorie analytique des Probabilités*, introduction, page LXXXIX and the following: first supplement page 29.

whom two pronounce themselves affirmatively, the third negatively. Without changing anything in the question, one is able to replace the sole judge by a tribunal B of three judges, of whom one affirms, and the opinions of the two others are unknown. We will be able, relative to tribunal B, to make the following three hypotheses:

1. The two judges with unknown opinions are of the same opinion as the first.
2. One of the two shares the opinion of the first, and the other does not share it.
3. Both contradict the first.

"The second hypothesis is exactly in the case of the tribunal A, the first is to the advantage of the tribunal B, or that which reverts to the same, to the advantage of a single judge, and the last, on the contrary, is to the advantage of the tribunal A; now, I see not why the first hypothesis would increase the probability of a single judge, no more than the last reduce it,

"In a tribunal of two hundred twelve judges, the majority of twelve votes show that one hundred twelve judges are in accord, but in a tribunal of twelve judges who pronounce with unanimity, one is certain only of the accord of twelve votes, and one does not know if, by bringing the number of judges to twelve hundred twelve, the two hundred that one would have added, would not be of contrary opinion to the first twelve judges.

"One accords the greatest confidence to an impartial and enlightened tribunal composed of twelve judges, who would decide unanimously; but if the tribunal were composed of two hundred twelve judges, of whom one would know the opinion of only twelve, agreed among them, one would expect in order to be settled, that the opinion of the majority is known. However, not knowing the opinion of two hundred judges, we are just in the case of the tribunal of twelve judges, who decide unanimously. Whence comes the great difference in the confidence which we accord to the same number of judges, equally truthful, and in the same situation relative to us? This difference, there is not at all; we are led into error, for want of having sufficiently studied the matter thoroughly. I will permit myself an observation.

"The decision of a tribunal of one thousand judges, for example, of whom five hundred decide a question affirmatively, and five hundred others decide it negatively, is null: the five hundred positive votes are destroyed by the five hundred negative votes, as would be destroyed two equal forces, contraries, and applied to the same point. We add one more affirmative vote; according to the received opinion, this additional vote will be reduced by the votes which destroy themselves, or will reduce the weight of five hundred affirmative votes which one precedes, for, having the additional vote, the five hundred affirmative votes destroyed the five hundred negative votes, and after the addition, they no longer destroy them, the negative votes prevail over the positives, since the difference of the five hundred one positive votes and five hundred negative votes is less than one vote. One senses that the weakening of the additional vote by the votes which, in some kind, no longer exist, would not know how to be admitted; one senses equally, that one more vote is able to add only to the force of those, to which one has added it.

“If it is true that one is brought to consider as null the decision of a numerous tribunal, rendered by a very feeble majority, and that on the contrary, one gives a great weight to a unanimous decision of the tribunal composed of a small number of judges, I believe that that which we sustain is rather a prejudice, than the good sense and the exact consideration of the matter.

“Besides, that which I just said decides not at all between the formula of Laplace and of Condorcet, and that which I propose in order to replace it; but I have an objection to make against the analysis of these celebrated geometers, an objection which must decide the thing.

“I have admitted, in this memoir, the same principles of analysis of the probabilities as those that Laplace and Condorcet have followed; it will not be therefore on these principles that will sustain my attention, but on the manner of employing them; now, I do not believe that it is permitted in the questions of the tribunals to represent the veracities of all the judges by one same letter. These veracities have each the same limits, but they must go from the first limit to the second, independently from one another; it will be necessary, consequently, to designate them each by a different letter, and one will have, instead of one alone, as many integrals to consider as there are judges.

“We suppose that the veracity of each judge is able to have only a certain determined number of values, for example, that each veracity is able to be only $\frac{1}{2}$ or $\frac{3}{4}$ or 1. According to Laplace and Condorcet, by taking for a veracity any one of the values that it is able to have, one must give, at the same time, the same value to all the others, so that, in the case of which there is concern, there would be only three combinations of the veracities, namely: all the veracities $= \frac{1}{2}$, or $= \frac{3}{4}$, or $= 1$. Now, it seems to me, that one must make all possible combinations of them, so that, instead of three combinations, there would be 3^n, n being the number of judges.”—

We will terminate this extract by the citation of some formulas contained in the memoir of Mr. Ostrogradsky.

A tribunal being composed of $m + n$ judges, m judges condemn an accused, and n acquit him one seeks the probability of the error in the case of the condemnation. Let x_1, x_2, x_3, $\ldots x_m$ be the veracities of the judges who condemn, and y_1, y_2, y_3, $\ldots y_n$ the veracities of those who acquit. If the veracities would be able to have only the preceding values, the sought probability would be

$$\frac{(1-x_1)(1-x_2)\cdots(1-x_m)y_1y_2\cdots y_n}{(1-x_1)(1-x_2)\cdots(1-x_m)y_1y_2\cdots y_n + x_1x_2\cdots x_m(1-y_1)(1-y_2)\cdots(1-y_n)};$$

but x_1, x_2, $\ldots x_m$, y_1, y_2, $\ldots y_n$ having each an infinity of different values, it is necessary yet to multiply the preceding expression by the probability

$$\frac{[(1-x_1)(1-x_2)\cdots(1-x_m)y_1y_2\cdots y_n + x_1x_2\cdots x_m(1-y_1)(1-y_2)\cdots(1-y_n)]dx_1dx_2\cdots dx_m dy_1dy_2\cdots dy_n}{\int[(1-x_1)(1-x_2)\cdots(1-x_m)y_1y_2\cdots y_n + x_1x_2\cdots x_m(1-y_1)(1-y_2)\cdots(1-y_n)]dx_1dx_2\cdots dx_m dy_1dy_2\cdots dy_n}$$

of the simultaneous existence of the veracities x_1, x_2, $\cdots$ x_m, y_1, y_2, $\cdots$ y_n this which will give

$$\frac{(1-x_1)(1-x_2)\cdots(1-x_m)y_1y_2\cdots y_n dx_1dx_2\cdots dx_m dy_1dy_2\cdots dy_n}{\int[(1-x_1)(1-x_2)\cdots(1-x_m)y_1y_2\cdots y_n + x_1x_2\cdots x_m(1-y_1)(1-y_2)\cdots(1-y_n)]dx_1dx_2\cdots dx_m dy_1dy_2\cdots dy_n}$$

for the portion of the probability of error of the tribunal, a portion due to the veracities x_1, x_2, $\cdots$ x_m, y_1, y_2, $\cdots$ y_n alone. The total probability will be evidently the sum of all the partial probabilities, relative to all the possible veracities. This sum is

$$\frac{\int(1-x_1)(1-x_2)\cdots(1-x_m)y_1y_2\cdots y_n dx_1dx_2\cdots dx_m dy_1dy_2\cdots dy_n}{\int[(1-x_1)(1-x_2)\cdots(1-x_m)y_1y_2\cdots y_n + x_1x_2\cdots x_m(1-y_1)(1-y_2)\cdots(1-y_n)]dx_1dx_2\cdots dx_m dy_1dy_2\cdots dy_n}$$

the integral of the numerator and of the denominator will be relative to all the values of x_1, x_2, $\cdots$ x_m, y_1, y_2, $\cdots$ y_n.

We designate the inferior limits of the quantities x_1, x_2, $\cdots$ x_m, y_1, y_2, $\cdots$ y_n respectively by x'_1, x'_2, $\cdots$ x'_m, y'_1, y'_2, $\cdots$ y'_n, and the superior limits of these same quantities respectively by x''_1, x''_2, $\cdots$ x''_m, y''_1, y''_2, $\cdots$ y''_n; the probability of the error of the tribunal will become

$$\frac{1}{1+\frac{x''_1+x'_1}{2-x''_1-x'_1}\cdot\frac{x''_2+x'_2}{2-x''_2-x'_2}\cdots\frac{x''_m+x'_m}{2-x''_m-x'_m}\cdot\frac{2-y''_1-y'_1}{y''_1+y'_1}\cdot\frac{2-y''_2-y'_2}{y''_2+y'_2}\cdots\frac{2-y''_n-y'_n}{y''_n+y'_n}}$$

It is remarkable that the preceding probability depends only on the sums of the extreme values of the veracities, that is on $x''_1+x'_1$, $x''_2+x'_2$, $\cdots$ Thus these sums remain the same whatever be besides x''_1, x'_1, x''_2, x'_2, $\cdots$; the probability of the error will not vary, if all the quantities $x''_1+x'_1$, $x''_2+x'_2$, $\cdots$ $y''_1+y'_1$, $y''_2+y'_2$, $\cdots$ are equal among themselves, and, by designating by z their common value, the probability of the error will be reduced to $\frac{1}{1+\left(\frac{z}{2-z}\right)^{m-n}}$. This case comprehends evidently the one, where the limits of all the veracities would be the same; one sees by the preceding equation that the probability of the error of the tribunal depends only on the difference $m-n$ between the number of judges who condemn and the one of the judges who acquit, that is, it depends only on the majority. By making $z=1$, the preceding probability will be reduced to $\frac{1}{2}$. The same fraction $\frac{1}{2}$ will represent also the probability of the validity of the judgment; thus in the case where the sum of the limits of the veracities of each judge is equal to unity, one is in a complete indecision on the value of a decision; it would revert to the same to give up to chance the lot of the accused, provided that one equalizes the chances for condemnation and for absolution. The decision of the tribunal will acquire a value only in the case where the sum of the extreme veracities overtake unity, and the more this sum approaches the superior limit 2, the more one must expect to see only decisions conformed to the truth.

We suppose in the last expression $z = \frac{3}{2}$; we will have $\frac{1}{1+3^{m-n}}$ for the probability of the error relative to this supposition: by admitting $\frac{1}{2}$ and 1 for the limits of the veracities, one satisfies evidently the equation $z = \frac{3}{2}$; therefore $\frac{1}{1+3^{m-n}}$ is the expression of the probability of the error of the tribunal, when the veracities of the judges are comprehended between the limits $\frac{1}{2}$ and 1. The formula of Laplace relative to this case is

$$\frac{\int_0^{\frac{1}{2}} x^m (1-x)^n \, dx}{\int_0^1 x^m (1-x)^n \, dx};$$

it differs much from the preceding.

We consider anew a tribunal of $m+n$ judges, of which m condemn and n acquit an accused; but one knows not who are the judges who condemn, and consequently one does not know who are those who absolve. Let $x_1, x_2, \ldots x_{n+m}$ be the veracities of the judges; x_1 is comprehended between the limits x_1' and x_1'', x_2 between the limits x_2' and x_2'', and so forth. We make

$$V = [x_1 + (1-x_1)y][x_2 + (1-x_2)y] \cdots [x_{m+n} + (1-x_{m+n})y];$$

the coefficient of y^m in the development of V expresses the probability that the tribunal is partitioned into two parts, the one of m and the other of n judges, and that the veracity is on the side of the n judges. We designate by P this coefficient. The coefficient Q of y^n in the development of V will express the probability of the same partition of the tribunal in the case, where the veracity will be on the side of the m judges; therefore

$$\frac{P}{P+Q}$$

is the probability of the error of the m judges under the hypothesis that each veracity has only a single value. But the veracities having each an infinity of different values, the probability that they are precisely equal to $x_1, x_2, \cdots, x_{n+m}$ is

$$\frac{(P+Q) dx_1 dx_2 \ldots dx_{n+m}}{\int (P+Q) dx_1 dx_2 \ldots dx_{n+m}};$$

the integral must be extended to all the values of $x_1, x_2, \cdots, x_{n+m}$.

By multiplying this probability by the preceding, one will obtain

$$\frac{P dx_1 dx_2 \ldots dx_{n+m}}{\int (P+Q) dx_1 dx_2 \ldots dx_{n+m}};$$

for the portion of the probability of the error of m judges, due to the veracities $x_1, x_2, \cdots, x_{n+m}$; each other combination of the veracities will furnish a similar portion; the sum

$$\frac{\int P dx_1 dx_2 \ldots dx_{n+m}}{\int (P+Q) dx_1 dx_2 \ldots dx_{n+m}};$$

of all the portions will be the probability sought of the error of m judges.

In order to effect conveniently the indicated integrations, we remark that by making $y = e^{x\sqrt{-1}}$, we will have

$$P = \frac{1}{2\pi}\int_{-\pi}^{+\pi} Ve^{-mx\sqrt{-1}}dx$$
$$Q = \frac{1}{2\pi}\int_{-\pi}^{+\pi} Ve^{-nx\sqrt{-1}}dx;$$

therefore the probability of the error will become

$$\frac{\int_{-\pi}^{+\pi} e^{-mx\sqrt{-1}} \int V dx_1 dx_2 \cdots dx_{m+n}}{\int_{-\pi}^{+\pi}(e^{-mx\sqrt{-1}} + e^{-nx\sqrt{-1}})dx \int V dx_1 dx_2 \cdots dx_{m+n}}$$

Now, it is evident that the numerator of this fraction is the coefficient of y^m in the development of

$$\frac{1}{2\pi}\int V dx_1 dx_2 \cdots dx_{m+n} = \frac{1}{2\pi} \cdot \frac{(x_1'' - x_1')(x_2'' - x_2') \cdots (x_{n+m}'' - x_{n+m}')}{2^{n+m}} [x_1'' + x_1' + y(2 - x_1'' - x_1')] [x_2'' + x_2' + y(2 - x_2'' - x_2')] \cdots [x_{n+m}'' + x_{n+m}' + y(2 - x_{n+m}'' - x_{n+m}')]$$

and the denominator is the sum of the coefficients of y^m and of y^n in the same development. Therefore, by designating by X the coefficient of y^m in the development of the product

$$[x_1'' + x_1' + y(2 - x_1'' - x_1')][x_2'' + x_2' + y(2 - x_2' - x_2')] \cdots\cdots\cdots [x_{m+n}'' + x_{m+n}' + y(2 - x_{m+n}'' - x_{m+n}')]$$

and by Y the coefficient of y^n in the same development, we will have, for the probability of the error of the judgment rendered by the majority of $m - n$ votes, the following expression:

$$\frac{X}{X+Y}.$$

The fraction

$$\frac{Y}{X+Y}$$

will express the probability of the validity of judgment. One sees that the preceding probabilities depend only on the sums $x_1''+x_1'$, $x_2''+x_2'$, $x_3''+x_3'$, $\cdots$ $x_{m+n}''+x_{m+n}'$ of the extreme veracities, and if one makes $x_1''+x_1' = x_2''+x_2' = \cdots = x_{m+n}''+x_{m+n}' = z$, one will find

$$X = \frac{1 \cdot 2 \cdots (m+n)}{1 \cdot 2 \cdots m \cdot 1 \cdot 2 \cdots n} z^n (2-z)^m$$
$$Y = \frac{1 \cdot 2 \cdots (m+n)}{1 \cdot 2 \cdots m \cdot 1 \cdot 2 \cdots n} z^m (2-z)^n$$

Therefore the probability of error becomes

$$\frac{1}{1+\left(\frac{z}{2-z}\right)^{m-n}}.$$

This expression coincides, as this must be, with that which one has found for the case, where one knows nominally the judges voting for, and the judges voting counter.