AEQUATIONUM QUARUNDAM POTESTATIS TERTIAE, QUINTAE, SEPTIMAE, NONAE, & SUPERIORUM, AD INFINITUM USQUE PERGENDO, IN TERMINIS FINITIS, AD INSTAR REGULARUM PRO CUBICIS QUAE VOCANTUR *CARDANI*, RESOLUTIO ANALYTICA.

AB. DE MOIVRE, R.S.S.
PHILOSOPHICAL TRANSACTIONS, VOL. 25, NO. 309, (1706/7) PP. 2368–2371.

Note: Of certain equations of the third, fifth, seventh, ninth, & higher Power, all the way to infinity, by proceeding, in finite terms, in the form of Rules for Cubics which are called by Cardano, Resolution by Analysis.

Let n be any Number, y an unknown quantity, or the Root sought of the equation, and let a be any known quantity at all, or as they call Homogeneous of Comparison: And also with these the relation among themselves may be expressed by the Equation

$$ny + \frac{nn-1}{2\times 3}ny^3 + \frac{nn-1}{2\times 3}\frac{nn-9}{4\times 5}ny^5 + \frac{nn-1}{2\times 3}\frac{nn-9}{4\times 5}\frac{nn-25}{6\times 7}ny^7, \text{ \&c.} = a$$

Out of this series the nature is evident, because if n is assumed any odd number (namely an integer, and it matters not either it be positive or negative) then the series of its own accord will be terminated, & the Equation becomes one out of the presented above, of which the Root is

(1) $$y = \frac{1}{2}\sqrt[n]{\sqrt{1+aa}+a} - \frac{\frac{1}{2}}{\sqrt[n]{\sqrt{1+aa}+a}}$$

or

(2) $$y = \frac{1}{2}\sqrt[n]{\sqrt{1+aa}+a} - \frac{\frac{1}{2}}{\sqrt[n]{\sqrt{1+aa}-a}}$$

or

(3) $$y = \frac{\frac{1}{2}}{\sqrt[n]{\sqrt{1+aa}-a}} - \frac{1}{2}\sqrt[n]{\sqrt{1+aa}-a}$$

or

(4) $$y = \frac{\frac{1}{2}}{\sqrt[n]{\sqrt{1+aa}-a}} - \frac{\frac{1}{2}}{\sqrt[n]{\sqrt{1+aa}-a}}$$

For the sake of an example, let there be of this Equation of the fifth power

$$5y + 20y^3 + 16y^5 = 4$$

Date: August 30, 2009.

Translated by Richard J. Pulskamp, Department of Mathematics and Computer Science, Xavier University, Cincinnati, OH. .

the Root to be discovered be, where in this case there will be $n = 5$ & $a = 4$. The Root will be in like manner

$$y = \frac{1}{2}\sqrt[5]{\sqrt{17}+4} - \frac{\frac{1}{2}}{\sqrt[5]{\sqrt{17}+4}},$$

which in common numbers is able to be explicated most expeditiously in this manner. There is $\sqrt{17}+4 = 8.1231$, of which the Logarithm 0.9097164, & of this the fifth part 0.1819433, to this the corresponding number is

$$1.5203 = \sqrt[5]{\sqrt{17}+4}.$$

Indeed of the one 0.1819433 the Arithmetic Complement is 9.8180567 to which corresponds the number

$$0.6577 = \frac{1}{\sqrt[5]{\sqrt{17}+4}}.$$

Therefore of these numbers the half difference $0.4313 = y$.

Here comes an Observation which in the place of the general Root, should not be assumed disadvantageous

$$y = \frac{1}{2}\sqrt[n]{2a} - \frac{\frac{1}{2}}{\sqrt[n]{2a}},$$

if when the number a with respect to unity, if large enough, as if the Equation may have been

$$5y + 20y^3 + 16y^5 = 682,$$

there will be $\log 2a = 3.1348143$, of which the fifth part 0.6269628, & corresponding to this the number 4.236. Moreover of the Arithmetic Complement 9.3730372 the number is 0.236 & of these numbers the half-difference $2 = y$.

However in addition, if in the preceding Equation the signs are alternately positive & negative, or what reverts to the same, if the series will have come of this way

$$ny + \frac{1-nn}{2\times 3}ny^3 + \frac{1-nn}{2\times 3}\frac{9-nn}{4\times 5}ny^5 + \frac{1-nn}{2\times 3}\frac{9-nn}{4\times 5}\frac{25-nn}{6\times 7}ny^7, \text{ \&c. } = a$$

the Root of this will be

$$y = \frac{1}{2}\sqrt[n]{a+\sqrt{aa-1}} + \frac{\frac{1}{2}}{\sqrt[n]{a+\sqrt{aa-1}}} \tag{1}$$

or

$$y = \frac{1}{2}\sqrt[n]{a+\sqrt{aa-1}} + \frac{1}{2}\sqrt[n]{a-\sqrt{aa-1}} \tag{2}$$

or

$$y = \frac{\frac{1}{2}}{\sqrt[n]{a+\sqrt{aa-1}}} + \frac{1}{2}\sqrt[n]{a-\sqrt{aa-1}} \tag{3}$$

or

$$y = \frac{\frac{1}{2}}{\sqrt[n]{a-\sqrt{aa-1}}} + \frac{\frac{1}{2}}{\sqrt[n]{a+\sqrt{aa-1}}} \tag{4}$$

But this to be Noted, that if the number $\frac{n-1}{2}$ will have been odd, the sign of the discovered Roots in it should be reversed.

Let be put the Equation $5y - 20y^3 + 16y^5 = 6$, whence $n = 5$ & $a = 6$. The Root will be

$$= \frac{1}{2}\sqrt[5]{6 + \sqrt{35}} + \frac{\frac{1}{2}}{\sqrt[5]{6 + \sqrt{35}}}.$$

Or seeing that $6 + \sqrt{35} = 11.916$, the logarithm of this will be 1.0761304 & the fifth part of it 0.2152561, indeed the Arithmetic Complement 9.7847439. The Logarithms of these numbers are 1.6415 & 0.6091 respectively, of which the half-sum $1.1253 = y$.

Indeed if it will have happened that a is less than unity, then the second form of the Root, that which is more convenient with the proposed, should be selected before the remaining ones. Thus if the Equation will have been

$$5y - 20y^3 + 16y^5 = \frac{61}{64},$$

there will be

$$y = \frac{1}{2}\sqrt[5]{\frac{61}{64} + \sqrt{\frac{-375}{4096}}} + \frac{1}{2}\sqrt[5]{\frac{61}{64} - \sqrt{\frac{-375}{4096}}}$$

And certainly if the fifth Root of the binomial may be able to be extracted in any manner, the good & possible Root will appear, even if the real one should assume the impossible in the expression. Indeed the fifth Root of the Binomial $\frac{61}{64} + \sqrt{\frac{-375}{4096}}$ is $\frac{1}{4} + \frac{1}{4}\sqrt{-15}$, & the same fifth Root of the Binomial $\frac{61}{64} - \sqrt{\frac{-375}{4096}}$ is $\frac{1}{4} - \frac{1}{4}\sqrt{-15}$, of which the half-sum of the Binomials $= \frac{1}{4} = y$.

But if that extraction either was not able to be completed, or even appeared too difficult, the thing is everywhere to be accomplished by the Table of natural sines in the following way.

To the Radius 1 let $a = \frac{61}{64} = 0.95112$ be the arc sin of a certain one, which hence will be $72^\circ\ 23'$ of which the fifth part (the same because $n = 5$) is $14^\circ\ 28'$; the sine of this $0.24981 = \frac{1}{4}$ approximately. Nor to be advanced otherwise in Equations of higher degree.