Correspondence
of
Nicholas Bernoulli and Montmort
on
Divine Providence*

Extracted from
Essay D'Analyse sur les Jeux de Hazard, 2nd Ed.

1713

Letter of Mr. (Nicholas) Bernoulli to Mr. de Montmort (pg. 373–374)

At London this 11 October 1712,

. . . Here is that which I have found on the occasion of your discoveries. I myself am going to make you part of one that I have made lately on the occasion of an argument for Divine Providence, which one has inserted into the Philosophical Transactions. One has already spoken to me of this argument in Holland without saying to me that one had printed some part. This is an argument drawing on the regularity which one observes between the infants of one & and the other sex who are born each year in London. One asserts that if chance should govern the world, it would be impossible that the numbers of males & females approach so near one another during many years in succession, as they have done during 80 years, & one gives for reason that in throwing a great number of tokens, for example, 10000 at random, it is very unlikely that half fall heads & half tails, & again much less probable that this happen a great number of times in sequence. As one has reiterated the same thing here to me, & as one has demanded my sentiment above, I have been obliged to refute this argument, & to prove that there is a great probability that the number of males & females happen each year between some limits again smaller than those which one has observed during 80 years in succession. You sense well, Sir, that it would be a ridiculous thing, if one wished to prove that it is more probable that the number of boys will be rightly equal to the number of girls; but that the ratio between the number of the ones & the others will approach more near to the ratio of equality, is that of which I believe that you are persuaded. I have found in examining the List of infants born in London from 1629 to 1710 inclusively, that there are more males than females; & that by taking an average, the ratio of males to females

*Translated by Richard J. Pulskamp, Department of Mathematics and Computer Science, Xavier University, Cincinnati, OH. Printed on September 9, 2009.

is very near the ratio of 18 to 17, a little greater; whence I conclude that the probability, in order that there is born a boy, is to the probability in order that there is born a girl about as 18 to 17, & that thus among 14000 infants, which is very nearly the number of infants who are born per year in London, there will be about 7200 males & 6800 females. Now the year where there is born the greatest number of males, with respect to the one of the females, has been the year of 1661, in which there is born 4748 males & 4100 females; & the year where there were born the smallest number of males with respect to the number of females, is the year 1703, in which there is born 7765 males & 7683 females. I say that these limits are so great, that one is able to wager at least more than 300 against 1 that among 14000 infants the number of males & females will fall between these limits rather than outside.

From NICHOLAS BERNOULLI *to* PIERRE DE MONTMORT *(pg. 388–393)*

At Paris this 23 January 1713.

I send you the List of Infants of each sex born in London from 1629 until 1710, with my demonstrations of that which I have written to you touching the argument by which one wishes to prove that it is a miracle that the numbers of children of each sex born in London are not more distant from one another during 82 years in succession, & that by chance it will be impossible during so long a time they should be always contained between the limits as small as those which one has observed in the List of 82 years. I claim that there is no reason to be astonished, and that there is a large probability in order that the number of males & females fall between some limits yet smaller than those one has observed. In order to prove this, I suppose that the number of all the infants who are born each year in London is 14000, among whom there will be born 7200 males & 6800 females, so the number of children of each sex should follow exactly the ratio 18 to 17, which expresses the ratio between the facility of the birth of a boy and that of the birth of a girl; or as the number of boys is sometimes greater, sometimes smaller than 7200, I take the limit: For example, in the year 1703, when the number of girls has been the nearest to that of the boys, there is born in this year 7765 males and 7683 females, that which in reducing the sum to 14000, makes 7037 males and 6963 females; the number of females has therefore surpassed the number 6800 by 163, & the number of males has been as much less than 7200. Now I will prove that there are great odds that among 14000 infants, the number of males will be neither greater nor lesser than 7200 by 163; that is to say, that the ratio of the males to the females will not be greater than 7363 to 6637, nor lesser than that of 7037 to 6963. To this end we imagine 14000 dice with 35 faces each, of which 18 are white and 17 black. You know that in the terms of the binomial $18 + 17$, raised to 14000, we will give all the possible cases in order to bring forth as many white faces with these 14000 dice as one would wish; namely, the first term of all of these cases in order to bring forth all white faces; the second, in order to bring forth one black face & 13999 white; the 3$^{\text{rd}}$, in order to bring forth two black faces & 13998 white, &c. So that the 6801$^{\text{st}}$ term will express all the cases in order to bring forth precisely 6800 black faces & 7200 white; the 6638^{th} term the cases in order to bring forth 6637 black faces & 7363 white; and the 6964^{th} term the cases in order to bring forth 6963 black faces and 7037 white. The concern is therefore to find what ratio there is between the sum of all the terms from the 6638th to the 6964th taken inclusively, & between the sum of all the other terms which are on this side of the 6638th, & beyond the 6964th. Now as these terms are seriously great, a singular artifice is necessary to find this ratio: here is how I myself let it be taken. Let generally instead of 14000 the number of all the infants be $= n$, the facility of the birth of a male & of a female as m to f, instead of the ratio 18 to 17; & instead of the limit 163, let there be taken some limit l; let also $p = \frac{n}{m+f}$, or $n = mp + fp$; in our example $mp = 7200$, & $np = 6800$. I seek firstly for an approximation very near the ratio of the term of which the index is $fp + 1$, to the term of which the index is $fp - l + 1$. By the law of progression of these terms, the term

$$fp + 1 = \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \cdots \frac{n - fp + 1}{fp} \times m^{n-fp} f^{fp},$$

& the term

$$fp - l + 1 = \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \cdots \frac{n - fp + l + 1}{fp - l} \times m^{n-fp+1} f^{fp-l},$$

therefore the ratio of the former to the latter is as

$$\frac{n - fp + l}{fp - l + 1} \times \frac{n - fp + l - 1}{fp - l + 2} \times \frac{n - fp + l - 2}{fp - l + 3} \times \cdots \frac{n - fp + 1}{fp} \times \left.\frac{f}{m}\right\rceil^{l} \text{ to } 1,$$

or in putting mp[1] in the place of $n - fp$ this ratio is as

$$\frac{mp + l}{fp - l + 1} \times \frac{mp + l - 1}{fp - l + 2} \times \frac{mp + l - 2}{fp - l + 3} \times \cdots \frac{mp + 1}{fp} \times \left.\frac{f}{m}\right\rceil^{l} \text{ to } 1;$$

I suppose that the factors of the first term of this ratio except the last $\left.\frac{f}{m}\right\rceil^{l}$; are in geometric progression and their logarithms in arithmetic progression; this supposition is very nearly the truth, especially when n is a large number; the sum therefore of all their logarithms will be

$$\frac{1}{2} l \times \overline{\log \frac{mp + l}{fp - l + 1} + \log \frac{mp + 1}{fp}},$$

that is to say the sum of the logarithms of the first & last factor, multiplied by the mean of the number of all the terms, to which if one adds the logarithm of $\left.\frac{f}{m}\right\rceil^{l}$, that is to say $l \times \log \frac{f}{m}$, one will have

$$\frac{1}{2} l \times \overline{\log \frac{mp + l}{fp - l + 1} + \log \frac{mp + 1}{fp}} + l \times \log \frac{f}{m},$$

or

$$\frac{1}{2} l \times \overline{\log \frac{mp + l}{fp - l + 1} + \log \frac{mp + 1}{mp} + \log \frac{fp}{mp}}$$

[1] Correction of the original from np.

for the logarithm of the ratio sought. And by consequence the ratio will be the same as

$$\overline{\frac{mp+l}{fp-l+1} \times \frac{mp+1}{mp} \times \frac{fp}{mp}}\Big]^{\frac{1}{2}l} \text{ to } 1.$$

If one wishes to approach nearer the true value, one would be able to divide this sequence of factors

$$\frac{mp+l}{fp-l+1} \times \frac{mp+l-1}{fp-l+2} \times \frac{mp+l-2}{fp-l+3} \times \&c$$

into many parts, & to suppose that the factors of each part are in geometric progression; but one has no need to do this; because all the values that one will find for these different assumptions will be very little different from one another; & when likewise for this first assumption I would make this ratio a little greater than it is, this excess would be hardly significanct with regard to that which I will disregard in the following.

If one takes at this time the terms which precede immediately the terms of index $fp+1$ & $fp-l+1$; namely those of which the index is fp & $fp-l$, the ratio of the former to the latter will be as

$$\frac{mp+l+1}{fp-l} \times \frac{mp+l}{fp-l+1} \times \frac{mp+l-1}{fp-l+2} \times \cdots \frac{mp+2}{fp-1} \times \frac{f}{m}\Big]^{l} \text{ to } 1;$$

& consequently greater than

$$\frac{mp+l}{fp-l+1} \times \frac{mp+l+1}{fp-l+2} \times \frac{mp+l-2}{fp-l+3} \times \cdots \frac{mp+1}{fp} \times \frac{f}{m}\Big]^{l}$$

or

$$\overline{\frac{mp+1}{fp-l+1} \times \frac{mp+p}{mp} \times \frac{fp}{mp}}\Big]^{\frac{1}{2}l} \text{ to } 1,$$

since each factor of the first sequence is greater than the corresponding factor of the second. For the same reason the term of which the index is $fp-1$ will be to the term, of which the index is $fp-l-1$, a greater ratio than the term fp to the term $fp-l$; & the term $fp-2$ will have to the term $fp-l-2$ a greater ratio than the term $fp-1$ to the term $fp-l-1$, & also consecutively backwards always from one term to the first. This is why if one divides all the terms which precede the term $fp+1$ into some classes, of which each contains a number equal to the terms expressed by l, in commencing to compute at the term of which the index is fp; the first term of the first class will be to the first term of the second class a greater ratio than the term $fp+1$ to term $fp-l+1$; & the second term of the first class will be to the second of the second class a ratio yet greater; & the third of the first class to the third of the second a ratio yet greater, & thus consecutively ; therefore also all the terms of the first class taken together will have to all the terms of the second class taken together a greater ratio than the term $fp+1$ to the term $fp-l+1$. And by the same reason all the terms of the second class will have to all the terms of the third class; *likewise*, all the terms of the third to those of the fourth, &c. a greater ratio than the term $fp+1$ to the term $fp-l+1$; that is to say as

$$\overline{\frac{mp+1}{fp-l+1} \times \frac{mp+1}{mp} \times \frac{fp}{mp}}\Big]^{\frac{1}{2}l} \text{ to } 1.$$

Therefore if one names

$$\overline{\frac{mp+1}{fp-l+1}\times\frac{mp+1}{mp}\times\frac{fp}{mp}}\Big]^{\frac{1}{2}l}=q;$$

& the sum of the terms of the first class $= s$, the sum of the terms of the second class will be smaller than $\frac{s}{q}$; & the sum of the third class smaller than $\frac{s}{qq}$, and that of the terms of the fourth class smaller than $\frac{s}{q^3}$, &c. Therefore the sum of all the classes, excepting the first, even when the number of classes would be infinite, will be smaller than this $\frac{s}{q}+\frac{s}{qq}+\frac{s}{q^3}+\frac{s}{q^4}$, continued to infinity, that is to say smaller than $\frac{s}{q-1}$; whence it follows that the sum of the first class, that is to say of all the terms which are between the term $fp+1$ & the term $fp-l+1$, comprehending of them also the term $fp-l+1$, will have to the sum of all the preceding a ratio greater than $q-1$ to 1, or in putting for q its value, that

$$\overline{\frac{mp+1}{fp-l+1}\times\frac{mp+1}{mp}\times\frac{fp}{mp}}\Big]^{\frac{1}{2}l}-1\text{ to }1;$$

consequently by putting m instead of f, & f instead of m; the sum of all the terms which are between the terms $fp+1$ & the term $fp+l+1$, by comprehending the term $fp+l+1$, will have to the sum of all the others following to the last a greater ratio than

$$\overline{\frac{fp+l}{mp-l+1}\times\frac{fp+1}{fp}\times\frac{mp}{fp}}\Big]^{\frac{1}{2}l}-1\text{ to }1.$$

Therefore finally the sum of all the terms from the term $fp-l+1$ to the term $fp+l+1$, taken inclusively, without counting even the term $fp+1$ which is in the middle, will be to the sum of all the other terms at least a greater ratio than the smaller of the two quantities

$$\overline{\frac{mp+l}{fp-l+1}\times\frac{mp+1}{mp}\times\frac{fp}{mp}}\Big]^{\frac{1}{2}l}$$

&

$$\overline{\frac{fp+l}{mp-l+1}\times\frac{fp+1}{fp}\times\frac{mp}{fp}}\Big]^{\frac{1}{2}l}$$

less the unit to the unit; *that which it was necessary to find.*

We apply this at this time to our example, where

$$n=14000, mp=7200, fp=6800, l=163,$$

& we will find

$$\begin{aligned}&\frac{1}{2}l\times\overline{\log\frac{mp+l}{fp-l+1}+\log\frac{mp+1}{mp}+\log\frac{fp}{mp}}\\&\quad=\frac{163}{2}\times\overline{\log.\frac{7363}{6638}+\log.\frac{7201}{7200}+\log.\frac{6800}{7200}}\\&\quad=\frac{163}{2}\times\overline{0.0450176+0.0000603-0.0248236}=1.6507254;\end{aligned}$$

the number of this logarithm is $44\frac{58}{100}$. In putting fp instead of mp, & mp instead of fp, we will find

$$\begin{aligned}&\frac{1}{2}l \times \overline{\log\,\frac{fp+l}{mp-l+1} + \log\,\frac{fp+1}{fp} + \log\,\frac{mp}{fp}}\\ &\quad= \frac{163}{2} \times \overline{\log.\frac{6963}{7038} + \log.\frac{6801}{6800} + \log.\frac{7200}{6800}}\\ &\quad= \frac{163}{2} \times \overline{-0.0046529 + 0.0000639 + 0.0248236}\\ &\quad= 1.6491199;\end{aligned}$$

the number of this logarithm is $44\frac{58}{100}$; whence I conclude that the probability that among 14000 infants the number of males will neither be greater than 7363, nor smaller than 7037, will be to the probability that the number of males falls outside of these limits in a ratio greater at least than $43\frac{58}{100}$ to 1. Therefore one is able already to wager with advantage that in 82 times the number of males will not fall three times outside of these limits. Now in examining the List of infants born during the 82 years in London, you will find that the number of males has been 11 times greater than 7363; namely in 1629, 39, 42, 46, 49, 51, 59, 60, 61, 69, 76; you will find also easily that one is able to wager more than 226 against 1 that the number of males will not fall in 82 years 11 times outside of these limits. You must note also that if I have taken another limit greater than 163, but yet smaller than the greatest that one finds in this List, I would have found a probability much greater than 43 to 1, that the number of infants of each sex will fall each year rather between this limit than outside. Therefore there is no reason at all to be astonished that the numbers of infants of each sex are not more distant from one another, this that I have wished to demonstrate. I myself remember that my late uncle demonstrated a similar thing in his tract *De Arte conjectandi*, which is being printed at present in Basel, namely, that if one wishes to discover by experiences repeated often the number of cases by which a certain event is able to happen or not, one is able to increase the observations in such a manner that finally the probability that we would find the true ratio that there is between the number of cases, may be greater than a given probability. When this book will be published we will see if in these types of matters I have found an approximation as correct as he. I have the honor to be with a perfect esteem,

Sir,

your very humble & very
obedient servant,

N. Bernoulli