Sur une question des probabilités. Extrait.*

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1. The question which is going to occupy us is one of the fundamental problems of this branch of analysis of hazards which, by departing from phenomena, goes back to the causes. One could enunciate it in different ways: here is the simplest.

A vase contains white and black marbles of which one knows the total number, but one does not know that which there is of each color. One removes from it a certain number and, after having counted among those the whites and blacks and having put them back into the vase, one demands the probability that the total of the white marbles will not deviate from some limits as one will wish to assign. Or rather, one demands the relation between the probability and the limits of which there is concern.

In order to imagine the importance of this question, let one be set in the place of one who would be charged to receive a great number of objects subject to certain conditions, and who, in order to be assured of these conditions, must give some time to each object. The suppliers of the army have often to fulfill charges of this kind. For them the marbles contained in the vase will represent the objects to receive, the whites for example the objects which, fulfilling the requisite conditions, are acceptable, and the blacks those which are not. The drawing of a certain number of objects, in order to be assured of their color, will return to the review of one part of the objects to receive, in order to recognize the quality of them. One will fix this part to five, six or seven percent that one will take at random out of the total; next, after having recognized and counted that which is able to be received, one will determine the probability, that the total of the acceptable objects does not deviate from the limits that one will be able to assign in advance. This determination will be made as if it concerned the white and black marbles in a vase. By taking conveniently as much for the limits as for the number of objects submitted to the review, the probability of which there is concern could differ from certitude as little as one wishes.

Thus, the question that we ourselves are proposed being to resolve, a supplier could serve himself with it in order to reduce, to about the twentieth part, a mechanical and most often very fatiguing work, as the review of a very great number of sacks of flour or of pieces of cloth. Seeing the importance of a similar reduction, it is astonishing that the question proper to bring it about has not been conveniently treated; for the solutions

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. October 23, 2011 that we have of them are not very exact and not very conformed to the principles of the analysis of hazards.

2. We enter into the matter, and in order to set us to carry it to the majority of readers, we will serve ourselves in this extract, only with the most elementary analysis. We will begin with a question different from that which we ourselves have proposed and incomparably simpler.

One is certain that a vase contains a given number of white and black marbles without mixing any other color. One does not know absolutely the proportion of the two colors. The vase is able to contain only some white marbles or only some black marbles, or the one and other color and in a ratio that we know nothing of it. But the total is known. One is equally certain that one will withdraw from the vase, or that one has already withdrawn from it, a given number of marbles that we will designate by l. One demands the probability that in this number l there will be n white marbles and m black.

This question is resolved by this principle, the simplest, which returns to the same definition of the probability or of its measure. In fact among the l marbles withdrawn, or to withdraw, there is able to be

$$0, \quad 1, \quad 2 \quad ,3, \ldots n, \quad l-1, \quad l$$

white marbles, therefore respectively

$$l, \quad l-1 \quad ,l-2, \quad l-3, \ldots m, \ldots 1, \quad 0$$

black marbles.

All these different hypotheses, in number $l+1$, being equally possible, the probability of each of them, and departing from that which we have seen, will be

$$\frac{1}{l+1}.$$

3. We resolve the same question by another process. The comparison of the two results will be useful to us.

We designate by s the total of marbles in the vase. Out of this number there is able to be

$$0, \quad 1, \quad 2, \quad 3, \ldots s$$

white marbles, therefore respectively

$$s, \quad s-1, \quad s-2, \quad s-3, \ldots 0$$

black marbles. All the $s+1$ hypotheses being equally possible, each will have

$$\frac{1}{s+1}$$

for measure of its probability. Thus, by admitting that there is in the vase x white marbles and y black marbles, that which requires that one have

$$x + y = s,$$

the probability of this hypothesis, as that of each other, will be

$$\frac{1}{s+1}.$$

We suppose now that the hypothesis of which there is concern is certain, that is in the vase there is found effectively x white marbles and y black marbles; and we see the probability that out of l marbles one will withdraw n white and m black.

We partition, by thought, the s marbles of the vase into groups, each of l marbles; there will be, as one knows, by the theory of combinations,

$$\frac{s(s-1)(s-2)\dots(s-l+1)}{1.2.3\dots l}$$

different groups, therefore as many *possible cases*; and as we have no place to believe that one will withdraw one of these groups rather than another, all these cases will be equally possible.

Now in order to have the *favorable* cases, note that you have x white marbles and y black, and that by partitioning the first into groups by n, the last into groups by m marbles, you will have respectively

$$\frac{x(x-1)(x-2)\dots(x-n+1)}{1.2.3\dots n}$$

and

$$\frac{y(y-1)(y-2)\dots(y-m+1)}{1.2.3\dots m}$$

groups. By combining them among themselves, there will come to you

$$\frac{x(x-1)(x-2)\dots(x-n+1)}{1.2.3\dots n}\cdot\frac{y(y-1)(y-2)\dots(y-m+1)}{1.2.3\dots m}$$

groups of n white marbles and of m black marbles. This number is also the one of the favorable cases. You will have therefore

$$\frac{1.2.3\dots l.x(x-1)(x-2)\dots(x-n+1)y(y-1)(y-2)\dots(y-m+1)}{1.2.3\dots n.1.2.3\dots m.s(s-1)(s-2)\dots(s-l+1)}$$

for the sought probability.

For brevity, we will make use of a known notation which serves to represent that which one calls the factorials or powers of the second order. By this notation a product such as

$$z(x-1)(z-2)\dots(z-k+1)$$

is represented by

$$[z]^k.$$

The letter z is able to be any number, but k is necessarily whole. This put, instead of

$$1.2.3\ldots l$$

one will write

$$[l]^l,$$

likewise

$$[x]^n$$

will be the same thing as

$$x(x-1)(x-2)\ldots(x-n+1) \text{ etc.}$$

Hence, the probability of the extraction of n white marbles and of m black marbles, under the hypothesis admitted and supposed certain, will become

$$\frac{[l]^l[x]^n[y]^m}{[n]^n[m]^m[s]^l}.$$

But the hypothesis being only probable, it is necessary to multiply the preceding probability by that of the hypothesis, that is by

$$\frac{1}{s+1}.$$

Then, the product

$$\frac{[l]^l[x]^n[y]^m}{[n]^n[m]^m[s+1]^{l+1}}$$

will be the composite probability, that the vase contains x white marbles and y black marbles and that, out of l marbles that one will withdraw from it, n will be of the first color, and m of the second.

If, now, we substitute for x and y all the positive whole numbers and zero, which satisfy the equation

$$x+y=s,$$

we will have the similar probabilities, relative to all the hypotheses that one is able to make out of the proportion of the white and black marbles in the vase. The sum of these probabilities is evidently the result that we seek, namely the probability, that a vase containing s marbles, as many white as black, and in a ratio completely unknown, out of l marbles that one will draw from it, n will be white and m black. On the other hand, the same probability being the fraction

$$\frac{1}{l+1},$$

the sum of which there comes to be question, will be equal to this fraction. Therefore, if we designate by S_ρ, placed before a function of x and y, a sum of the values of this function, relative to all the x and y which, satisfying the equation

$$x+y=\rho,$$

are positive integers or zero, we will have

$$S_s \frac{[l]^l [x]^n [y]^m}{[n]^n [m]^m [s+1]^{l+1}} = \frac{1}{l+1}.$$

But as under the sums that S_s indicates, there are only variables x and y, we will be able to write the preceding equation under this form

$$\frac{[l]^l}{[n]^n [m]^m [s+1]^{l+1}} S_s [x]^n [y]^m = \frac{1}{l+1}$$

and we will deduce

$$\text{(I)} \qquad S_s [x]^n [y]^m = \frac{[n]^n [m]^m [s+1]^{l+1}}{[l+1]^{l+1}}.$$

One is able to regard this equation as a whole small theorem of the calculus in finite differences. It had been very easy to demonstrate it by the principle of this calculus, but then one would have been able to believe that we ourselves are deviated from elementary analysis.

We will remark in passing that, in order to have made use of the notation of the factorials, and despite that we will serve ourselves in the following with some considerations relative to finite differences, we do not believe to overtake the principles of the simplest algebra. Because the first notions of the factorials, in the same way as the elements of the calculus of finite differences, especially when the concern will be only of rational functions, are able to be carried back to these principles.

In the sum

$$S_s [x]^n [y]^m$$

there are some elements which are zero and which are able to not be counted. The elements of which there is concern are first those which correspond to the values of x, smaller than n; next, those where y is smaller than m. Thus, without changing the value, one is able to understand the sum

$$S_s [x]^n [y]^m$$

only with the values of x and y which, satisfying the equation

$$x + y = s,$$

are respectively greater than $n-1$ and $m-1$. In order that we may serve ourselves with the admitted notation S_s, we replace x and y by $n+x$ and $m+y$: the new quantities x and y will be all those which, satisfying the equation

$$x + y = s - l,$$

remain integers and positive without excepting zero. Thus we will have

$$\text{(I)} \qquad S_s [x]^n [y]^m = S_{s-l} [n+x]^n [m+y]^m = \frac{[n]^n [m]^m [s+1]^{l+1}}{[l+1]^{l+1}}.$$

4. We suppose now that one has withdrawn from the vase l marbles, that one has found, in this number, n whites and m blacks, and that one demands the probability that, in $s-l$ marbles not exited, there is found x white and y black.

This is the question that we ourselves have proposed. We have enunciated it, in the preamble, a little differently, and in a manner more conformed to the spirit of the analysis of hazards. Because, rigorously speaking, the enunciation of the questions of this kind must exclude no possible hypothesis a priori; that is before the fact was observed. But, in demanding that which remains in the vase, after the extraction of the n white marbles and of the m blacks, we make the fact of which there is concern intervene, by excluding the hypothesis relative to the numbers smaller than n, for the first color, and smaller than m, for the second. Now, this diminution of the possible hypotheses impairs the probability of it a priori, that which could be able to lead sometimes, in some cases different from the one that we treat, to some inexact results. In the actual case there is no error to fear; one could even be able to redress the inexactitude over the probabilities of the hypotheses a priori, by admitting that the numbers of white and black marbles, left in the vase, are able to be negatives; but of similar hypotheses they do not appear very natural. Thus, we will seek the probability that the total of the white marbles is x, and y the one of the blacks.[1]

Our problem depends on a known principle, by which some facts supposed certain, or even already observed one reascends to the probability of the hypotheses that one will have made in order to explicate them. The principle of which there is concern, in the particular case where a priori all the hypotheses are equally admissible, returns to that which it follows.

"The probability of a hypothesis is equal to the probability of the fact, drawn from that hypothesis supposed certain, divided by the sum of the similar probabilities relative to all the hypotheses."

We have, for the observed fact, the extraction of n white marbles and of m black. Before it takes place, all the hypotheses that one would have been able to admit on the proportion of the white and black marbles, would have one same probability. (Nr. 3)

$$\frac{1}{s+1}.$$

Thus we are able to serve ourselves with the principle that we just enunciated. Now we have seen (Nr. 2) that the probability of the observed fact under the hypothesis admitted, that is that of the extraction of n white marbles and of m black, by admitting that the vase contains x of the first color in it and y of the second, is

$$\frac{[l]^l[x]^n[y]^m}{[n]^n[m]^m[s]^l}.$$

We divide this probability by the sum

$$S_s\frac{[l]^l[x]^n[y]^m}{[n]^n[m]^m[s]^l} = \frac{[l]^l]}{[n]^n[m]^m[s]^l}S_s[x]^n[y]^m$$

[1]It had been simpler to not change at all the enunciation of the preamble. We have gone otherwise into a particular end which is superfluous to say.

of the similar probabilities, relative to all the hypotheses which are able to explicate the observed fact, we will have the chance of the admitted hypothesis, this chance will be

$$\frac{[x]^n[y]^m}{S_s[x]^n[y]^m}$$

but

$$S_s[x]^n[y]^m = \frac{[n]^n[m]^m[s+1]^{l+1}}{[l+1]^{l+1}}$$

therefore the probability of our hypothesis will become

$$\frac{[l+1]^{l+1}[x]^n[y]^m}{[n]^n[m]^m[s+1]^{l+1}}.$$

If we attribute to the quantities x and y some numerical values, the preceding formula will give the probability of these values. The calculation of it will be very easy by the aid of logarithms. However when the numbers n and m will be very great, the calculation without being awkward, would become fatiguing by its length. One will shorten it by serving oneself with the following formula which we give without demonstration,

$$\begin{aligned}\log[b]^{b-a} =& \frac{(2b+1)\log(2b+1)-(2a+1)\log(2a+1)}{2} \\ &- 0.73532\,44775\,67233\,02286(b-a) \\ &+ \frac{\mu}{12}\left[\frac{1}{2a+1}-\frac{1}{2b+1}\right] \\ &- \frac{7\mu}{360}\left[\frac{1}{(2a+1)^2}-\frac{1}{(2b+1)^2}\right] \\ &+ \frac{31\mu}{1260}\left[\frac{1}{(2a+1)^5}-\frac{1}{(2b+1)^5}\right] \\ &- \frac{127\mu}{1680}\left[\frac{1}{(2a+1)^7}-\frac{1}{(2b+1)^7}\right] \\ &+ \ldots\ldots\ldots\ldots \\ &+ (-1)^{i+1}\frac{(2^{2i-1}-1)B_i\mu}{(2i-1)2i}\left[\frac{1}{(2a+1)^{2i-1}}-\frac{1}{(2b+1)^{2i-1}}\right] \\ &+ (-1)^{i+2}\frac{(2^{2i+1}-1)B_{i-1}\mu}{(i+1)(\overline{2a+1,2b+1})^{2i+2}}(b-a).\end{aligned}$$

and for the usage that one will make it is not necessary to possess the demonstration of it. One has designated: by μ, the modulus of the ordinary logarithms, so that

$$\log\mu = 9.63778\,43113\,00536\,78913,$$

by $(\overline{2a+1,2b+1})$, a number comprehended between $2a+1$ and $2b+1$, and by B_i, the number of Bernoulli of n° i. Here is, according to Euler, the first fifteen of these numbers

$$\begin{array}{ccccc}
\frac{1}{6}, & \frac{1}{30}, & \frac{1}{42}, & \frac{1}{30}, & \frac{5}{66}, \\
\frac{691}{2730}, & \frac{7}{6}, & \frac{3617}{510}, & \frac{43867}{798}, & \frac{174611}{330}, \\
\frac{854513}{138}, & \frac{236364091}{2730}, & \frac{8553103}{6}, & \frac{23749461029}{870}, & \frac{8615841276005}{14322}.
\end{array}$$

5. We examine the change that the probability experiences

$$\frac{[l+1]^{l+1}[x]^n[y]^m}{[n]^n[m]^m[s+1]^{l+1}}$$

when one will attribute to x successively the values

$$n, \quad n+1, \quad n+2, \dots s-m$$

and to y the corresponding values

$$s-n, \quad s-n-1, \quad s-n-2, \dots m.$$

It is clear that it will suffice for us to examine the march of the factor

$$[x]^n[y]^m$$

which alone varies with x and y.

The difference between two consecutive values of this factor is

$$[x+1]^n[y-1]^m - [x]^n[y]^m$$

or else

$$[x]^{n-1}[y-1]^{m-1}(ny - mx - m).$$

As long as it will be positive, the product

$$[x]^n[y]^m$$

and hence the probability

$$\frac{[l+1]^{l+1}[x]^n[y]^m}{[n]^n[m]^m[s+1]^{l+1}},$$

will increase with x. But, on the contrary, they will diminish when x will increase, all the time that this difference will be negative. It follows from it that the greatest probabilities, that is the most probable hypotheses, will correspond to the value of x for which the difference of which there is concern will pass from the positive to the negative. Now, as the factor

$$[x]^{n-1}[y-1]^{m-1}$$

is always positive, the sign of the difference will be the one of its other factor

$$ny - mx - m$$

the one here has its greatest value

$$n(s-l)-m$$

when $x=n$, $y=s-n$; next, it diminishes without ceasing in measure as x increases, and it diminishes from l for each unit increase of x, its smallest value

$$-n(s-l+1)$$

corresponds to $x=s-m$, $y=m$, it is evidently negative. Thus the factor of which there is concern changes necessarily in sign by passing from the positive to the negative, and changes only one time. It will be likewise for it for the entire difference

$$[x+1]^n[y-1]^m-[x]^n[y]^m$$

Therefore, unless that here, in passing from the positive to the negative, it not become zero, the probability will have only one maximum, that is that it will have only one hypothesis more probable than all the others. But if the difference of which there is concern could become zero, then it would have two probabilities equal between them and superior to all the others. Namely that there would be two hypotheses equally probable and more probable than the others.

In order to determine x to which corresponds the change of the sign of the factor

$$ny-mx-m$$

we replace there the quantity y by its value $s-x$, there will come

$$ns-lx-m$$

or else

$$n(s+1)-l(x+1).$$

It is clear that by taking for $x+1$ the greatest integer e contained in

$$\frac{n(s+1)}{l}$$

one will have a positive result; but one will obtain a negative from it, by making x likewise equal to the integer of which there is concern. Thus the change of sign of the difference holds for

$$x=e$$

and holds only for this value. It follows next that the greatest probability corresponds to $x=e$, that is the most probable hypothesis, out of the number x of white marbles and the one y of black marbles, is this here

$$x=e, \quad y=s-e.$$

Or else, in order to not introduce a new letter e, the most probable hypothesis corresponds to the values x and y, respectively equal to the greatest integers contained in

$$\frac{n(s+1)}{l} \text{ and } \frac{m(s+1)}{l}.$$

The probability of the other hypotheses diminishes in measure as they deviate from this, so that the most probable hypotheses group themselves about the most probable of all.

We note however that if the quotients

$$\frac{n(s+1)}{l}, \quad \frac{m(s+1)}{l}$$

were integers, and they would be so evidently at the same time, then the expression

$$n(s+1) - l(x+1),$$

and, next, the difference between two consecutive probabilities, would vanish for

$$x = \frac{n(s+1)}{l} - 1$$

there would be therefore two equally probable hypotheses, and of which the probability would surpass that of all the other hypotheses.

The two most probable hypotheses would correspond the one to

$$x = \frac{n(s+1)}{l} - 1, \quad y = \frac{m(s+1)}{l}$$

the second to

$$x = \frac{n(s+1)}{l}, \quad y = \frac{m(s+1)}{l} - 1$$

The other hypotheses would be so much more probable as they would approach more to these two here, and in measure as they would deviate from it, their probabilities would proceed by diminishing.

If one wished to consider the $s - l$ marbles left in the vase, instead of the total s, it would be only to set $x - n$ and $y - m$ in the place of x and y. Thus, one will have the most probable hypothesis, out of the number of white marbles left in the vase, by making this number equal to the greatest integer contained in

$$\frac{n(s+1)}{l} - n$$

or in

$$\frac{n(s-l+1)}{l}$$

this which gives, for the number of black marbles, the greatest integer contained in

$$\frac{m(s-l+1)}{l}.$$

The two hypotheses equally probable and surpassing all the others in probability, when they would take place, would correspond the one to

$$\frac{n(s-l+1)}{l} - 1$$

white marbles and

$$\frac{m(s-l+1)}{l}$$

black marbles; the other to

$$\frac{n(s-l+1)}{l}$$

white marbles and

$$\frac{m(s-l+1)}{l}-1$$

black marbles.

6. The probability that any one of many hypotheses will take place, is the sum of the probabilities of these hypotheses. Thus the probability that the total of the white marbles is one of the numbers

$$a, \quad a+1, \quad a+2, \quad a+3, \dots c$$

will be obtained by making the sum of the expressions

$$\frac{[l+1]^{l+1}[x]^n[y]^m}{[n]^n[m]^m[s+1]^{l+1}}$$

relative to all the values of x, from a to c inclusively, and to the corresponding values of y, that is from $y=s-a$ to $y=s-c$ inclusively. Now, by making

$$s-c=b$$

it is easy to be assured that the sum of which there is concern returns to that of the expressions

$$\frac{[l+1]^{l+1}[a+x]^n[b+y]^m}{[n]^n[m]^m[s+1]^{l+1}}$$

relative to all the values of x and y which, being positive integers, or zero, satisfy the equation

$$x+y=s-a-b=c-a,$$

or, by designating for brevity $s-a-b$ or $c-a$ by q, to that here

$$x+y=q$$

therefore the probability in question will be expressed by

(A) $$\frac{[l+1]^{l+1}S_q[a+x]^n[b+y]^m}{[n]^n[m]^m[s+1]^{l+1}}$$

and in order to have it the concern will be only to find the sum

$$S_q[a+x]^n[b+y]^m$$

this which will be made in the instant, if the quantities a and b would not surpass respectively the exponents n and m of the factorials; for one would have then

$$S_q[a+x]^n[b+y]^m = S_s[x]^n[y]^m.$$

But most often a will be greater than n, and b greater than m, the equation which precedes therefore is not able to take place; this which obliges us to develop the binomial factorials

$$[a+x]^n \qquad [b+y]^m$$

by aide of the known formulas, for the binomials of this kind, and which are demonstrated as easily as the binomial of Newton. These formulas are

$$[a+x]^n = [a]^n + [n]^1[a]^{n-1}[x]^1 + \frac{[n]^2}{[2]^2}[a]^{n-2}[x]^2 + \frac{[n]^3}{[3]^3}[a]^{n-3}[x]^3 + \cdots + \frac{[n]^i}{[i]^i}[a]^{n-i}[x]^i + \cdots + [x]^n$$

$$[b+y]^n = [b]^m + [m]^1[b]^{m-1}[y]^1 + \frac{[m]^2}{[2]^2}[b]^{m-2}[y]^2 + \frac{[m]^3}{[3]^3}[b]^{m-3}[y]^3 + \cdots + \frac{[m]^k}{[k]^k}[b]^{m-k}[y]^k + \cdots + [y]^m$$

We will have been able only to have cited these developments, but we are going to demonstrate them in favor of the readers not geometers to which this extract is destined.

It is first easy to be assured that not only the factorial

$$[a+x]^n$$

but a rational and any integer function of x is able to be set under this form

$$A + A_1[x]^1 + A_2[x]^2 + \cdots + A_i[x]^i + \cdots A_n[x]^n$$

A, A_1, $A_2, \ldots A_n$ being some numerical coefficients depend on the function. We make consequently

$$[a+x]^n = A + A_1[x]^1 + A_2[x]^2 + \cdots + A_i[x]^i + \cdots A_n[x]^n$$

and we suppose that x varies from unity; we take i times in sequence the finite differences of this equation. In order to arrive there, the reader will note that the finite difference of a factorial as

$$[a+x]^q$$

in the admitted hypothesis, that is when x varies from unity, is

$$[a+x+1]^q - [a+x]^q = q[a+x]^{q-1}.$$

In regard to this remark, one will find successively

$$n[a+x]^{n-1} = A_1 + 2A_2[x]^1 + 3A_3[x]^2 + \cdots + iA_i[x]^{i-1} + \cdots + nA_n[x]^{n-1}$$
$$[n]^2[a+x]^{n-2} = [2]^2A_2 + [3]^2A_3[x]^1 + [4]^2A_4[x]^2 + \cdots + [i]^2A_i[x]^{i-2} + \cdots + [n]^2A_n[x]^{n-2}$$
$$[n]^3[a+x]^{n-3} = [3]^3A_3 + [4]^3A_4[x]^1 + \cdots + [i]^3A_i[x]^{i-3} + \cdots + [n]^3A_n[x]^{n-3}$$
$$\cdots\cdots$$
$$[n]^i[a+x]^{n-i} = [i]^iA_i + [i+1]^iA_{i+1}[x]^1 + \cdots + [n]^iA_n[x]^{n-i}$$

We make $x = 0$ in the last of these equations, there will come

$$[n][a]^{-i} = [i]^i A_i$$

whence

$$A_i = \frac{[n][a]^{n-i}}{[i]^i}.$$

By setting, in this expression, for i successively

$$0, \quad 1, \quad 2, \quad 3, \ldots \quad n$$

one will find the quantities

$$A_0, \quad A_1, \quad A_2, \quad A_3, \ldots \quad A_n$$

and one will encounter the exactitude of the factorial binomial cited above. And it is clear that the other factorial

$$[b+y]^m$$

is susceptible of being treated in the same manner and will furnish some similar results.

Supposing that the reader has formed a clear idea of the binomial factorials, we write them, for brevity, by aid of the summation sign in the same way as follows

$$[a+x]^n[b+y]^m = \sum_{k=0}^{k=m} \sum_{i=0}^{i=n} \frac{[m]^k[n]^i}{[k]^k[i]^i} [a]^{n-i}[b]^{m-k}[x]^i[y]^k$$

whence

$$S_q[a+x]^n[b+y]^m = \sum_{k=0}^{k=m} \sum_{i=0}^{i=n} \frac{[m]^k[n]^i}{[k]^k[i]^i} [a]^{n-i}[b]^{m-k} S_q[x]^i[y]^k$$

Now, by virtue of formula (I),

$$S_q[x]^i[y]^k = \frac{[i]^i[k]^k[q+1]^{k+k+1}}{[i+k+1]^{i+k+1}}$$

therefore

$$S_q[a+x]^n[b+y]^m = \sum_{k=0}^{k=m} \sum_{i=0}^{i=n} \frac{[m]^k[n]^i[a]^{n-i}[b]^{m-k}[q+1]^{i+k+1}}{[i+k+1]^{i+k+1}}$$

By substituting this expression into formula (A), the probability that this formula represents will become

$$\frac{[l+1]^{l+1} \sum_{k=0}^{k=m} \sum_{i=0}^{i=n} \frac{[m]^k[n]^i[a]^{n-i}[b]^{m-k}[q+1]^{i+k+1}}{[i+k+1]^{i+k+1}}}{[n]^n[m]^m[s+1]^{l+1}}$$

7. The formula that we just found is able to be considerably simplified, either by effecting the summation relative to one of the two indices, i or k, or by the considerations that we are going to expose.

It is convenient first to distinguish the smallest of the two numbers m and n. We suppose that the first is it. We note next that the formula to simplify represents the probability that the total of the white marbles is one of the numbers

$$a, \quad a+1, \quad a+2, \ldots q+a.$$

We make $a=0$, we will have the probability that the total of which there is concern not surpass q, and this probability will be incomparably simpler than the preceding. In fact, for $a=0$ the quantity

$$[a]^{n-i},$$

or rather this here

$$[0]^{n-i},$$

is zero, all the time that i differs from n, and it becomes unity, when $i=n$; it follows from it that the sum relative to i is reduced to its last term, that is to the one which corresponds to $i=n$, and hence the probability will become

$$\frac{[l+1]^{l+1}}{[m]^m[s+1]^{l+1}} \sum_{k=0}^{k=m} \frac{[m]^k[b]^{m-k}[q+1]^{n+k-1}}{[n+k+1]^{n+k+1}}.$$

This is the probability, that the total of the white marbles not surpass q, or that the total of the black marbles is not smaller than $b=s-q$. If we change q into p and b into a, this last letter having a signification totally different from that just now, we will have

$$\frac{[l+1]^{l+1}}{[m]^m[s+1]^{l+1}} \sum_{k=0}^{k=m} \frac{[m]^k[b]^{m-k}[p+1]^{n+k-1}}{[n+k+1]^{n+k+1}}$$

for the probability, that the total of the white marbles is at most p, or that the one of the black marbles is at least $a=s-p$.

Therefore, by supposing $q>p$, the difference

$$\frac{[l+1]^{l+1}}{[m]^m[s+1]^{l+1}} \sum_{k=0}^{k=m} \frac{[m]^k}{[n+k+1]^{n+k+1}} \left([b]^{m-k}[q+1]^{n+k+1} - [a]^{m-k}[p+1]^{n+k+1}\right)$$

will represent the probability, that the total of white marbles is greater than p, but not surpass q, that is that it will be one of the numbers

$$p+1, \quad p+2, \quad p+3, \ldots q.$$

Now

$$
\begin{aligned}
[l+1]^{l+1} &= [l+1]^{m}[n+1]^{n+1} \\
[n+k+1]^{n+k+1} &= [n+k+1]^{n}[n+1]^{n+1} \\
[a]^{m-k} &= \frac{[a]^{m}}{[a-m+k]^{k}} \\
[b]^{m-k} &= \frac{[b]^{m}}{[b-m+k]^{k}} \\
[p+1]^{n+k+1} &= [p+1]^{n+1}[p-n]^{k} \\
[q+1]^{n+k+1} &= [q+1]^{n+1}[q-n]^{k}
\end{aligned}
$$

therefore the preceding probability will become

$$
\frac{[l+1]^{m}[b]^{m}[q+1]^{n+1}}{[m]^{m}[\rho+1]^{l+1}} \sum_{k=0}^{k=m} \frac{[m]^{k}}{[n+k+1]^{k}} \frac{[q-n]^{k}}{[b-m+k]^{k}}
- \frac{[l+1]^{m}[a]^{m}[p+1]^{n+1}}{[m]^{m}[\rho+1]^{l+1}} \sum_{k=0}^{k=m} \frac{[m]^{k}}{[n+k+1]^{k}} \frac{[p-n]^{k}}{[a-m+k]^{k}}
$$

It is convenient to examine the succession of terms of the finished series

$$
\sum_{k=0}^{k=m} \frac{[m]^{k}}{[n+k+1]^{k}} \frac{[q-n]^{k}}{[b-m+k]^{k}}
$$

and

$$
\sum_{k=0}^{k=m} \frac{[m]^{k}}{[n+k+1]^{k}} \frac{[p-n]^{k}}{[a-m+k]^{k}}
$$

It is clear that it suffices to consider one of them alone, for example the first, for the two series are of like nature.

We make

$$
P_k = \frac{[m]^{k}}{[n+k+1]^{k}} \frac{[q-n]^{k}}{[b-m+k]^{k}}
$$

we will have

$$
P_{k+1} - P_k = \frac{[m]^{k+1}}{[n+k+2]^{k+1}} \frac{[q-n]^{k+1}}{[b-m+k+1]^{k+1}} - \frac{[m]^{k}}{[n+k+1]^{k}} \frac{[q-n]^{k}}{[b-m+k]^{k}}
$$

namely

$$
P_{k+1} - P_k = P_k \left(\frac{m-k}{n+k+2} \frac{q-n-k}{b-m+k-1} - 1 \right)
$$

or else

$$
P_{k+1} - P_k = P_k \frac{(l+2)(q+2) - (s+3)(k+n+2)}{(n+k+2)(b-m+k+1)}.
$$

One will judge, in each particular case, by the sign of the quantity

$$(l+2)(q+2)-(s+3)(k+n+2),$$

if the terms of the series that we consider proceed increasing and to what value of k they will increase. For they will finish necessarily by diminishing, since k becomes m, the quantity

$$(l+2)(q+2)-(s+3)(k+n+2),$$

obtains a negative value

$$-(l+2)(s-q+1).$$

This quantity

$$(l+2)(q+2)-(s+3)(k+n+2),$$

diminishes from $s+3$ when k increases from unity, that is when one passes from one term to the following. It becomes negative, in the greater part of the cases, from the first terms, and one time negative, it will no longer change sign; therefore, most often, by departing from a term little deviated from the origin of the series, this here will diminish continually.

In the practical applications, one will fix the values of p and q in a manner to have regard only to the hypotheses which enjoy a certain probability, and which, as we have seen, group themselves about the most probable hypothesis. This here corresponds to the greatest integer contained in

$$\frac{n(s+1)}{l},$$

one will take the number q greater than this integer, and p smaller. But it is not necessary that they deviate from it equally. By fixing the number of hypotheses that one judges appropriate to consider, that is by fixing the difference $q-p$, one will make rather

$$p=\frac{n(s+1)-n(q-p)}{l}$$
$$q=\frac{n(s+1)-m(q-p)}{l}$$

for the most probable hypotheses will be very nearly between these limits.

8. The sums:

$$1+\sum_{k=1}^{k=m}\frac{[m]^k[x-n]^k}{[n+k+1]^k[y-m+k]^k}$$

and

$$1+\sum_{k=1}^{k=m}\frac{[m]^k[p-n]^k}{[n+k+1]^k[q-m+k]^k}$$

are able to be calculated, by aid of logarithms rather easily, and especially if one is taken conveniently and that m is not a very great number. We have replaced q, b, a, respectively by x, y, q.

In order to execute the calculation of which there is concern, one will partition a sheet of paper into ten vertical columns, sufficiently large in order to contain, each, nine to twelve digits. At the top of the first column, one will set the general term

$$\frac{[m]^k[p-n]^k}{[n+k+1]^k[q-m+k]^k},$$

of the second sum that, for brevity, we will designate by P_k, and one will set it in order to note that this column will contain the values of P_k relative to the different indices k. At the top of the second column, on the same line as P_k and to the same purpose, one will write Log P_k. Next to the third, fourth, fifth, tenth column one will set successively, always on the same line,

$$\text{Log }(q-m+k),\quad \log(p-n),\quad \log m,\quad \text{Log}(n+k+1),$$
$$\log(x-n),\quad \text{Log}(y-m+k),\quad \text{Log }\ X_k,\text{ and } X_k.$$

One will underscore all these indices and one will have a kind of table as is here

P_k	LogP_k	Log$(q-m+k)$	$\log(p-n)$	$\log m$	Log$(n+k+1)$	$\log(x-n)$	$\log(y-m+k)$	LogX_k	X_k

For brevity, one designates by X_k the quantity

$$\frac{[m]^k[x-n]^k}{[n+k+1]^k[y-m+k]^k}$$

and the characteristics log and Log represent the logarithms of Briggs and the arithmetic complement of these logarithms.[2]

This being, in the fifth column, marked by $\log m$, under the horizontal stroke one will write the logarithm of the number m, that one will find in the tables, and immediately below, the one of the number $m-1$. One will take care to underscore the last. Next, leaving enough space in order to be able later to write a line, one will set the logarithm of the number $m-2$, and one will underscore it. One will defer anew one line and one will write, by underscoring it, the logarithm of $m-3$. One will continue likewise until one arrives to the logarithm of unity that one will write also by underscoring it.

Here is the beginning of the fifth column.

$\log(p-n)$	$\log m$	Log$(n+k+1)$
	1.301 03000	
	<u>1.278 75360</u>	
	<u>1.255 27251</u>	
	<u>1.230 44892</u>	
	<u>1.204 11998</u>	

[2]*Translator's note*: The logarithm of Briggs is the common logarithm. That is, $\log x = \log_{10} x$. Log $x = 10 - \log x$.

One has made $m = 20$ and one is served with table to eight decimals.

Just as one has written in the fifth column the logarithms of

$$m, \quad m-1, \quad m-2, \dots 2, \quad 1$$

one will write in the sixth the arithmetic complements

$$\text{Log}(n+2), \quad \text{Log}(n+3), \quad \text{Log}(n+4), \dots \log(n+m), \quad \log(n+m+1).$$

Thus one will set, immediately, without the horizontal stroke, on the same line as the logarithm of m, the arithmetic complement of the logarithm of $n+2$; next, immediately below, the arithmetic complement of $\log(n+3)$. One will underscore the last and leaving the place for one line, one will write the arithmetic complement $\text{Log}(n+4)$, one will underscore it etc.

Absolutely in the same manner one will write, in the seventh column, the m logarithms of the numbers

$$x-n, \quad x-n-1, \quad x-n-2, \dots x-n-m+1,$$

and in the eighth the m arithmetic complements

$$\text{Log}(y-m+1), \quad \text{Log}(y-m+2), \quad \text{Log}(y-m+3), \dots \text{Log}\, y$$

next, without changing anything to the process, one will set into the third column m arithmetic complements

$$\text{Log}(q-m+1), \quad \text{Log}(q-m+2), \quad \text{Log}(q-m+3), \dots \text{Log}\, q$$

and in the fourth the m logarithms

$$\log(p-n), \quad \log(p-n-1), \quad \log(p-n-2), \dots \log(p-n-m+1).$$

By underscoring the numbers in the different columns, thus it has been explained, take care that one same horizontal trace passes in all the columns, and underscore immediately all the corresponding numbers. One would be able to rule a sheet of paper conveniently for this object.

Note that in each column you have to write only the logarithms, or their arithmetic complement, of the numbers which follow immediately, this which is a great convenience.

This put, by commencing with any one of the six columns, one will add the first two numbers written above the one immediately under the other. One will write the sum under the trace which passes below these numbers, and as soon as it is written, one will add the number which is found immediately below, by writing the new sum under the trace which underscores this number; to this new sum one will add the number which will be found immediately below, and one will write the result in the place preserved for that below the last number. One will continue the addition in the same manner until all the column will be exhausted. Here is the commencement of these additions for the fifth column.

$\log(p-n)$	$\log m$	$\text{Log}(n+k+1)$
	1.301 03000	
	1.278 75360	
	2.579 78360	
	1.255 27251	
	3.835 05611	
	1.230 44892	
	5.065 50503	
	1.204 11998	
	6.269 62501	
	......	

One will make absolutely the same addition and in the same order in all six columns.

The operations of which there is concern achieved, one will commence anew. One will add the first four numbers which are found under the first horizontal trace in the fifth, sixth, seventh and eighth column, and after having taken off from the sum ten units, one will write it in the ninth column. One will add next the four numbers which are found immediately below the second horizontal trace, in the same columns one will take off three tens from the sum, and one will set the result in the ninth column under the one which is found already; one will continue the addition until that which one has added, by four, all the numbers which are under the horizontal traces in the fifth, sixth, seventh and eighth columns, one will add only these numbers, without touching those which are below horizontal traces, and after each addition, one will subtract from the sum twice as many tens, less one, as there are units in the No. of a horizontal trace under which were the numbers added; thus, for example, the last four numbers are found under the trace No. m. One will subtract from their sum $(2m-1)10$. One will write all the results, the one under the others, in the ninth column, so that after the operation this column will contain m numbers.

That which one will have done with the numbers of the fifth, sixth, seventh and eighth columns, one will do with those which are in the third, fourth, fifth and sixth columns, and one will write the results in the second column, this which will make arrive in this column m numbers, that is as many as in the ninth.

Considering the numbers that we just described in the second and the ninth column as the arithmetic complements of the logarithms, seek the corresponding numbers and write the result relative to the ninth column, in the tenth, and in the first, those which correspond to the second column. Next add all the results of the tenth column together, and those of the first together, you will have the two sums

$$\sum_{k=1}^{k=m} X_k$$

and

$$\sum_{k=1}^{k=m} P_k$$

By adding the unit to each sum, one will have the values of

$$1 + \sum_{k=1}^{k=m} X_k$$

and

$$1 + \sum_{k=0}^{k=m} P_k.$$

We designate these values respectively by Y and Q and we find the logarithms

$$\log Y$$

and

$$\log Q$$

by the tables.

All this done, we return to the probability that we have in view. By introducing the letters Y and Q, it will become

$$\frac{[l+1]^m [y]^m [x+1]^{n+1}}{[m]^m [s+1]^{l+1}} Y - \frac{[l+1]^m [q]^m [p+1]^{n+1}}{[m]^m [s+1]^{l+1}} Q.$$

It is easy to calculate each term of this difference, or rather the logarithm of each term. In fact, the one of the first is

$$\log[l+1]^m + \log[y]^m + \log[x+1]^{n+1} - \log[m]^m - \log[s+1]^{n+1} + \log Y$$

Now from these five logarithms one comes to find the last $\log Y$; the fourth

$$\log[m]^m$$

is found in the fifth table. It is the last number of this table. The sixth and the eighth tables will furnish $\log[l+1]^m$ and $\log[y]^m$, because

$$\log[l+1]^m = 10m - \text{the last number of the 6th table,}$$
$$\log[y]^m = 10m - \text{the last number of the 8th table.}$$

There will remain therefore to calculate only

$$[x+1]^{n+1} \text{ and } [s+1]^{l+1}$$

this which will be made by the formula of No. 4.

The logarithm of the term

$$\frac{[l+1]^m [q]^m [p+1]^{n+1}}{[m]^m [s+1]^{l+1}} P$$

will require only the calculation of

$$\log[p+1]^{n+1}$$

all the other logarithms are known, or are found with ease. In fact, the only thing that one has not yet calculated, beyond $\log[p+1]$, is $\log[q]^m$; now we have $\log[q]^m = 10m-$ the last number of the third table. While for the $\log[p+1]^{n+1}$ one will find it by the formula of No. 4.

9. The probability

$$\frac{[l+1]^m[y]^m[x+1]^{n+1}}{[m]^m[s+1]^{l+1}}\sum_{k=0}^{k=m}\frac{[m]^k[x-n]^k}{[n+m+1]^k[y-m+k]^k}$$
$$-\frac{[l+1]^m[q]^m[p+1]^{n+1}}{[m]^m[s+1]^{l+1}}\sum_{k=0}^{k=m}\frac{[m]^n[p-n]^k}{[n+m+1]^k[y-m+k]^k}$$

is able to take this other form

$$\frac{[x-1]^{l+1}}{[s+1]^{l+1}}\sum_{k=0}^{k=m}\frac{[l+1]^k[y]^k}{[k]^k[x-l+k]^k}-\frac{[p+1]^{l+1}}{[s+1]^{l+1}}\sum_{k=0}^{k=m}\frac{[l+1]^k[q]^k}{[k]^k[p-l+k]^k}$$

but this here is less convenient for the calculation, than the preceding, because the sums which are found contained there have generally some considerable values. In fact, by designating

$$\frac{[l+1]^k[y]^k}{[k]^k[x-l+k]^k}$$

by Y_k, we will have

$$Y_{k+1}-Y_k = Y_k\left(\frac{(l-k+1)(y-k)}{(k+1)x-l+k+1}-1\right)$$

or else

$$Y_{k+1}-Y_k = \frac{(l+2)(y+1)-(s+3)(k+1)}{(k+1)(x-l+k+1)}Y_k$$

the numerator $(l+2)(y+1)-(s+3)(k+1)$ will be positive all the time that k will differ from m; it follows that the terms of the sum

$$\sum_{k=0}^{k=m}\frac{[l+1]^k[y]^k}{[k]^k[n-l+k]^k}$$

will proceed increasing by departing from the first which is unity.

Thus we will hold ourselves to the first form. However if the number m were very considerable, the sums

$$\sum_{k=1}^{k=m}\frac{[m]^k[x-n]^k}{[n+k+1]^k[y-m+k]^k}$$

and

$$\sum_{k=1}^{k=m}\frac{[m]^k[p-n]^k}{[n+k+1]^k[q-m+k]^k}$$

which are contained there would be composed of a great number of terms, and the calculation would become painful. It would be necessary then to recur to the process that we give in the memoir, but which requires the use of the definite integral, and which belongs to transcendental analyses.

10. We apply the formula of No. 8 to an example.

We suppose that the vase contains 10000 marbles, and that having withdrawn 100 of these marbles, one has found of the 80 white and 20 black.

The greatest integers contained in

$$\frac{n(s+1)}{l} \text{ and } \frac{m(s+1)}{l}$$

that is in

$$\frac{80.10001}{100} \text{ and } \frac{20.10001}{100}$$

are 8000 and 2000; thus the most probable hypothesis corresponds to 8000 white marbles and 2000 black marbles. We determine the probability that one of the 200 hypotheses, neighboring the most probable, will take place. If one would wish that these hypotheses be the most probable of all the others, it would be necessary to take very nearly

$$x = 8040 \text{ therefore } y = 1960$$
$$p = 7840 \text{ therefore } q = 2160$$

but, by a reason that is useless to explicate, we do not admit this hypothesis. We will make

$$x = 8100 \text{ therefore } y = 1900$$
$$p = 7900 \text{ therefore } q = 2100$$

The sought probability will become

$$\frac{[101]^{20}[1900]^{20}[8101]^{81}}{[20]^{20}[10001]^{101}}Y - \frac{[101]^{20}[2100]^{20}[7901]^{81}}{[20]^{20}[10001]^{101}}P$$

in order to have Y and P, we execute the calculation which is found in the adjoined table.

The first and the last columns of this table furnish

$$\sum_{k=1}^{k=m} X_k = 5.551887$$
$$\sum_{k=1}^{k=m} P_k = 3.616786$$

therefore

$$Y = 6.551887$$
$$Q = 3.616786$$

and

$$\log Y = 0.816\ 36640$$
$$\log Q = 0.664\ 33964$$

Moreover, by the fifth and sixth columns

$$\log[20]^{20} = 18.386\ 12463$$
$$\log[101]^{20} = 200 - 160.788\ 88772 = 39.211\ 11228$$

and by columns eight and three

$$\log[1900]^{20} = 200 - 134.468\ 50679 = 65.531\ 49321$$
$$\log[2100]^{20} = 200 - 133.595\ 02961 = 66.404\ 97039$$

there remains to find the three logarithms

$$\log[7901]^{81}, \quad \log[8101]^{81}, \quad \log[10001]^{101}$$

Now the formula of No. 4 gives

$$\log[7901]^{81} = \frac{15803 \log 15803 - 15641 \log 15641}{2}$$
$$- 0.735\ 32447\ 76.81 + \frac{27m}{2.15641.15803}$$
$$= 315.531\ 54589,$$
$$\log[8101]^{81} = \frac{16203 \log 16203 - 16041 \log 16041}{2}$$
$$- 0.735\ 32447\ 76.81 + \frac{3m}{5347.10802}$$
$$= 316.417\ 35463,$$
$$\log[10001]^{101} = \frac{20003 \log 20003 - 19801 \log 19801}{2}$$
$$- 0.735\ 32447\ 76.81 + \frac{101m}{6.19801.20003}$$
$$= 403.784\ 35109.$$

It was useless to take more terms of this formula, because that which has been neglected attacks only the decimals of an order quite more elevated than the one where we have stopped ourselves.

This put, we have

$$\begin{aligned}
\mathrm{Log}\frac{[101]^{20}[1900]^{20}[8101]^{81}}{[20]^{20}[10001]^{101}}Y =&10 + 39.211\,11228 + 65.531\,49321 + 316.417\,35463\\
&+ 0.816\,36640 - 18.386\,12463 - 403.784\,35109\\
=&9.805\,85080,\\
\mathrm{Log}\frac{[101]^{20}[2100]^{20}[7901]^{81}}{[20]^{20}[10001]^{101}}Q =& 10 + 39.211\,11228 + 66.404\,97039 + 315.531\,54589\\
&+ 0.664\,33964 - 18.386\,12463 - 403.784\,35109\\
=&9.641\,49248.
\end{aligned}$$

By passing from logarithms to numbers, one finds, for the sought probability,

$$0.639515 - 0.438019 = 0.201496.$$

TABLEAU

P_k	$\text{Log}P_k$	$\text{Log}(q-m+k)$	$\log(p-n)$	$\log m$	$\text{Log}(n+k+1)$	$\log(x-n)$	$\text{Log}(y-m+k)$	$\text{Log}X_k$	X_k
0.916539	9.96215082	6.68172792	3.89320675	1.30103000	8.08618615	3.90417437	6.72561120	10.01700172	1.039924
		6.68151927	3.89315121	1.27875360	8.08092191	3.90412021	6.72538038		
0.787947	9.89649681	13.36324719	7.78635796	2.57978360	16.16710806	7.80829458	13.45099158	10.00617782	1.014327
		6.68131073	3.89309567	1.25527251	8.07572071	3.90406605	6.72514968		
0.633719	9.80189643	20.04455792	11.67945363	3.83505611	24.24282877	11.71236063	20.17614126	9.96638677	0.925522
		6.68110229	3.89304011	1.23044892	8.07058107	3.90401188	6.72491910		
0.475411	9.67706882	26.72566021	15.57249374	5.06550503	32.31340984	15.61637252	26.90106036	9.89634774	0.787676
		6.68089394	3.89298455	1.20411998	8.06550155	3.90395771	6.72468865		
0.331565	9.52056884	33.40655415	19.46547829	6.26962501	40.37891139	19.52033022	33.62574901	9.79461563	0.623183
		6.68068570	3.89292898	1.17609126	8.06048075	3.90390353	6.72445831		
0.214168	9.33075553	40.08723985	23.35840727	7.44571627	48.43939214	23.42423375	40.35020732	9.65954948	0.456614
		6.68047755	3.89287341	1.14612804	8.05551733	3.90384934	6.72422810		
0.127571	9.10575186	46.76771740	27.25128068	8.59184431	56.49490947	27.32808309	47.07443542	9.48927229	0.308512
		6.68026951	3.89281782	1.11394335	8.05060999	3.90379514	6.72399801		
0.069726	8.84339253	53.44798691	31.14409850	9.70578766	64.54551946	31.23187823	53.79843343	9.28161878	0.191258
		6.68006156	3.89276223	1.07918125	8.04575749	3.90374094	6.72376804		
0.034766	8.54115506	60.12804847	35.03686073	10.78496891	72.59127695	35.13561917	60.52220147	9.03406650	0.108160
		6.67985371	3.89270664	1.04139269	8.04095861	3.90368673	6.72353820		
0.015706	8.19606671	66.80790218	38.92956737	11.82636160	80.63223556	39.03930590	67.24573967	8.74364273	0.055417
		6.67964597	3.89265103	1.00000000	8.03621217	3.90363252	6.72330847		
0.006376	7.80457588	73.48754815	42.82221840	12.82636160	88.66844773	42.94293842	73.96904814	8.40679589	0.025515
		6.67943832	3.89259542	0.95424251	8.03151705	3.90357829	6.72307887		
0.002303	7.36236918	80.16698647	46.71481382	13.78060411	96.69996478	46.84651671	80.69212701	8.01921261	0.010452
		6.67923077	3.89253980	0.90308999	8.02687215	3.90352406	6.72284939		
0.000731	6.86410189	86.84621724	50.60735362	14.68369410	104.72683693	50.75004077	87.41497640	7.57554820	0.003763
		6.67902332	3.89248418	0.84509804	8.02227639	3.90346983	6.72262003		
0.000201	6.30298382	93.52524056	54.49983780	15.52879214	112.74911332	54.65351060	94.13759643	7.06901249	0.001172
		6.67881597	3.89242855	0.77815125	8.01772877	3.90341559	6.72239079		
0.000047	5.67010836	100.20405653	58.39226635	16.30694339	120.76684209	58.55692619	100.85998722	6.49069886	0.000310
		6.67860872	3.89237291	0.69897000	8.01322827	3.90336134	6.72216167		
0.000009	4.95328826	106.88266525	62.28463926	17.00591339	128.78007036	62.46028753	107.58214889	5.82842017	0.000069
		6.67840157	3.89231726	0.60205999	8.00877392	3.90330708	6.72193267		
0.000001	4.13484100	113.56106682	66.17695652	17.60797338	136.78884428	66.36359461	114.30408156	5.06449377	0.000012
		6.67819452	3.89226161	0.47712125	8.00436481	3.90325282	6.72170379		
0.000000	3.18678319	120.23926134	70.06921813	18.08509463	144.79320909	70.26684743	121.02578535	4.17093650	0.000001
		6.67798756	3.89220595	0.30103000	8.00000000	3.90319855	6.72147504		
0.000000	2.05800670	126.91724890	73.96142408	18.38612463	152.79320909	74.17004598	127.74726039	3.09664009	0.000000
		6.67778071	3.89215028	0.00000000	7.99567863	3.90314427	6.72124640		
0.000000	0.62361632	133.59502961	77.85357436	18.38612463	160.78888772	78.07319025	134.46850679	1.71670939	0.000000
3.616786					Sum				5.551886