PROBLEMS TO SOLVE
FIRST PROBLEM
ON THE GAME OF TRIEZE

PIERRE RENARD DE MONTMORT
EXTRACTED FROM
ESSAY D'ANALYSE SUR LES JEUX DE HAZARD
2ND EDITION OF 1713, §218, P. 278

To determine generally what is in the game the advantage of the one who holds the cards. One will find the explanation of the rules of this game on pages 130 & 131. See the letters of 10 April & of 19 November 1711. (page 278)

Extract of the letter from Jean Bernoulli to M. de Montmort
At Basel this 17 March 1710 (page 298)

The four problems that you propose at the end of your treatise are interesting; but the first seems to me insoluble because of the length of the calculation that it would demand, & that the human lifespan would not suffice to accomplish it:

Extract of the letter of Nicolas Bernoulli to M. de Montmort
From Basel this 26 February 1711 (pages 308–309)

I have not yet attempted the general solution of the problem on the game of Treize, because it seems to me almost impossible; this is also why I was greatly astonished by that which you say, that you have found $\frac{690568237871898 97}{2413478176215356 25} A$ for the advantage of the one who holds the cards; but in examining the thing a little more closely, I had the thought, that you perhaps have resolved generally this problem only in the supposition, that the one who holds the cards having won or lost, the game would conclude; that which confirms to me in this thought, is that I have found for this hypothesis a general formula, which applied to the particular case of 52 cards, gives for the advantage of the one who holds the cards this fraction $\frac{99177450342464537}{3362451227815680 00} A$ which is a little greater than yours, but which has for denominator a number composed of nearly the same factors as the one of yours, this which makes me believe that you have made an error of calculation in the application of your formula: here is mine of which I come to speak.

$$S = \frac{1}{1} - \frac{n-p}{1.2 \times n-1} + \frac{n-2p}{1.2.3 \times n-2} - \frac{n-3p}{1.2.3.4 \times n-3} + \&\text{c.}$$

up to the a term which is $= 0$; by p I intend the number of times that each different card is repeated, & by n the number of all cards. I have also calculated the case for 4 cards, of which you speak, & I have found $\frac{130225}{172279} = \frac{56908325}{75285923}$ as you; but it is apropos to observe here, that according to the rules of this game there, it is not necessary to suppose that the game is complete, when the one who has the hand just loses, because then he is obligated to

cede the hand to another, & the game continues; this is why the advantage of the one who holds the cards being diminished by the disadvantage that he had in losing the hand, will be in the aforesaid case only $\frac{130225}{344558}$ the half of that which had been found. If one assumes that there were many players against the one who has the hand, & that their number is $= n$, his advantage will be $\frac{130225}{344558} \times n$, & the one of the other players $\frac{130225}{344558} \times$ either $n - 2$, or $n - 4$, or $n - 6$, &c. according to the rank that each occupies by relation to the right of the one who holds the card. This remark extends itself on all of the players in which the hand passes from one to the other; also in your first case of Lansquenet I have found that the advantage of Pierre is only $\frac{161}{1006} A$, the disadvantage of Paul — $\frac{163}{4024} A$, & the one of Jacques $\frac{481}{4024} A$.

Extract of the letter of M. de Montmort to Nicolas Bernoulli
At Montmort 10 April 1711.

(page 315) It is necessary, Sir, that you have badly copied your general formula for Treize, because I am not able to find in it my count: here is mine. Let n be the number of cards, p the number of times that each different card is repeated, let also $\frac{n}{p} = m$ & $n - m = q$, one will have the sought lot

$$= p^m - mp^{m-1} \times q + 1 + \frac{m.m-1}{1.2}p^{m-2} \times q + 1.q + 2$$
$$- \frac{m.m-1.m-2}{1.2.3}p^{m-3} \times q + 1.q + 2.q + 3$$
$$+ \frac{m.m-1.m-2.m-3}{1.2.3.4}p^{m-4} \times q + 1.q + 2.q + 3.q + 4 - \&\text{c.}$$

the whole divided by as many products of the numbers $n - 1.n - 2.n - 3.n - 4$ &c. as there are units in $\frac{n}{p}$.

Note, 1. That it is necessary to take as many terms of this sequence as m expressed in units. 2. That it is necessary to change all the signs of this sequence when m is an even number.

Also I find that the lot of the one who holds the cards at the beginning of the game is $\frac{310404641408725522}{241347817621535625}$, and his advantage is $\frac{690568237871 89897}{241347817621535625}$. I do not believe that there is an error in this calculation; but surely there is none at all in the method.

(pages 317–318) You say, Sir, that you have calculated the case of four cards, *page* 64, <See section 114> & that you have found as me $\frac{130225}{172279}$; but you add that according to the rules of this game it is not necessary to suppose that the game be complete, when the one who has the hand just loses; because now, say you, he is obligated to cede the hand to another. This is why the advantage of the one who holds the cards being diminished by the disadvantage that he has in losing the hand, will be only $\frac{130225}{344558}$ the half of that which had been found, & you add next; if one supposes that there are many players against the one who has the hand, & that their number be n, his advantage will be $\frac{130225}{344558} \times n$, & the one of the other players $\frac{130225}{344558} \times n - 2$, $\frac{130225}{344558} \times n - 4$ &c. according to the rank that each occupies. You extend next this remark on Lansquenet, & and it seems that your series of notices to apply to all sorts of games. For me I believe myself to have some reasons to think otherwise: I am going to expose them to you. Firstly, in regard to Treize, it is certain that the one who quits the hand is not at all obligated to continue to play, & moreover, he is not obligated to wager the same sum in the game; on the contrary it happens that in this game those who noticed, as it is easy to discover by practice, that the advantage is for the one who holds the cards, they keep everything when they hold the hand, & wager little money in the game when they do not have the hand. There is yet to remark that in this game the wagers

increase or diminish without ceasing likewise the number of players, & that in Lansquenet the number of players is able to decrease from one hand to the other of the same player. In such a way that in my note one is able to say nothing useful & certain on these games, when in taking the part to determine at each move the advantage or the disadvantage of the one who holds the hand with respect to a determined number of wagers of the players. If I make to enter in Lansquenet the consideration of the expectation that the one who holds the cards has to make the hand: this had been only by elegance, because in the foundation that one is only correct when supposing that the number of players will be always the same amount as when Pierre will have the hand, this which is uncertain. It suffices it seems to me in order to be educated, as perfectly as it is possible, of the chances of these games, for example of Lansquenet, knowing that by ratio to such number of players & of wagers there is as much advantage & disadvantage for each of the players, according as the different places that they occupy.

Extract of the letter of Mr. Nicolas Bernoulli to Mr. de Montmort
At Basel this 10 November 1711 (page 323).

You have reason to say that you have not found your count in my formula for Treize, because an error is slipped there; it is necessary to put

$$S = \frac{1}{1} - \frac{n-p}{1.2.n-1} + \frac{n-p.n-2p}{1.2.3.n-1.n-2} - \frac{n-p.n-2p.n-3p}{1.2.3.4.n-1.n-2.n-3} + \&\text{c.}$$

instead of

$$S = \frac{1}{1} - \frac{n-p}{1.2.n-1} + \frac{n-2p}{1.2.3.n-2} - \frac{n-3p}{1.2.3.4.n-3} + \&\text{c.}$$

This error, to that which I myself can remember, comes from that which by making the calculation I have put on the table on these last factors of the terms of each fraction, in order to indicate the law of the progression which there is among the terms of this series; whence it happens that next no more remembering the true solution, I have allowed to escape the other factors. You will see that this formula thus corrected will agree exactly with yours. The number

$$\frac{69056823787189897}{241347817621535625}$$

which you give for the case $n = 52$ & $p = 4$ is not yet correct, it is necessary according to your formula & mine

$$\frac{69056823706869897}{241347817621535625} = \frac{7672980411874433}{26816424180170625}.$$

The method of which I am being served in order to find this formula is the same as that of which I was being served once in my Latin Remarks for the resolution of the particular case of $p = 1$.

Extract of the letter of Nicolas Bernoulli to M. de Montmort
At Basel this 10 November 1711 (pages 327–328)

I am surprised, Sir, to see your objections against my remarks on the games in which the hand turns from one to another; it seems to me that you are much wrong to oppose me some things which are also as much against you as against me; because if you are in a state to suppose, for example, at Lansquenet, that the number of players & of the wagers are always the same, & that the game continues as long as Pierre will have the hand, why would there not be permitted to me to suppose again the same thing, the same after Pierre will have lost the hand? You say that one is able to say nothing useful & certain on these

games, because the number of wagers & of the players are always able to vary there: this is true, & this is also the reason why one must make a certain hypothesis to which one can take oneself in the calculation. I have therefore made this hypothesis, namely that one continues to play when one just loses the hand, because it is more natural & more conforming to that which happens ordinarily, than yours which supposes that the game continues as long as Pierre will have the hand, this which is a condition which is being scarcely practiced among the players, especially when they know that there is advantage to have the hand. But you oppose me still when, for example at Treize, the one who quits the hand is no longer obligated to continue the game, to which I respond that an honest man must be held obligated to it, although one is not expressly agreed to that; because it is certain that ordinarily one begins the game with the plan to make a great enough number of games, & not to end immediately after the first move, this which engages the players tacitly to continue the game during a certain time. It will not be permissible to quit the game after having had the advantage of the hand, at least one does not wish to pass in order for a man who thinks rather of grabbing the money of the others than to amuse them. You see by this, Sir, that you would not have done badly to take into consideration, not only the advantage that one has in conserving the hand; but also the disadvantage that one has in losing it.

Extract of the letter of M. de Montmort to Mr. Nicolas Bernoulli
At Paris 1 March 1712 (pages 344)

Your formula for Treize is quite correct. I myself have well doubted that the error of the preceding could result only from some inadvertance in transcription. The idea that I myself have made of your infallibility in Geometry has not permitted to suspect that you had been able to deceive yourself at the base of a method.