METHOD

MR. NICOLE
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METHOD

In order to determine the lot of as many players as we will wish, & the advantage that one has over the others, when they play to who will win the most parts in a number of determined parts.

In the Memoir that I read, it is some days, I have determined the lot of two players, & the advantage of the one over the other, for whatever number of parts[1] as there be. I make use in this Memoir of the analytic method, & by examining all the equations that the nature of the different questions furnish, I have shown in what manner they lead to the solution in each case. The comparison of the resultant quantities in each solution of different cases, next reveal the law according to which the quantities grow, & give the general solution for whatever number of games. 29 March 1730

In the Memoir that I give today, I avail myself likewise first of the same analytic method: but the different cases which one is obliged to examine, becoming soon quite composite, & thence the number of the equations of which it is necessary to make use becoming very great, I abandon this method, which has given the solution only of some particular cases, & give for them another much more simple, & which satisfies in all the possible cases what one can propose on this matter. This new way to proceed, furnishes yet another utility, this is a general method to raise a multinomial composed of as many parts as one will wish, to any power, much more simple, & which demands considerably less calculation than the ordinary methods.

PROBLEM I

Three players, of whom the forces are among themselves, as the quantities p, q, m, play or wager to who will win the most times a determined number of parts. We demand the lot of each of these players, & the advantage of the strongest player over each of the others.

SOLUTION.

If we name a the money which is in the game, or the stake of the three players, & if we suppose that they play to one part,

the lot of the 1st player will be $\frac{\overset{1.0.0}{p\times a}+\overset{0.1.0}{q\times 0}+\overset{0.0.1}{m\times 0}}{p+q+m} = \frac{ap}{p+q+m}$.

That of the second $\frac{aq}{p+q+m}$

That of the third $\frac{am}{p+q+m}$.

Where it is necessary to note that the numbers $1.0.0$, $0.1.0$ & $0.0.1$ which are written above each term of the quantity which expresses the lot of the first player, indicate the number of parts that each player has won: for example, $1.0.0$ expresses that the first player has won

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[1] *Translator's note*: *parties*, that is subgames or parts of games.

one part, & the two others win none of them, which must be understood for the rest of this Memoir: 3.2.1 will express likewise that the first player has won 3 parts, the second 2 parts, & the third one part.

The unknowns s, x, y, z, t, r,&c. express here the lot of the first player, in the different states indicated by the numbers of which we just spoke, or, that which is the same thing, the part of the money which is in the game, which belongs to this player relatively to each state.

If one plays to two parts

The lot of the 1st is

$$s = \frac{p \times \overset{1.0.0}{x} + q \times \overset{0.1.0}{y} + m \times \overset{0.0.1}{z}}{p+q+m};$$

in order to determine the value of s, we have

$$x = \frac{p \times \overset{2.0.0}{a} + q \times \overset{1.1.0}{\tfrac{1}{2}a} + m \times \overset{1.0.1}{\tfrac{1}{2}a}}{p+q+m} = \frac{ap + \frac{1}{2}aq + \frac{1}{2}am}{p+q+m},$$

$$y = \frac{p \times \overset{1.1.0}{\tfrac{1}{2}a} + q \times \overset{1.2.0}{0} + m \times \overset{0.1.1}{0}}{p+q+m} = \frac{\frac{1}{2}ap}{p+q+m},$$

$$\& \, z = \frac{p \times \overset{1.0.1}{\tfrac{1}{2}a} + q \times \overset{0.1.1}{0} + m \times \overset{0.0.2}{0}}{p+q+m} = \frac{\frac{1}{2}ap}{p+q+m}.$$

Whence we deduce

$$s = \frac{app + \frac{1}{2}apq + \frac{1}{2}apm + \frac{1}{2}apq + \frac{1}{2}apm}{(p+q+m)^2} = \frac{app + apq + apm}{(p+q+m)^2}.$$

The lot of the second is therefore

$$\frac{aqq + apq + aqm}{(p+q+m)^2}.$$

And that of the third is

$$\frac{amm + amq + apm}{(p+q+m)^2}.$$

Which are among themselves as pa, qa, ma.

If one plays to three parts

The lot of the 1st is

$$s = \frac{p \times \overset{1.0.0}{x} + q \times \overset{0.1.0}{y} + m \times \overset{0.0.1}{z}}{p+q+m};$$

in order to determine s, we have

$$x = \frac{p \times \overset{2.0.0}{a} + q \times \overset{1.1.0}{u} + m \times \overset{1.0.1}{t}}{p+q+m},$$

$$u = \frac{p \times \overset{2.1.0}{a} + q \times \overset{1.2.0}{0} + m \times \overset{1.1.1}{\tfrac{1}{3}a}}{p+q+m} = \frac{ap + \frac{1}{3}am}{p+q+m}$$

$$\& \, t = \frac{p \times \overset{2.0.1}{a} + q \times \overset{1.1.1}{\tfrac{1}{3}a} + m \times \overset{1.0.2}{0}}{p+q+m} = \frac{ap + \frac{1}{3}aq}{p+q+m},$$

therefore

$$x = \frac{app + 2apq + 2apm + \frac{2}{3}aqm}{(p+q+m)^2}.$$

We have also

$$y = \frac{p \times \frac{ap + \frac{1}{3}am}{p+q+m} + \overset{0.2.0}{q \times 0} + \overset{0.1.1}{m \times r}}{p+q+m}$$

$$r = \frac{\overset{1.1.1}{p \times \frac{1}{3}a} + \overset{0.2.1}{q \times 0} + \overset{0.1.2}{m \times 0}}{p+q+m} = \frac{\frac{1}{3}ap}{p+q+m},$$

therefore

$$y = \frac{app + \frac{2}{3}apm}{(p+q+m)^2}.$$

We will find also

$$z = \frac{\overset{1.0.1}{p \times \frac{ap + \frac{1}{3}aq}{p+q+m}} + \overset{0.1.1}{\frac{q \times \frac{1}{3}ap}{p+q+m}} + \overset{0.0.2}{m \times 0}}{p+q+m} = \frac{app + \frac{2}{3}apq}{(p+q+m)^2}.$$

If therefore we substitute for x, y & z, the values which we just found, we will have the value of s

for the lot of the 1st player $\frac{ap^3 + 3appq + 3appm + 2apqm}{(p+q+m)^3}$

for that of the second $\frac{aq^3 + 3aqqp + 3aqqm + 2apqm}{(p+q+m)^3}$

for that of the third $\frac{am^3 + 3ammq + 3ammp + 2apqm}{(p+q+m)^3}$

If one plays to four parts

The lot of the 1st player will be

$$s = \frac{\overset{1.0.0}{p \times x} + \overset{0.1.0}{q \times y} + \overset{0.0.1}{m \times z}}{p+q+m};$$

in order to determine s, we will have all the following equations

$$x = \frac{\overset{2.0.0}{p \times u} + \overset{1.1.0}{q \times t} + \overset{1.0.1}{m \times r}}{p+q+m},$$

$$u = \frac{\overset{3.0.0}{p \times a} + \overset{2.1.0}{q \times k} + \overset{2.0.1}{m \times l}}{p+q+m},$$

$$k = \frac{\overset{3.1.0}{p \times a} + \overset{2.2.0}{q \times \frac{1}{2}a} + \overset{2.1.1}{m \times a}}{p+q+m} = \frac{ap + \frac{1}{2}aq + am}{p+q+m},$$

$$l = \frac{\overset{3.0.1}{p \times a} + \overset{2.1.1}{q \times a} + \overset{2.0.2}{m \times \frac{1}{2}a}}{p+q+m}.$$

Therefore

$$u=\frac{app+2apq+2apm+2aqm+\frac{1}{2}aqq+\frac{1}{2}amm}{(p+q+m)^2},$$

$$t=\frac{\overset{2.1.0}{p\times\frac{ap+\frac{1}{2}aq+am}{p+q+m}}+\overset{1.2.0}{q\times g}+\overset{1.1.2}{m\times h}}{p+q+m},$$

$$g=\frac{\overset{2.2.0}{p\times\frac{1}{2}a}+\overset{1.3.0}{q\times 0}+\overset{1.2.1}{m\times 0}}{p+q+m}=\frac{\frac{1}{2}ap}{p+q+m},$$

$$h=\frac{\overset{2.1.1}{p\times a}+\overset{1.2.1}{q\times 0}+\overset{1.1.2}{m\times 0}}{p+q+m}=\frac{ap}{p+q+m},$$

therefore

$$t=\frac{app+apq+2apm}{(p+q+m)^2},$$

$$r=\frac{\overset{2.0.1}{\overline{p\times ap+aq+\frac{1}{2}am}}+\overset{1.1.1}{q\times ap}}{(p+q+m)^2}+\frac{\overset{1.0.2}{m\times f}}{p+q+m},$$

$$f=\frac{\overset{2.0.2}{p\times\frac{1}{2}a}+\overset{1.1.2}{q\times 0}+\overset{1.0.3}{m\times 0}}{p+q+m}=\frac{\frac{1}{2}ap}{p+q+m},$$

therefore

$$r=\frac{app+2apq+apm}{(p+q+m)^2}$$

&

$$x=\frac{ap^3+3appq+3appm+6apqm+\frac{1}{2}apqq+\frac{3}{2}apmm}{(p+q+m)^3}.$$

In order to determine y, we have these equations

$$y=\frac{\overset{1.1.0}{p\times\frac{app+apq+2apm}{(p+q+m)^2}}+\overset{0.2.0}{q\times d}+\overset{0.1.1}{m\times e}}{p+q+m},$$

$$d=\frac{\overset{1.2.0}{p\times\frac{\frac{1}{2}ap}{p+q+m}}+\overset{0.3.0}{q\times 0}+\overset{0.2.1}{m\times 0}}{p+q+m}=\frac{\frac{1}{2}app}{(p+q+m)^2},$$

$$e=\frac{\overset{1.1.1}{p\times\frac{ap}{p+q+m}}+\overset{0.2.1}{q\times 0}+\overset{0.1.2}{m\times 0}}{p+q+m}=\frac{app}{(p+q+m)^2},$$

therefore

$$y=\frac{ap^3+\frac{3}{2}appq+3appm}{(p+q+m)^3},$$

& to determine z, we have

$$z = \frac{p \times \overset{1.0.1}{\frac{app+2apq+apm}{(p+q+m)^2}} + q \times \overset{0.1.1}{\frac{app}{(p+q+m)^2}} + m \times \overset{0.0.2}{b}}{p+q+m},$$

$$b = \frac{p \times \overset{1.0.2}{X} + q \times \overset{0.1.2}{0} + m \times \overset{0.0.3}{0}}{p+q+m},$$

$$X = \frac{p \times \overset{2.0.2}{\tfrac{1}{2}a} + q \times \overset{1.1.2}{0} + m \times \overset{1.0.3}{0}}{p+q+m} = \frac{\frac{1}{2}ap}{p+q+m},$$

therefore

$$b = \frac{\frac{1}{2}app}{(p+q+m)^2},$$

&

$$z = \frac{ap^3 + 3appq + \frac{3}{2}appm}{(p+q+m)^3}.$$

If therefore we substitute for x, y & z their values, we will have s, or

the lot of the 1st player $= \frac{ap^4+4ap^3q+4ap^3m+12appqm+3appqq+3appmm}{(p+q+m)^4}$.
That of the 2nd $\frac{aq^4+4aq^3p+4aq^3m+12apqqm+3appqq+3aqqmm}{(p+q+m)^4}$.
That of the 3rd $\frac{am^4+4apm^3+4aqm^3+12apqmm+3appmm+3aqqmm}{(p+q+m)^4}$.

When one plays to one part

The lot of the 1st player is $\frac{ap}{p+q+m}$.
That of the 2nd $\frac{aq}{p+q+m}$
That of the 3rd $\frac{am}{p+q+m}$

When one plays to two parts

The lot of the 1st is $\frac{app+apq+apm}{(p+q+m)^2}$.
That of the 2nd $\frac{aqq+apq+aqm}{(p+q+m)^2}$.
That of the 3rd $\frac{amm+apm+aqm}{(p+q+m)^2}$.

When one plays to three parts

The lot of the 1st is $\frac{ap^3+3appq+3appm+2apqm}{(p+q+m)^3}$.
That of the 2nd $\frac{aq^3+3aqqp+3aqqm+2apqm}{(p+q+m)^3}$.
That of the 3rd $\frac{am^3+3ammq+3ammp+2apqm}{(p+q+m)^3}$.

When one plays to four parts

The lot of the 1st is $\frac{ap^4+4ap^3q+4ap^3m+3appq^2+3appm^2+12appqm}{(p+q+m)^4}$.
That of the 2nd $\frac{aq^4+4aq^3p+4aq^3m+3appqq+3aqqm^2+12apqqm}{(p+q+m)^4}$.
That of the 3rd $\frac{am^4+4apm^3+4aqm^3+2appmm+3aqqm^2+12apqmm}{(p+q+m)^4}$.

If $p = 6$, $q = 5$, $m = 4$, the lots will be

for one part	6.	5.	4.
for two	90.	75.	60.
or	6.	5.	4.
for three	1428.	1115.	832.
for four	22100.	16981.	11754.

REMARK.

If we wished to seek the lot of these three players, for 5, 6, 7, 8, &c. parts, the number of the equations which it would be necessary to examine by this method would become quite considerable; it would be necessary to examine again a quite great number, if in place of three players, we supposed four, five, six, &c. because these equations express the different events which can happen in the course of the game, the number of these events will be accordingly greater, as there will be a greater number of players, & as they will play to a greater number of parts. In all these compound cases, the way of the equations is too long & too painful. Here is a method which satisfies all the cases, whatever be the number of players, & whatever be the number of parts that one must play.

PROBLEM II

Let, for example, four players, of whom the strengths are expressed by the quantities p, q, m, r. *We demand the lot of each of these players, & the advantage of one over the other, when they agree to play to eight parts: it suffices to order to win the fund of the game, to win at least one part more of these eight than any of the other players.*

SOLUTION.

We know that $\frac{p}{p+q+m+r}$ expresses the probability that the first player has of winning the 1st part, that $\frac{pp}{(p+q+m+r)^2}$ expresses that which he has of winning the first two, & finally $\frac{p^8}{(p+q+m+r)^8}$ expresses the probability that he has of winning the eight parts. If we add to this quantity the probability that the same player has of winning seven of these parts, any one of the three other players winning one of them.

If we add next to these two quantities, the probability that the same player has of winning six parts, one of the other three players winning two, or two of these three players winning one each.

If to this sum we add next the probability that the same player has of winning five parts, any one of the three other players winning three, or two or one of them.

Then the probability that the same player has of winning four of them, any one of the other three players winning four, three, two or one of them.

And finally if we add next the probability that this same player has of winning three parts, any one of the three other players winning three, two or one of them, & that this same player has of winning two parts, each of the other three players winning two of them.

⟨Then⟩ it is clear that the sum formed by the addition of all these parts, will express the lot of this 1st player, or the right that he has to the money which is in the game: because this sum is formed of all the possible ways that this player has of winning, or all that which is in the game, when he wins one part more than any of the other players, or the half of that which is in the game, when one other player wins as many parts has him, or finally the third or the fourth of that which is in the game, when two or three of the other players win as many parts as him.

Now, if is evident that the numbers which express in how many ways to take eight things, 8 by 8, 7 by 7, 6 by 6, 5 by 5, 4 by 4, 3 by 3, & 2 by 2, express also the number of ways that this player has of winning eight parts, or seven, six, five, four, three, two.

Now, everyone knows that the seventh perpendicular band of the arithmetic triangle of Mr. Pascal furnishes all these numbers, 1, 8, 28, 56, 70, 56, 28, 8. There remains nothing more than to multiply these numbers by those which express all the varieties which can happen to the three other players, for the number of parts which they can win, relatively to each case of the first player, & which multiplies each of these cases. If therefore we name a the money which is in the game,

1°. We will have $\frac{1\times p^8\times \mathrm{a}}{(p+q+m+r)^8}$ in order that this player wins the eight parts, $\frac{8\times p^7(q+m+r)}{(p+q+m+r)^8}$ in order that he wins seven parts, each of the other players winning one of them: because it is clear that each of the other players can win one in eight ways, namely, either the 1st part, or the 2nd part, 3rd, 4th . . . 8th.

2°. We will have $\frac{28\times p^6(qq+mm+rr)}{(p+q+m+r)^8}$ in order that this player winning six of them, one of the others win two of them, because 28 expresses all the ways of winning six parts of eight, & on each of these ways, each of the other players can win the two other parts.

3°. We will have $\frac{28p^6\times 2\times (qm+qr+mr)}{(p+q+m+r)^8}$ in order that this player winning six of them, two of the three other players each win one of them: because it is clear that these two others can be the 2nd & 3rd, the 2nd & the 4th, or the 3rd & the 4th, & that in each case there are two ways.

4°. We will have also $\frac{56p^5(q^3+m^3+r^3)}{(p+q+m+r)^8}$ in order that this player winning five parts, any one of the three others win three of them.

5°. Then $\frac{56p^5\times 3\times (qq\times (m+r)+mm\times (q+r)+rr\times (q+m))}{(p+q+m+r)^8}$, since this player has 56 ways to win five parts of the eight, & each of the others has three ways to win two parts of the remaining three.

6°. Then $\frac{56p^5\times 6qrm}{(p+q+m+r)^8}$ in order that this player wins five parts of the eight, each of the three others winning one of them: because three things can be combined in six ways.

7°. We will have also $\frac{70p^4\times (q^4+m^4+r^4)}{(p+q+m+r)^8}$ in order that this player winning four parts, any one of the three others wins four of them also.

8°. Then $\frac{70p^4\times 4\times (q^3\times (m+r)+m^3\times (q+r)+r^3\times (q+m))}{(p+q+m+r)^8}$, in order that any one of the three others win three of them: because there are four ways in order that this happen, four things being able to be taken 3 by 3 in four ways.

9°. We will have next $\frac{70p^4\times 6\times (qq\times (mm+rr)+mmrr)}{(p+q+m+r)^8}$ in order that any two of the three players each win two of them.

10°. Next $\frac{70p^4\times (6\times (qq\times 2mr)+(mm\times 2qr)+(rr\times 2qm))}{(p+q+m+r)^8}$ in order that any one of the three others win two of them, the two remaining each winning one of them: because there are six ways to take four things 2 by 2, & the two remaining players can change in two ways.

11°. We will have also $\frac{56p^3\times 10\times (q^3\times (mm+rr)+m^3\times (qq+rr)+r^3\times (qq+mm))}{(p+q+m+r)^8}$ in order that this player winning three parts, any one of the other three win also three of them, each of the remaining winning two of them: because there are ten ways to take five things 3 by 3.

12°. Then $\frac{56p^3\times(10q^3\times2mr+10m^3\times2qr+10r^3\times2qm)}{(p+q+m+r)^8}$ in order that this player winning three parts, any one of the three others win three of them also, while the remaining two each win one of them: now there are ten ways to take five things 3 by 3, & two ways to arrange two of them.

13°. Then $\frac{56p^3\times(10qq\times(3mmr+3rrm)+10m^2\times3rrq)}{(p+q+m+r)^8}$ in order that this player winning three parts of them, any two of the other three players each win two of them, while the remaining player wins one of them: now there are ten ways to take five things 2 by 2, & three ways to take the remaining three also 2 by 2.

14°. We will have finally $\frac{28pp\times15qq\times6mm\times1rr}{(p+q+m+r)^8}$ in order that this player winning two parts of the eight, the three others each win also two of them: because there are fifteen ways to take six things 2 by 2, six ways to take four things 2 by 2, & one way to take the remaining two 2 by 2.

It is evident that this is all the ways that this player has to win, since in every other way to distribute the eight parts, this player will win less of them than some one of the other players.

There remains nothing more than to distinguish among all these cases which are those which make this player win all the money in the game; & which are those which make him win only the half or the third, or the fourth: now it is clear that he wins all, when he has taken more parts than any of the other players; that he wins only the half, when one other player takes as many parts as him; that he wins only the third of that which is in the game, when two other players win as many parts as him; & finally the fourth of that which is in the game, when the three other players take as many parts as him. The lot of this player will be therefore

$$\frac{\begin{array}{l} ap^8+8ap^7\times(q+m+r)+28ap^6\times(qq+mm+rr)+56ap^6(qm+qr+mr) \\ +56ap^5\times(q^3+m^3+r^3)+168ap^5\times(qqm+qqr+m^2q+m^2r+r^2q+r^2m) \\ +336ap^5qmr+35ap^4\times(q^4+m^4+r^4)+280ap^4\times(q^3m+q^3r+m^3q+m^3r+r^3q+r^3m) \\ +420ap^4\times(q^2m^2+q^2r^2+m^2r^2)+840ap^4\times(qqmr+m^2qr+r^2qm)+280ap^3\times \\ (q^3m^2+q^3r^2+m^3q^2+m^3r^2+r^3q^2+r^3m^2)+560ap^3\times(q^3mr+m^3qr+r^3qm) \\ +1680ap^3\times(qqm^2r+qqr^2m+m^2rrq)+630appqqmmrr \end{array}}{(p+q+m+r)^8}$$

COROLLARY I.

It is clear that if in this formula, we put q in place of p, & p in place of q, it will be changed into another compound quantity, which will express the lot of the player, of whom the strength or the ability is expressed by q. Because the same reasoning which has been made for the first player, must be made for each of the other players; thus by substituting next successively for p the quantities m & r, & reciprocally, we will have the lots of the two other players of whom the strengths are represented by m & r.

COROLLARY II.

The compound quantity which has been found for the lot of the first player, & which expresses in the course of the eight parts all the events which are favorable to him, this quantity, I say, being added to the three similar quantities, which result from the substitution which has been made, which express in the course of the eight parts, all the events favorable to the three other players, & which are contrary to the first, the sum which will come from them will be equal to unity or to the money which is in the game. Because each of these quantities being a fraction which expresses the part of this money belongs to each player, according to the right which he has to that share of the game, it is necessary that all these reassembled portions be equal to the total. Now as each of these fractions has a common denominator, which in this example is the eighth power of $p + q + m + r$, it follows that the four numerators taken together, must also be equal to this eighth power. The same reasoning will always take place, whatever be the number of players, & the quantity of parts that one plays.

COROLLARY III.

If we name A that which has been found for the lot of the 1st player, & B, C, D, for the lots of the other players, found by the successive substitution of q, m, r, in the place of p, the advantage of the 1st player over the 2nd will be $A - B$, over the 3rd $A - C$, & over the 4th $A - D$; & consequently his total advantage will be $3A - B - C - D$. Whence it follows that the advantage of the 2nd will be $3B - A - C - D$, that of the 3rd will be $3C - A - B - D$, & that of the 4th will be $3D - A - B - C$; some of these quantities will be negatives, & then they will express the disadvantage of the player to which they belong.

REMARK.

If we pay attention to that which has been done in order to find all the terms which compose the lot of the first player in the example that we have proposed, we will see that in all the possible cases that one can propose on this matter, that is to say, whatever be the number of players of which the strengths are p, q, m, r, s, t, &c. & whatever be the number of parts that they must play, for example 20, we will see, I say, that the lot of the first player will be composed of all the terms of the twentieth power of $p + q + m + r + s+$&c. in which the letter p has more dimensions, or as many as some one, or as all the others q, m, r, s, t, &c. The first of these terms is p^{10}, & the last is $p^4q^4m^4r^4s^4$, of which the coefficient must be made by these numbers

$$\frac{20.19.18.17}{1.2.3.4} \times \frac{16.15.14.13}{1.2.3.4} \times \frac{12.11.10.9}{1.2.3.4} \times \frac{8.7.6.5}{1.2.3.4} \times \frac{4.3.2.1}{1.2.3.4}.$$

The 1st factor expresses in how many ways one can take 20 things 4 by 4.

The 2nd the remaining 16, 4 by 4.

The 3rd the remaining 12, 4 by 4.

The 4th the remaining 8, 4 by 4.

And the 5th the remaining 4, 4 by 4.

And their product $2845 \times 1820 \times 495 \times 70 \times 1$ expresses the number of ways in which each of the five players can win four parts, & in this case each of the five players must get back $\frac{1}{5}$ of that which is in the game.

The term of the middle is the one which expresses the number of ways that the first player has of winning 12 parts, the other five players winning either 8, or 7, 6, 5, 4, 3, 2, & 1 of them by all the possible ways.

It will be likewise for the other terms of which we do not give the calculation here, that we will find, if we wish, by following the same rules as in the example resolved.

COROLLARY.

We see by the second corollary & by the following, that to seek the lot of the first player among many, of which the strengths are p, q, r, s, t, &c who play a number n of parts; it is to seek in the multinomial $p + q + m + r + s + t+$&c. raised to the power n, all the terms where p has more dimensions, or at least as many as any of the other letters q, m, r, &c. & that this quantity being found, we find the lot of the other players, by substituting successively for p the other letters q, m, r, s, &c. It is therefore also evident that the quantities found by these substitutions, will represent also successively in the same multinomial all the terms where the letters q, m, r, s, &c. will have more dimensions, or at least as many as all the other letters, & that thus the same method as we have followed, can serve to raise any multinomial to such power as we will wish, & that it suffices for this to find all which belongs to one of the parts of which the multinomial is composed.

EXAMPLE.

We demand the sixth power of $a+b+c+d$. In order to find it, it suffices to seek all the terms of this power where the letter a has more or as many dimensions as each of the other letters b, c, d. There terms are

$$\begin{aligned}
&a^6 + 6a^5 \times (b+c+d) + \frac{6.5}{1.2} \times a^4 \times (bb+cc+dd)\\
&+ \frac{6.3}{1.2} \times a^4 \times 2 \times (bc+bd+cd)\\
&+ \frac{6.5.4}{1.2.3} \times a^3 \times \frac{1}{2} \times (b^3+c^3+d^3)\\
&+ \frac{6.5.4}{1.2.3} \times a^3 \times \frac{3.2}{1.2} \times (bbc+bbd+ccb+ccd+ddb+ddc)\\
&+ \frac{6.5.4}{1.2.3} \times a^3 \times 1.2.3 \times bcd + \frac{6.5}{1.2} \times aa \times \frac{4.3}{1.2} \times (bbcc+bbdd+ccdd) \times \frac{1}{3}\\
&+ \frac{6.5}{1.2} \times aa \times \frac{4.3}{1.2} \times 2 \times (bbcd+ccbd+ddbc) \times \frac{1}{2}.
\end{aligned}$$

If in all these terms which express the parts of the sixth power, in which the letter a dominates, we substitute successively for a the quantities b, c, d, & reciprocally for b, c, d, the quantity a, we will have all the terms of this sixth power where the letters b, c, d, dominate, & by reassembling all these parts, we will have the sixth power demanded.