ON MOIVRE'S *DOCTRINE OF CHANCES*

MONTMORT AND NICHOLAS BERNOULLI

Extract from the letter from Mr. (Nicolas) Bernoulli to Mr. de M. . .
At Basel this 2 June 1712
Essay d'Analyse sur Les Jeux de Hazard, 2nd ed. p. 351

. . .I know not at all of news of the sciences, except that Mr. de Moivre who is member of the Society in England, had printed at London a Book on Chances. As I believe that you will be curious to have this Book when it will be printed, & as I hope to pass from Holland into England, I will try to procure for you a copy of it.

Extract from the letter from Mr. de M. . . to Mr. N. Bernoulli
At Montmort this 5 September 1712
Essay d'Analyse sur Les Jeux de Hazard, 2nd ed. pp. 362-369

I have received at the beginning of the month of August the Book of Mr. Moivre, the Author had addressed it for me to Mr. l'Abbé Bignon[1] who has had the kindness to send it to me. Out of that which you have sent me, & out of the manner of which the Author speaks in the Preface, I myself expected entirely another thing; I expected to find the solution of the four Problems that I proposed at the end of my Book, or at least the solution of some one of the four, & some novelties of this kind proper to understand the routes that I have opened; but you find that his work is limited nearly entirely to resolve in a more general manner than I have done, the simplest & easiest questions which are in my Book; for example, the five Problems of Mr. Huygens what I have treated summarily only because of their extreme facility, in comparison to the greater part of the other Problems which are resolved in my Book. You will find finally that the questions that he treats, which are not at all resolved, are in our Letters; so that I do not believe that he has in this Work, moreover very well, nothing new for us, & nothing which is able to give us pleasure by the uniqueness, if this is not the way to find what is often new, & always good and ingenious. Here are some remarks that I have cast in haste on the paper these days past, when I worked to render account of this Work to Mr. l'Abbé Bignon who had demanded of me his sentiment. You know without doubt that this illustrious Abbé, who is in France the Protector of the Sciences & of the Scholars, has an expanse of knowledge well beyond the ordinary limits, a very great taste for all that which is of the resort of the mind, & much ardor to contribute to the perfection of the Sciences.

The first Problem is a particular case of the general formula

$$\overline{m-1}^{p-q} \times p \cdot \frac{p-1}{2} \cdot \frac{p-2}{3} \cdot \frac{p-3}{4} \cdot \frac{p-4}{5} \cdot \text{\&c.}$$

of which I have made part to Mr. your Uncle in my Letter of 15 November 1710. This formula gives the number of chances which there are to bring forth precisely a certain

Translated by Richard J. Pulskamp, Department of Mathematics and Computer Science, Xavier University, Cincinnati OH. August 25, 2009.

[1] *Translator's note*: Jean-Paul Bignon (1662-1743) Librarian of the king.

number q of six, with a certain number p of dice, of which the number of faces is m. In the case resolved by Mr. Moivre the question is to find how many chances there are to bring forth no six with 8 dice, or to bring forth one only of them (because it is the same thing to cast eight times in sequence one die, or to cast eight of them at one time,) one has therefore by my formula by taking the denominations of the Author, who calls $a+b$ that which I call m, & n that which I name p,

$$\frac{b^n + nb^{n-1}}{\overline{a+b}^n}$$

for the lot of the one who holds that part &

$$\frac{1}{\overline{a+b}^n} \times \frac{n.n-1}{1\cdot 2}b^{n-2} + \frac{n.n-1.n-2}{1\cdot 2\cdot 3}b^{n-3} + \frac{n.n-1.n-2.n-3}{1\cdot 2\cdot 3\cdot 4}b^{n-4}$$
$$+\frac{n.n-1.n-2.n-3.n-4}{1\cdot 2\cdot 3\cdot 4\cdot 5}b^{n-5} + \frac{n.n-1.n-2.n-3.n-4.n-5}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}b^{n-6}+$$
$$+\frac{n.n-1.n-2.n-3.n-4.n-5.n-6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}b^{n-7} + \frac{n.n-1.n-2.n-3.n-4.n-5.n-6.n-7}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}b^{n-8}$$
$$=\frac{\overline{a+b}^n - b^n - nb^{n-1}}{\overline{a+b}^n}$$

for the one who holds the contrary part, thus as the Author has found.

The second Problem is not different from the one which one finds resolved on page 177[2] of my Book, but in this that one makes $a = b$; I have supposed this formula in the solution in many other problems.

The third Problem is found resolved word for word in my Letter[3] of the first March 1712.

The fourth Problem is a kind parallel to the preceding, & as I have already remarked in my Letter of the first March 1712, all the similar questions where the question is, in supposing equal the lots of the Players, to determine what is the facility that each of the Players has to win in a game, where one will have more points than the other, have no difficulty than that which one finds in the resolution of the equalities; because it is always the same method to suppose the expression of the lot of each of the Players $= \frac{1}{2}$ when there are two Players, $= \frac{1}{3}$ when there are three, $= \frac{1}{4}$ when there are four.

The 5th Problem is resolved on page 144[4] of my Book, & the formula is the same in it.

The Lemma which follows is quite curious, but it is drawn from page 141[5] of my Book, & I have sent the solution of it to Mr. your Uncle in a very general formula. My Letter is from 15 November 1710.

Problems 6 & 7 are a very ingenious extension of the 5th Problem. I do not know if the limits marked by the Author are perfectly correct. I would well wish to know if one could at all have by another way the solution of this Problem.

Problem 8 is resolved in the same manner as on page 175[6] of my Book, & next in my Letter of the eighth of June 1710; but I swear that I am not at all content with these solutions; it is a great fault, it seems to me, to be obliged to examine in detail what are in one

[2]See page 244 of the second edition. *Translator's note*: This and all other page references are to the 1st edition.

[3]*Translator's note*: The problems are introduced in the letter of 10 November 1711 (pp. 332-334) and resolved in the letter of 1 March 1712 (pp. 341-344).

[4]Page 231 of the 2nd Edition.

[5]Page 46 of the 2nd edition.

[6]Page 242 of the 2nd edition.

same term the arrangements of letters favorable to Pierre, Paul & Jacques, an inconvenience which is not found at all in the case of two Players, & which it would be necessary to try to surmount in the case of many.

Problem 9 is the last of the five proposed by Mr. Huygens. I myself have noticed in reading it that Mr. Moivre had observed the fault that I have made by putting a false enunciation at the head of this Problem. I have indicated to Mr. your Uncle that which has given place to this mistake. The Problem which follows is quite well resolved, Mr. your Uncle has given the same solution of it in the Letter which he has given me the honor to write me, dated on 17 March 1710.

The 11^{th} which is the 2^{nd} of the five proposed by Mr. Huygens is resolved otherly than in my Book page 158;[7] this comes from that we have differently understood the enunciation: I do not know who of we two has taken the true sense of the Author. I have found that the number of black casts being b, & the number of white tickets a, the number of Players q, the numbers interposed from q to q of order a, of the figurate numbers, page 80,[8] will give the lots of the Players. This remark which has appeared to me curious gives the facility in order to find the particular formulas, proper to shorten the calculation, & without which it would be impossible to find the lots of the Players, when a & b are large numbers, I have found that the number of Players, q, being 3, as in the Problem of Mr. Huygens, the formula

$$\begin{aligned}&\frac{1}{6} \times n \times p^3 - 3npp - \frac{n \times \overline{n-1}}{1.2} \times 9pp + 2np + \frac{n.n-1}{1.2} \times 45p\\ &+ \frac{n.n-1.n-2}{1.2.3} \times 54p - 1 \times \overline{n-1} \times 10 + \frac{n-1.n-2}{1.2} \times 46\\ &+ \frac{n-1.n-2.n-3}{1.2.3} \times 63 + \frac{n-1.n-2.n-3.n-4}{1.2.3.4} \times 27,\end{aligned}$$

divided by

$$\frac{p.p-1.p-2.p-3}{1.2.3.4},$$

gave the lots of the Players, I suppose $p = b + q$, & $n = \frac{p}{q}$.

It is necessary to remark that in order to find by this formula the lot of Paul, one must, 1°, understand by n a quantity equal to the quotient of $p-1$ divided by q. 2° That it is necessary to substitute everywhere $p-1$, & its powers in the place of p, & of its powers, & for the lot of Jacques, 1°, understand by n a quantity equal to the quotient of $p-2$ divided by q. 2° To substitute everywhere $p-2$, & its powers in the place of p, & of its powers. Thus supposing, for example, three Players, 58 black tokens and four white, one will have all in one coup the lots of Pierre, of Paul & of Jacques as these three numbers 198345, 185745, 173755. I myself is served in order to find this formula from the method that I have given in order to find the sum of the figurate numbers interposed as one will wish, & raised to any power: this method will furnish easily some formulas for all the similar cases.

I know not why the Author has given the labor to resolve in Propositions 12, 13, & 14 of his Book the Problems posed by Mr. Huygens which are already resolved in mine; because besides these Problems are too facile in order to stop anew; the Author has well seen by the Corollary on page 157[9] that the way of infinite series was not at all unknown to me, & that it was used. If this Author had wished to push this matter, & to teach us some new things,

[7]Page 219 of the 2nd edition.

[8]Page 2 of the 2nd edition.

[9]Page 217 of the 2nd edition.

he had been able to seek the sum of the infinite series that one finds in this Corollary: this is in what consists all the difficulty of these sorts of questions.

I have observed by design that this research was not easy; & as there is found an infinity of series in which the exponents have their 2nd, 3rd, 4th, &c. difference constant; this discovery would be a great extension, & one of extreme utility.

The Author has given the 4th Problem of Mr. Huygens a sense different from the one that I give to it, also it is found differently; for me I believe to have taken the true, & it would be necessary, it seems to me, that the word minimum must be found in the enunciation, in order that the one of Mr. Moivre was preferable. Although it may be, nothing is more indifferent, each solution is only a particular example of my Proposition 13, page 94.[10]

Problem 15 is our Problem of the Pool of which I have sent you the solution in my Letter of 10 April 1711, I have been quite surprised to find this very risky Corollary by the Author: *Si plures sint collusores, ratio sortium eadem ratiocinatione invenietur*. You have made me understand, Sir, that the application of this Problem to the case of four & of five & of six Players was infinitely more difficult than that which is limited to three Players. The way of the infinite series that Mr. Moivre employs, & which is also employed in my Letter, is easy for three Players, but absolutely impractical for many Players.

Problems 16 & 17 are only two very simple cases of one same Problem, it is nearly the only which has escaped me of all those which I find in this Book. Although the Author may make profession in the Preface to generalize all, it seems to me that he would have been able to render the Problem more curious & of a greater extent, by supposing that the number of bowls of Pierre is n, & the number of bowls of Paul m, here is that which I have found by reading the Problem of the Author. Let A be the lot of Paul when one point is lacking to him, & one point to Pierre; B his lot when two points are lacking to him & one point to Pierre; C his lot when three points are lacking to him & one point to Pierre, &c. one has

$$A = \frac{m}{m+n},$$

$$B = \frac{m.m-1 + m \times n \times A}{m+n.m+n-1},$$

$$C = \frac{m.m-1.m-2 + m.m-1 \times n \times A + m \times n \times m+n-2 \times B}{m+n.m+n-1.m+n-2}$$

$$D = \frac{m.m-1.m-2.m-3 + m.m-1.m-2 \times n \times A + m.m-1 \times n \times m+n-3 \times B}{m+n.m+n-1.m+n-2.m+n-3} + \frac{m \times n \times m+n-3 \times m+n-2 \times C}{m+n.m+n-1.m+n-2.m+n-3}$$

$$E = \frac{m.m-1.m-2.m-3.m-4 + m.m-1.m-2.m-3 \times n \times A +}{m+n.m+n-1.m+n-2.m+n-3.m+n-4} + \frac{m.m-1.m-2.n \times m+n-4 \times B + m.m-1 \times n \times m+n-4 \times m+n-3 \times C}{m+n.m+n-1.m+n-2.m+n-3.m+n-4} + \frac{m \times n \times m+n-4 \times m+n-3 \times m+n-2 \times D}{m+n.m+n-1.m+n-2.m+n-3.m+n-4}$$

$$F = \&c.$$

One could again render the Problem more general, by supposing the forces:: $a : b$, it seems to me that the solution of it would be more difficult.

[10] Page 26 of the 2nd edition.

Problem 18 is a Problem of combinations, & has much in relation with the first; also the solution of each is drawn easily from my formula, & also from Proposition 30, page 136,[11] which is much more general, & gives how many chances in order to bring forth precisely certain faces, this which does not give the formula of the Author; for example, if one wishes to know how many chances to bring forth in eight coups one ace & one deuce only, I find that there is 229376 against 1450240, & generally, p, being the number of dice, q the number of different points that one must bring forth, f the number of faces of each die, the formula is

$$q.q-1.q-2.q-3.q-4.\text{ \&c.}\times \underset{q}{\overset{p}{\boxed{}}}\times \overline{f-q}^{\,p-q}.$$

I have again found that if one demands how many different coups which are able with eight dice to give precisely one ace & one deuce, neither more nor less, the number is 84; & generally the number of faces being f, the rank $f-1$ of the figurate numbers will give how many chances to bring forth precisely one ace. The rank $f-2$ of the figurate numbers will give how many chances to bring forth precisely one ace & one deuce. The rank $f-3$ of the figurate numbers will give how many chances in order to bring forth precisely one ace, one 2 & one 3, &c.

Thus one finds, for example, that playing with some dice which would have each twelve faces, there is one way in order to bring forth ace & deuce with two dice, six ways with three dice, 55 with four dice, 220 with five dice, 715 with six dice, 2002 with seven dice, 5005 with eight dice, &c. these numbers 1, 10, 55, 220, 715, 2002, 5005, &c. belong to the order $f-2$, of the figurate numbers which in this case is the 10th.

Problem 19 has much in relation with the 5th Problem; however the Author employs another method, it appears to me quite well invented, although it has perhaps the fault to not give at all a solution exact enough.

The rest of the Book contains seven propositions on a matter extremely curious to which I have the first thought; namely how long must a game endure where one plays always by reducing, this which I explain in my Book page 178,[12] & better yet in my last Corollary page 184[13] where I give this series

$$\frac{1}{4}+\frac{3^1}{4^2}+\frac{3^2}{4^3}+\frac{3^3}{4^4}+\frac{3^4}{4^5}+\text{\&c.}$$

in order to determine the odds that the game will end in less than 3, 5, 7, 9, 11, 13, &c. points to infinity. I end this Corollary with these words: *one will find without much effort some similar formulas for the other cases, & the research of it will appear curious*. The truth is however that this problem is not at all entirely easy, even with the help of the particular formula for the case of three games. I see with pleasure that Mr. Moivre is come at the end of this Problem in whole, & that his solution accords perfectly with ours. I am much in pain to know how this scholarly Geometer is arrived to this method to raise $a+b$ to the power n, to subtract the extreme terms from this product, & to multiply next the rest by the square of $a+b$, & thus in sequence as many times as there are of units in $\frac{1}{2}d$. a solution of this nature surprises me, & the more that the Author who had supposed equal the number of chances for Peter & for Paul coming to suppose it in any ratio, is obliged to take another route; instead that according to you & according to me the method is the

[11] Page 44 of the 2nd edition.
[12] Page 277 of the 2nd edition.
[13] Page 276 of the 2nd edition.

same for the general & particular solution; this does not prevent that I regard highly this discovery, & in general all his Work, in which I am pleased to have given the occasion, in opening first the course. I appears to me first quite singular that he has filled it with some things of which we ourselves have conversed in our Letters; but it is natural that having made his Work out of mine, & wishing to push these matters, the same ideas have come to him as to us. I would have only wished, & it seems to me that equity demands it, that he had recognized with frankness that which I had right to claim in his Work. I am obliged to him of some very honest expressions of which he is served in his Preface in speaking of me & of my Book; but I know not in truth on what he is based when he says

> "Huguenius primus, quod sciam, regulas tradidit ad istius generis Problematum solutionem quas nuperrimus Autor Gallus variis exemplis pulchre illustravit, sed non videntur viri clarissimi ea simplicitate ac generalitate usi fuisse, quam natura rei postulabat: etenim dum plures quantitates incognitas usurpant, ut varias collusorum conditiones repraesentent, calculum suum nimis perplexum reddunt; dumque collusorum dexteritatem semper aequalem ponunt, doctrinam hanc ludorum intra limites nimis arctos continent."[14]

I am not able to conjecture for what reasons this Author made these reproaches to me, & what motive carries him to pronounce against me, leaving me only the merit to have applied to some examples the supposed rules of Mr. Huygens, I call from this judgment to the Geometers who will read this that Mr. Huygens & Mr. Pascal, of whom the Author speaks not at all, have given on this matter.

Extract from the letter from Mr. (Nicolas) Bernoulli to Mr. de M. . .
At Brussels this 30 December 1712
Essay d'Analyse sur Les Jeux de Hazard, 2nd ed. p. 378-380 & 386-387.

I am very glad that you have received the Book of Mr. de Moivre *De Mensura Sortis*. It is true that nearly all the Problems which are proposed are resolved either in your Book or in our Letters. As I know that Mr. de Moivre awaited with impatience the judgment that you would make on his Book, I have taken the liberty to send to him the principals of your remarks, I will touch here some of them.

The 1st Problem is one particular case of the 2nd which is resolved on page 177[15] of your Book in the case of $a = b$; & thus I would prefer more to deduce the solution of the 1st Problem from the general formula which expresses all the terms of the binomial $a + b$ raised to any power, than from the formula

$$\overline{m-1}^{p-q} \times \frac{p.p-1.p-2}{1.2.3} \text{ \&c.}$$

The limits that the Author gives for Problems 5, 6, 7 are correct enough when q is a great number; we know besides that one can wager with advantage to bring forth sonnez in 25 coups, & that he would have disadvantage to undertake it in 24 coups; now we multiply

[14]Huygens first, who I know, has related the rules to the solution of Problems of this kind which in recent times the French author with various examples has illustrated beautifully; but the illustrious men do not appear to have used that simplicity and generality, as the nature of the thing demanded: indeed while they use many unknown quantities, in order that they may represent the various conditions of the players, they render their calculation exceedingly intricate; and while they always set the skill of the players equal, they keep this theory of games within exceedingly narrow limits.

[15]Page 244 of the 2nd edition.

35 which is the value of q by 0.693 which is the first limit marked by the Author, & we will find 24.255.

I have not at all examined the formulas which you gave on the occasion of the 11th Problem, the remark that you make that one can find the general solution of this Problem, by seeking the sums of the interposed figurate numbers as one will wish is very correct; I would have resolved this Problem by the same manner, or by that which has served me to find the Problems on Pharaon or Bassette which are only a particular case of the one considered generally here. I do not believe that one is able to find the sum of the series similar to those that you have given in the Corollary on page 157.[16]

You have quite well resolved the Problems of the balls for the case where one point is lacking to one of the Players, & to the other any number of points. I myself remember that Mr. de Moivre has said to me when I was in London, that he had the general solution of this Problem. [17] Corollary 3, *Si dexteritates*, &c. is not more difficult than when on supposes the forces equals; I have found that if the number of balls of Paul is m, the number of balls of Pierre n, their forces as a to b, the probability that Paul will win in a single turn a given number of points q precisely neither more nor less, will be

$$\frac{nb}{\overline{m-q}\cdot a+nb}\times\frac{ma}{ma+nb}\times\frac{\overline{m-1}\cdot a}{\overline{m-1}\cdot a+nb}\times\frac{\overline{m-2}\cdot a}{\overline{m-2}\cdot a+nb}\times\frac{\overline{m-3}\cdot a}{\overline{m-3}\cdot a+nb}\times\text{\&c.}$$

it is necessary to take as many products as there are units in $q+1$.

Your formula

$$q.q-1.q-2.\ \text{\&c.}\times\underset{q}{\overset{p}{\square}}\times\overline{f-q}^{\,p-q},$$

or more simply $\overline{f-q}^{\,p-q}$ multiplied by as many products $p.p-1.p-2.p-3.$ &c. as there are units in q, in order to express the number of cases in order to bring forth with any number of dice, a determined number of different faces neither more nor less, is very correct; but this Problem is not the same as the 18th Problem of Mr. de Moivre; because when one proposes to bring forth, for example in eight coups, an ace & a deuce, one has also won when one brings forth many times an ace and a deuce; now these cases are excluded in your Problem.

The property that you observed in the horizontal bands, that they serve in order to express all the different coups that there are to bring forth a determined number of different faces neither more nor less is very good; one deduces it easily from proposition 32 of your Book.[18]

The method of Mr. de Moivre for the duration of the games when one plays by reducing is very natural & based on this that it is always necessary to subtract the cases for which it can happen that one of the Players wins the écus of the other; the method when the number of écus of the one & the other is equal, is not different from that which he employs when the number of écus is unequal, but in this that one makes in the first two operations immediately, because of the inequality that there is on both sides. As you greatly wish to see my method & my demonstrations for the pool,. . .

⟨Here Bernoulli demonstrates his method to solve Waldegrave's Problem.⟩

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16 Page 218 of the 2nd edition.

17 I have sent it to Mr. J. Bernoulli in a Letter of 20 September 172. See here page 248.

18 Page 35 of the 2nd edition.

[386] . . .Here is, Sir, all that which I have to communicate to you on my method for the pool, I hope that you will be content with it. I have at no point communicated this method to Mr. de Moivre, I believe that if he had seen it he would have recognized that that which he has employed in his Book for the case of three Players, is completely useless for the case of a greater number of Players, & that thus his methods do not always have the advantage to be as general as he thinks. I do not know if Mr. de Moivre has had plan in his Preface[19] to bring so much reproach on you as you believe; for me I hold the methods which you have given in your Book sufficient enough to resolve all the general Problems of Mr. Moivre, most of which differ from yours only in the generality of the algebraic expressions, & I am persuaded that Mr. Moivre himself will do the justice to admit to you that you have pushed this material much further than Mr. Huygens & Mr. Pascal have done, who have given only the first elements of the science of chance, & that after them you have been the first who has published some general methods for this calculus. A Jesuit named Caramuel, who I have cited in my Thesis, has wished to push these matters, & even critique Mr. Huygens in the Treatise which he named ΚΥΒΕΙΑ, & which he has inserted in his grand Works of Mathematics; but as all that which he gives is only a heap of paralogisms, I count it for nothing.

Extract from the letter from Mr. de M. . . to Mr. N. Bernoulli
At Paris this 20 August 1713
Essay d'Analyse sur Les Jeux de Hazard, 2nd ed. p. 400

I agree with the remarks that you have made in your Letter of 5 September 1712 on the subject of those that I had sent you on the Book of Mr. Moivre.

[19]From the preface to De mensura sortis: "Huygens was the first that I know who presented rules for the solution of this sort of problems, which a French author has very recently well illustrated with various examples; but these distinguished gentlemen do not seem to have employed that simplicity and generality which the nature of the matter demands: moreover, while they take up many unknown quantities, to represent the various conditions of gamesters, they make their calculation too complex; and while they suppose that the skills of gamesters is always equal, they confine this doctrine of games within limits too narrow."