THE GAME OF PHARAON

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Extract from the Preface of 1717.

It is by the Help of that Theorem so contracted, that I have been able to give a compleat Solution of the Problems of Pharaon *and* Bassette*, which was never done before me: I own that some great Mathematicians had already taken the pains of calculating the advantage of the Banker, in any circumstance either of Cards remaining in his Hands, or of any number of times that the Card of the Ponte is contained in the Stock: But still the curiosity of the Inquisitive remained unsatisfied; The Chief Question, and by much the most difficult, concerning* Pharaon *or* Bassette*, being, What it is that the Banker gets* per Cent. *of all the Money adventured at those Games? which now I can certainly answer is very near Three* per Cent. *at* Pharaon*, and three fourths* per Cent. *at* Bassette*, as may be seen in my 33rd Problem, where the precise Advantage is calculated.*

The Game of Pharaon. pp. 77–82

The Calculation for *Pharaon* is much like the preceding, the reasonings about it being the same; it will therefore be sufficient to lay down the Rules of the Play, and the Scheme of Calculation.

Rules of the Play.

First, the Banker holds a Pack of 52 Cards.

Secondly, he draws the Cards one after the other, and lays them down at his right and left-hand alternately.

Thirdly, the Ponte may at his choice set one or more Stakes upon one or more Cards, either before the Banker has begun to draw the Cards, or after he has drawn any number of couples.

Fourthly, the Banker wins the Stake of the Ponte, when the Card of the Ponte comes out in an odd place on his right-hand; but loses as much to the Ponte when it comes out in an even place on his left-hand.

Fifthly, the Banker wins half the Ponte's Stake, when it happens to be twice in one couple.

Sixthly, when the Card of the Ponte being but once in the Stock, happens to be the last, the Ponte neither wins nor loses.

Seventhly, the Card of the Ponte being but twice in the Stock, and the last couple containing his card twice, he then loses his whole Stake.

PROBLEM XIV.

To find at Pharaon *the Gain of the Banker in any circumstance of Cards remaining in the Stock, and of the number of times that the Ponte's Cards is contained in it.*

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This problem having four Cases, that is, when the Ponte's Card is *once*, *twice*, *three*, or *four* times in the Stock; we shall give the Solution of these four cases severally.

SOLUTION of the *first* Case.

The Banker has the following number of Chances for winning and losing.

1	1	Chance for winning	1
2	1	Chance for losing	1
3	1	Chance for winning	1
4	1	Chance for losing	1
5	1	Chance for winning	1
*	1	Chance for losing	0

Wherefore, the Gain of the Banker is $\frac{1}{n}$, supposing n to be the number of Cards in the Stock.

SOLUTION of the *second* Case.

The Banker has the following Chances for winning and losing.

1	$n-2$	Chances for winning	1
	1	Chance for winning	y
2	$n-2$	Chance for losing	1
3	$n-4$	Chances for winning	1
	1	Chance for winning	y
4	$n-4$	Chance for losing	1
5	$n-6$	Chances for winning	1
	1	Chance for winning	y
6	$n-6$	Chance for losing	1
7	$n-8$	Chances for winning	1
	1	Chance for winning	y
8	$n-8$	Chance for losing	1
*	1	Chance for winning	1

The Gain of the Banker is therefore $\frac{\overline{n-2}.y}{n.n-1} + \frac{2}{n.n-1}$, or $\frac{\frac{1}{2}n+1}{n.n-1}$ supposing $y = \frac{1}{2}$.

The only thing that deserves to be explained here, is this; how it comes to pass, that whereas at *Bassette*, the first number of Chances for winning was represented by $n-1$, here 'tis represented by $n-2$; to answer this, it must be remembered, that according to the Law of this Play, if the Ponte's Cards come out in an odd place, the Banker is not thereby entitled to the Ponte's whole Stake: for if it so happens that his Card comes out again immediately after, the Banker wins but half of it; therefore the number $n-1$ is divided into two parts, $n-2$ and 1, whereof the first is proportional to the Probability which the Banker has for winning the whole Stake of the Ponte, and the second is proportional to the Probability of winning the half of it.

SOLUTION of the *third* Case.

The number of Chances which the Banker has for winning and losing, are as follow:

1	$\begin{cases} n-2\times n-3 \\ 2\times n-2 \end{cases}$	Chances for winning Chances for winning	1 y
2	$n-2\times n-3$	Chances for losing	1
3	$\begin{cases} n-4\times n-5 \\ 2\times n-4 \end{cases}$	Chances for winning Chances for winning	1 y
4	$n-4\times n-5$	Chances for losing	1
5	$\begin{cases} n-6\times n-7 \\ 2\times n-6 \end{cases}$	Chances for winning Chances for winning	1 y
6	$n-6\times n-7$	Chances for losing	1
7	$\begin{cases} n-8\times n-9 \\ 2\times n-8 \end{cases}$	Chances for winning Chances for winning	1 y
$*$	2×1	Chances for losing	1

Wherefore the Gain of the Banker is $\frac{3y}{2.n-1}$, or $\frac{3}{4.n-1}$ supposing $y=\frac{1}{2}$.

The number of Chances for the Banker to win, is divided into two parts, whereof the first expresses the number of Chances he has for winning the whole Stake of the Ponte, and the second for winning the half of it.

Now for determining exactly those two parts, it is to be considered, that in the first couple of Cards that are laid down by the Banker, the number of Chances for the first Card to be the Ponte's is $n-1\times n-2$; also, that the number of Chances for the second to be the Ponte's, but not the first, is $n-2\times n-3$: wherefore the number of Chances for the first to be the Ponte's, but not the second, is likewise $n-2\times n-3$. Hence it follows, that if from the number of Chances for the first Card to be the Ponte's, viz. from $n-1\times n-2$, there be subtracted the number of Chances for the first to be the Ponte's, and not the second, viz. $n-2\times n-3$, there will remain the number of Chances for both first and second Cards to be the Ponte's, viz. $2\times n-2$, and so for the rest.

SOLUTION of the *fourth* Case.

The number of Chances which the Banker has for winning and losing, are as follow:

1	$\begin{cases} n-2\times n-3\times n-4 \\ 3\times n-2\times n-3 \end{cases}$	for winning for winning	1 y
2	$n-2\times n-3\times n-4$	for losing	1
3	$\begin{cases} n-4\times n-5\times n-6 \\ 3\times n-4\times n-5 \end{cases}$	for winning for winning	1 y
4	$n-4\times n-5\times n-6$	for losing	1
5	$\begin{cases} n-6\times n-7\times n-8 \\ 3\times n-6\times n-7 \end{cases}$	for winning for winning	1 y
6	$n-6\times n-7\times n-8$	for losing	1
7	$\begin{cases} n-8\times n-9\times n-10 \\ 3\times n-8\times n-9 \end{cases}$	for winning for winning	1 y
8	$n-8\times n-9\times n-10$	for losing	1
$*$	$\begin{cases} 2\times 1\times 0 \\ 3\times 2\times 1 \end{cases}$	for winning for winning	1 y
	$2\times 1\times 0$	for losing	1

Wherefore the Gain of the Banker, or the Loss of the Ponte, is $\frac{2n-5}{n-1.n-3}y$ or $\frac{2n-5}{2\times n-1.n-3}$, supposing y to be $=\frac{1}{2}$.

A Table for PHARAON.

N	1	2	3	4
52	∗∗∗	∗∗∗	∗∗∗	∗50
50	∗∗∗	94	65	48
48	48	90	62	46
46	46	86	60	44
44	44	82	57	42
42	42	78	54	40
40	40	74	52	38
38	38	70	49	36
36	36	66	46	34
34	34	62	44	32
32	32	58	41	30
30	30	54	38	28
28	28	52	36	26
26	26	46	33	24
24	24	42	30	22
22	22	38	28	20
20	20	34	25	18
18	18	30	22	16
16	16	26	20	14
14	14	22	17	12
12	12	18	14	10
10	10	14	12	8
8	8	11	9	6

It will be easy, from the general expressions of the Losses, to compare the disadvantage of the Ponte at *Bassette* and *Pharaon*, under the same circumstances of Cards remaining in the hands of the Banker, and of the number of times that the Ponte's Card is contained in the Stock; but to save that trouble, I have thought fit here to annex a Table of the Gain of the Banker, or Loss of the Ponte, for any particular circumstance of the Play, as it was done for *Bassette*.

The numbers of the foregoing Table, as well as those of the Table for *Bassette*, are sufficiently exact to give at first view an idea of the advantage of the Banker in all circumstances, and the Method of using it is the same as that which was given for *Bassette*. It is to be observed at this Play, that the least disadvantage of the Ponte, under the same circumstances of Cards remaining in the Stock, is when the Card of the Ponte is but twice in it, the next greater when three times, the next when once, and the greatest when four times.

PROBLEM XXXIII. (pp. 105–107)

To find at Pharaon*, how much it is that the Banker gets* per Cent. *of all the Money that is adventured.*

HYPOTHESIS.

I suppose *first*, that there is but one single Ponte; *Secondly*, that he lays his Money upon one single Card at a time; *Thirdly*, that he begins to take a Card in the beginning of the Game; *Fourthly*, that he continues to take a new Card after the laying down of every couple: *Fifthly*, that when there remain but six Cards in the Stock, he ceases to take a Card.

SOLUTION.

When at any time the Ponte lays a new Stake upon a Card taken as it happens out of his Book, let the number of Cards already laid down by the Banker be supposed equal to x, and the whole number of Cards equal to n.

Now in this circumstance the Card taken by the Ponte has past four times, or three times, or twice, or once, or not at all.

First, If it has passed four times, he can be no loser upon that account.

Secondly, If it has passed three times, then his Card is once in the Stock: now the number of Cards remaining in the Stock being $n-x$, it follows by the first case of the xiii$^{\text{th}}$[1] Problem, that the Loss of the Ponte will be $\frac{1}{n-x}$: but by the Remark belonging to the xx$^{\text{th}}$ Problem, the Probability of his Card's having passed three times precisely in x Cards is $\frac{x.x-1.x-2.n-x+4}{n.n-1.n-2.n-3}$: now supposing the Denominator equal to S, multiply the Loss he will suffer, if he has that Chance, by the Probability of having it, and the product $\frac{x.x-1.x-2.4}{S}$ will be his absolute Loss in that circumstance.

Thirdly, If it has passed twice, his absolute Loss will, by the same way of reasoning, be found to be $\frac{x.x-1.n-x+2.6}{2S}$.

Fourthly, If it has passed once, his absolute Loss will be found to be $\frac{x\times n-x.n-x-2.3}{S}$.

Fifthly, If it has not yet passed, his absolute Loss will be $\frac{n-x.n-x-2.2n-2x-5}{2S}$.

Now the Sum of all these Losses of the Ponte will be $\frac{n^3-\frac{9}{2}nn+5n-3x-\frac{3}{2}xx+3x^3}{S}$, and this is the Loss he suffers by venturing a new Stake after any number of Cards x are passed.

But the number of Couples which at any time are laid down, is always one half of the number of Cards that are passed; wherefore calling t the number of those Couples, the Loss of the Ponte may be expressed thus $\frac{n^3-\frac{9}{2}nn+5n-6t-6tt+24t^3}{S}$.

Let now p be the number of Stakes which the Ponte adventures; let also the Loss of the Ponte be divided into two parts, viz. $\frac{n^3-\frac{9}{2}nn+5n}{S}$, and $\frac{-6t-6tt+24t}{S}$.

And since he adventures a Stake p times; it follows that the first part of his loss will be $\frac{pn^3-\frac{9}{2}pnn+5pn}{S}$.

In order to find the second part, let t be interpreted successively by $0, 1, 2, 3$, &c. to the last term $p-1$; then in the room of $6t$ we shall have a Sum of Numbers in Arithmetic Progression to be multiplied by 6; in the room of $6tt$ we shall have a Sum of Squares, whose Roots are in Arithmetic Progression, to be multiplied by 6; and in the room of $24t^3$ we shall have a Sum of Cubes, whose Roots are in Arithmetic Progression, to be multiplied by 24.

These several Sums being collected according to the Rule given in the second Remark on the x$^{\text{th}}$ Problem, will be found to be $\frac{6p^4-14p^3+6pp+2p}{S}$ and therefore the whole Loss of the Ponte will be $\frac{pn^3-\frac{9}{2}pnn+5pn+6p^4-14p^3+6pp+2p}{S}$.

Now this being the Loss which the Ponte sustains by adventuring the Sum p, each Stake being supposed equal to Unity, it follows that the Loss *per Cent.* of the Ponte, is the quantity above-written multiplied by 100, and divided by p, which considering that S has been supposed equal to $n\times n-1\times n-2\times n-3$, will make it to be

$$\frac{2n-5}{2.n-1.n-3}\times 100+\frac{\overline{p-1}\times\overline{6pp-8p-2}}{n.n-1\times n-2.n-3}\times 100;$$

[1]This should refer to the xiv$^{\text{th}}$ Problem.

let now n be interpreted by 52, and p by 23; and the Loss *per Cent.* of the Ponte, or Gain *per Cent.* of the Banker, will be found to be 2.99251; that is $2^{£} - 19^{\text{sh.}} - 10^{\text{d.}}$ *per Cent.*

By the same Method of process, it will be found that the Gain *per Cent.* of the Banker at *Bassette* will be

$$\frac{3n-9}{n.n-1.n-2} \times 100 + \frac{4p.p-1.p-2}{n.n-1.n-2.n-3} \times 100.$$

Let n be interpreted by 51, and p by 23; and the foregoing expression will become 0.790582 or $15^{\text{sh.}} - 9\frac{1}{2}^{\text{d.}}$. The consideration of the first Stake which is adventured before the Pack is turned being here omitted, as being out of the general Rule; but if that case be taken in, the Gain of the Banker will be diminished, and be only 0.76245, that is $15^{\text{sh}}. - 3^{\text{d}}.$ very near; and this is to be estimated as the Gain *per Cent.* of the Banker, when he takes but half Face.

Now whether the Ponte takes one Card at a time, or several Cards, the Gain *per Cent.* of the Banker continues the same: whether the Ponte keeps constantly to the same Stake, or some time doubles or triples it, the Gain *per Cent.* is still the same: whether there be one single Ponte or several, his gain *per Cent.* is not thereby altered. Wherefore the Gain *per Cent.* of the Banker, upon all the Money that is adventured at *Pharaon*, is $2^{£} - 19^{\text{sh.}} - 10^{\text{d.}}$ and at *Bassette* $15^{\text{sh}}. - 3^{\text{d}}.$