CRITICAL EXAMINATION

Of a Problem of probability of Mr. DANIEL BERNOULLI*,*
and solution of another Problem analogous to Bernoulli's[*]

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1. Since the Mathematicians have known the law, that their power is to extend their calculations as far as about chance, and to foresee in some way the events, assigning to each one the corresponding degree of probability, and giving a certain and determined value to hope and to fear, that man ought to have on them, it is not lacking among them who apply with all the attention to this most useful part of man to know; and we are debtors to *Moivre*, to *Montmort*, to *Huygens*, to *Bernoulli*, and to other excellent talents of the true principles, on which it is founded, and of the rules, that constitute the fairness in those contracts, which enter the risk and the dependency from that case which is called, fortuitousness, accident, fortune. Of this specie are games, of hazard especially, assurances, annuities, tontines, and other speculations of similar temper, that men have been drawn from the beginning on account of not leaving untried to improve their circumstances, and to render their state more comfortable.

2. But because also there have been many such problems made that the Authors have solved, and there are already several rules that we have, equipped with some generality, under which are reduced many others that have affinity with those, of which the solution has been given; so it is remote that nowadays the matter can be said exhausted, that one has not ceased to consider the events of chance under new aspects, and imagine new contracts, and new games are invented, often times by appeal alone to investigate the rules, with which they would have to be regulated in such and such case the deposits of the players and the agreements of the contracting parties, so that you save justice entirely and some damage not take part. And indeed the correspondence proportionate to the expectations of the useful events was done, equalizing the lots on both sides, and fortune is forced, so to speak, to be right in the distribution of its favors.

3. Among these inventors of new problems of probability the great Geometrician and Doctor Mr. *Daniel Bernoulli* has desired yet to be numbered, last of the three illustrious Siblings, who have so much enriched Mathematics with their sublime discoveries, of whom, three months ago, literary Europe has wept the death that followed after a long career of merits, and after that he had already so abundantly supplied with

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his works to the immortality of his name. The Problem, that is proposed in Volume XIV of the Commentary of the New Academy of *Petersburg*, is one of the more compound, if its extension is considered in all, but it becomes simple, if we limit ourselves to the first hypothesis. I feel the same as he, which speaks to us thus.

4. *Let there be two, three, or many urns, in which one at a time tickets in fixed and in equal number are supposed restored, but the tickets of one and any urn were distinguished from tickets of the remaining urns by its peculiar color at the beginning; next hereafter the tickets successively, nevertheless by lot, are permutationd by this rule that with any trial from the urns individually one ticket is extracted, and then is transferred into the urn next in order, moreover those which were extracted from the urn positioned in the last place, are restored into the first; thus with these having been assumed and with the number of permutations, given by the aforementioned method of causes, the number of tickets of any color whatsoever is sought which probably will be contained in any urn whatever, but as often as an extraction had been made from the urns individually simultaneously, and simultaneously with it, by what I said, any transposed ticket whatsoever into the next urn only, I indicate that entire operation with the name of one permutation. . . .*

5. *However much easy the calculation may be for two urns I will apply it yet on account of the connection, which will be considered with the following: therefore let there be in the first urn n white tickets and just as many black in the other urn, there will be according to the well-known rules of combinations and probabilities, after the first permutation, the number of white tickets remaining in the first urn will be held* $= n-1$*, after the second permutation*

$$= \frac{(n-1)(n-2)}{n} + 1,$$

after the third permutation

$$\frac{(n-1)(n-2)^2}{nn} + \frac{n-2}{n} + 1;$$

after the fourth

$$\frac{(n-1)(n-2)^3}{n^3} + \frac{(n-2)^2}{nn} + \frac{n-2}{n} + 1;$$

after the fifth

$$\frac{(n-1)(n-2)^4}{n^4} + \frac{(n-2)^3}{n^3} + \frac{(n-2)^2}{nn} + \frac{n-2}{n} + 1;$$

and thus hereafter; therefore there will be deduced, if generally the number of permutations made will have been r and there will be put for the sake of brevity $\frac{n-2}{n} = m$ *the number of white tickets remaining in the first urn will be*

$$= \frac{1-m^{r-1}}{1-m} + (n-1)m^{r-1} = \frac{1}{2}n(1+m^r).$$

Hence the distribution of the remaining tickets is understood through one another. From this first hypothesis he then passes to the other more composed ones, and finds for these also the corresponding formulas, in which he at last discovers a law of progression that makes to establish the general law for determining the number of the colored tickets that are left in the urns after any number of permutations.

6. Not departing from the first supposition of the two urns, since the Author does not mention it, by which road he has come to find again his formulas, assembly stopped me for some time, without being able to guess from which reason and by which calculation he came to give it. Finally it came into my head, that there could be some analogy with his problem of the tickets and the two urns to another problem of two barrels A, B, of equal capacity, the first which is full of wine and the other of water. Raising from the first, and second equal measures, then the permutation made, it is clear that, where the measure is called 1, and n the amount of the fluid in each barrel, there remains in barrel A, $n-1$ of wine and 1 of water, the contrary happening precisely in barrel B. For second permutation then reflect, that in 1 measure of mixture, that comes from vase A, must be mixed with all the wine in that same proportion, that the the entire mixture observes with the wine in the barrel, I find, that $\frac{n-1}{n}$ expresses the quantity of wine drawn from A the second time, so that the residual wine in A at the midpoint of the operation becomes $n-1-\frac{(n-1)}{n}=\frac{(n-1)^2}{n}$. Passing then to the vase B, as soon as the wine in it is 1, the analogy $n:1::1:\frac{1}{n}$ lets us know, that $\frac{1}{n}$ is quantity of wine drawn from B, which in order to finish the entire operation is poured into A. Therefore, the second permutation executed, I will have in A the quantity of wine $\frac{(n-1)^2}{n}+\frac{1}{n}$, that is, $\frac{n^2+(n-2)^2}{2n}$, a formula equivalent to Bernoulli's $\frac{(n-1)(n-2)}{n}+1$.

7. Replicating a similar reason for the third, fourth etc. permutation, it will be found for the third wine in $A=n^3+\frac{(n-2)^3}{2nn}$; for the $4^{\text{th}}=\frac{n^4+(-2)^4}{2n^3}$ etc., so that for the indefinite num. r of permutations, it will make the wine residual in $A=\frac{n^r+(n-2)^r}{n2^{r-1}}$, that is (substituting m in place of $\frac{n-2}{n}$)$=\frac{n}{2}(1+m^r)$, which is exactly the law of our Celebrated Author.

8. Assuring myself in this way of the identity of my formulas with those of the num. 5, I have suspected, that Bernoulli first solving the problem of the barrels, and afterwards turning to the other of the tickets, he has argued thus. Distribute the black and white tickets, that are in the urns A, B after the first permutation, in such way.

$$\begin{array}{lcl} & A & \\ \text{white} & n-1 & :\text{black } 1 \\ & B & \\ \text{black} & n-1 & :\text{white } 1 \end{array}$$

And the second one is undertaken. Since in A there are $n-1$ white ones, and 1 black, the probable part of the white ones which I take is $\frac{n-1}{n}$, and the probable part of the black ones is $\frac{1}{n}$, because these two united parts know the only ticket that I extract; and how much probable one of the white ones must be to how much probable one of the black ones, as the number of the white ones to the number of the black ones which are in the urn, in which ratio stand exactly the parts $\frac{n-1}{n}$, $\frac{1}{n}$. Because, $\frac{n-1}{n}$ subtracted from the number $n-1$ of white ones which were in the urn after the half of the 2^{nd} operation, it will make probable that there remain in A $n-1-\left(\frac{n-1}{n}\right)$ white tickets. With similar reason, for the other half of the operation, it is found, that probably the $\frac{1}{n}$ part of white ones from the urn B is mined, which put in A constitutes the probable number of white tickets in A, executed that is the 2nd permutation, $= n-1-\left(\frac{n-1}{n}\right)+\frac{1}{n} = \left(\frac{(n-1)(n-2)}{n}\right)+1$.

9. Continuing then to the consecutive permutations, they turn out the other formula and the rule of Bernoulli, so that there will no other difference between the problem of the two fluids and that of the tickets, that the formulas in the first one represent the sure and necessary amounts of wine that remain in the barrel, whereas in the second with the same formulas it expresses alone how much probable the one of the white tickets which remain in the urn.

10. Of the justness of this or similar reasoning, that Bernoulli himself can have made, makes me doubt a Corollary under it, and that he himself notes in the course of his dissertation with these words. *Because m is always smaller than unity the term m^r vanishes, if r be an exceedingly great number and then the number of white tickets remaining in the first urn is simply $= \frac{1}{2}n$; the situation is asymptotic, toward which, while the permutations happen, it is approaching more and more, ...* Therefore, said I, if the white tickets in the first urn will be three, and one black; and in second you have three black and one white, you will want infinite permutations, so that I can hope to have two white ones in the first urn, that are precisely the half of all the white that we have. This consequence appears to me hard to have to admit that the waiting time for the facility of drawing one of the three white ones from the first urn, and simultaneously one of the three black ones from the second one, I would not have doubted to bet, that I would have reduced it to half after a single permutation.

11. It increased then my doubt, indeed I finished by convincing myself of the different nature of the two problems, and of the error that Bernoulli had fallen with this simple reflection. The problem of the two fluid demands, that continuing to permutation, the wine of the first barrel should necessarily diminish; and the first permutation executed, the cases are never possible, that with the subsequent returns to having the quantity of wine $n-1$ in it, much less all the wine: on the contrary in the problem of the tickets it is possible that in the various permutations now the number of white tickets grows, now decreases, and after some operations it could also probably succeed, that there remained in the first urn $n-1$ white tickets, or all the white ones returned to it. Now all these cases not being able to be embraced, as is obvious, from the formula of the fluids, because the problem excludes them, it can never be, that one same formula governs both questions, and while it with justice expresses the residual ones of wine in the first barrel, can it apply also to express the probable remainders of the white tickets in the first urn?

12. The use of the general and incontrovertible principle, that governs such probability problems also demonstrates more clearly the inapplicability of the aforesaid formula. It is certain, by consent of all the Geometers, that to establish the degree of probability applicable to one such and determined event, wants you first of all to have under the eye all the possible events that belong to the problem; and to then assign to each event the number of the combinations, which lead to it. The proportion, that will pass between the combinations of the given event, and the sum of those which belong to the others, will serve to determine degree of probability, so that if these numbers of

combinations prove equal, that event will be able to be bet evenly and to say that he has called completely probable.

13. That put, substituting balls for tickets in the problem of Bernoulli, which is the same one, we treat the inverse problem, and we seek how many favorable combinations, and how many contrary they have had in order to leave in urn A a given number of white balls after having completed whatsoever given number of permutations; of which for me will always be the first that is made, when we have already in urn A $n-1$ white with one black, and $n-1$ black balls with one white in urn B.

14. First of all however I premise the following most general

LEMMA

In A let there be $n-p$ white balls p black; and in B $n-p$ black balls, p white balls. We extract a ball from A, another from B and permutation once in the Bernoulli way, it can happen, that there are found in A $n-p-1$ white balls, or $n-p$, or finally $n-p+1$. Outside of these three cases, no other is possible, as is clear. Now I say, that in order to obtain in A $n-p-1$ white balls, I will have $(n-p)^2$ favorable combinations; for $n-p$ white balls, $2p(n-p)$ favorable combinations; finally p^2 favorable combinations for $n-p+1$ white balls. You thus arrange the balls of the urns.

		A	
white balls	$n-p$		black p
		B	
black	$n-p$		white p

If from A I extract a white one, and from B a black one, at the end of the permutation I have in A $n-p-1$ white balls. But this can be obtained, where any of the $n-p$ white ones of A is matched with any of the $n-p$ black ones of B; and the number of these favorable combinations is $(n-p)^2$. Therefore etc. Similarly because in A $n-p$ white balls remain, also the permutation made, it is necessary either that I raise one black from A, and another black from B, or a white one from A and another white one from B. But for this we have many ways, how many are born from matching the number of $n-p$ white which are in A with the number of p white which are in B, and by matching the p black ones of A with the $n-p$ black of B. Therefore $p(n-p)+p(n-p)$ that is $2p(n-p)$ will make the useful number of combinations for the event of $n-p$ white ones in urn A. Finally to have $n-p+1$ white balls, make one of the jobs that one of the p black ones of A is associated with one of the p white ones of B; that it can be made in number p^2 of ways. Therefore p^2 will make the favorable cases of this third event; and the demonstration remains completed.

15. We will gather from the present Lemma, that is called from the three cases of white balls, which can take place in the permutation, they will be to me the contrary combinations, that favor the other two events. Next from the permutation I call, for example, $n-p$ white balls in A, a case that has $2p(n-p)$ useful combinations. Therefore the combinations, that are contrary to me, they are $(n-p)^2$, and p^2, which take the other two cases $n-p-1$, $n-p+1$ and the probability, that I have predicted the name event, to the opposite one stands as $2p(n-p):(n-p)^2+p^2$; which proportion we find also with this other very simple reason. Since n is the entire number of the white balls,

and equally n the entire number of black ones in the two urns, n^2 will make the number of all the possible combinations. Subtracted therefore from the n^2 combinations favorable to a given case of white ones, you will have the contrary. So that the contrary to the case of $n-p$ white ones, that has $2p(n-p)$ of gains, will make $n^2-2p(n-p)$, a formula identical with the aforementioned $(n-p)^2+p^2$. So that given the number of favorable combinations, the number of the adverse ones is had immediately; and thus to the contrary.

16. This last way to find the number of the contrary combinations, given that of the favorable ones to a certain event, for a single permutation, can be extended still to the case of any number of permutations, reasoning thus. In all there are n^2 combinations of balls that belong to each permutation in particular; and each of the combinations, that admits the first permutation, can combine with each combination received from the second permutation. Therefore the total number of the combinations, that concern two permutations, will make $n^2 \times n^2 = n^4$. Going forward with the discourse, for three permutations, we will find the number of the combinations to be $n^2 \times n^2 \times n^2 = n^6$; for four n^8, and generally n^{2m} for number m of permutations. Therefore put that N is the number, *v.g.* of the combinations contrary to the event ϕ in m permutations, it will make $n^{2m}-N$ the number of the favorable ones, and if N will make the number of the favorable, $n^{2m}-N$ will make that of the contrary.

17. An other advantage we draw from the Lemma, and from the following §§, that it consists in the way to trace the numbers of combinations either favorable or contrary to an event for a given number of permutations $= m$. Let α, β, γ, $\ldots\, \phi$ be the possible events. We consider the Problem of the two urns in two different ways. Either they can gamble, that at least in one of the m permutations the event ϕ will be had in A, or the same event is called after having executed all the permutations. In the first case, the combinations of every event α, β, γ, $\ldots$ for all the possible successions from one to the other we must combine with those of the event ϕ, put in all those places, from second, third etc. permutation, in which it can enter: and the aggregate of the produced of these same combinations will give the number of favorables to ϕ. Here is an example. Supposed $n-1$ white balls in A, 1 black, and the contrary in B, I ask for two permutations in order to have in A either in the first or in the second $n-2$ white balls, and I seek, how much probable is this event.

18. Make in the Lemma $p-1$, we will see at once, that the events of the first permutation are not able to be that three white balls in A $\begin{array}{ll} 1.^\circ & n-2 \\ 2.^\circ & n-1 \\ 3.^\circ & n \end{array}$. If under the hypothesis of the result $n-2$, in the second permutation a white one is extracted, and a white one is put, also after the second permutation it has $n-2$ white ones. Therefore these two events must be coupled and write $\begin{array}{c} n-2 \\ n-2 \end{array}$. The first one belongs to the first permutation and the other second one. The hypothesis of the first event $n-2$ maintained, it can happen, that in the second permutation I raise black, and put a white one. In such case the white ones pass from the state $n-2$ to the state $n-1$, and must thus be written $\begin{array}{c} n-2 \\ n-1 \end{array}$; and with this the hypothesis of the first event $n-2$ remains exhausted, because neither n nor any other number exceeding or lacking by two units

can succeed to $n-2$, respecting the same number $n-2$.

19. We accept now the other hypothesis of the first event $n-1$. For this it is clear, that it can be followed by the event $n-2$, and by the event n, and by the new event $n-1$. These two last cases are not for us, because, not finding either in the first, or in the second permutation, the number $n-2$, they are contrary. It will have therefore to take account of the single union of the events $n-1$, $n-2$, and be written $\begin{array}{c} n-1 \\ n-2 \end{array}$.

20. The third hypothesis supposes the first event n, which it can never hold since $n-2$ is necessarily the event following $n-1$. Therefore only three alone are the useful matings that can result from the two permutations $\begin{array}{ccc} n-2; & n-2; & n-1, \\ n-2; & n-1; & n-2 \end{array}$; there remains now that we find the numbers of the combinations. In order to do this, we must remember, that the original state of the white balls in A was $n-1$. Therefore desiring, that after the first permutation the white become $n-2$, by force of the Lemma we have $(n-1)^2$ favorable combinations. From the state $n-2$ the white pass in the second permutation to the new state $n-2$. Since the Lemma instructs us that in order to have replicated $n-p$ white balls in A we have $2p(n-p)$ propitious combinations, it becomes clear, that $4(n-2)$ will express the number of the various ways, with which the state $n-2$ can return in the second permutation. For the two successive events $n-2$ we join laterally in column the numbers of the corresponding combinations $\begin{array}{cc} n-2 & (n-1)^2 \\ n-2 & 4(n-2) \end{array}$, and thus argue. In number of ways $(n-1)^2$ the transition can be made from the original state of $n-1$ white balls to the state $n-2$. But to each of the ways $(n-1)^2$ corresponds a number of ways $4(n-2)$, in order to pass from the state $n-2$ of the first permutation to the new state $n-2$ of the second one. Therefore, composing, in order to pass from the original state $n-1$ to the two successive states $n-2$, $n-2$, we have ways or useful combinations, that will be expressed with the product of the respective combinations, that is with $4(n-1)^2(n-2)$. A similar reasoning will make us know, that to the second matching $\begin{array}{c} n-2 \\ n-1 \end{array}$ there correspond $4(n-1)^2$ useful combinations; and to the third $\begin{array}{c} n-1 \\ n-2 \end{array}$ useful combinations $2(n-1)^3$. Therefore the total number of combinations that take the event $n-2$ either in the first or the second permutation will be $= 4(n-1)^2(n-2) + 4(n-1)^2 + 2(n-1)^3$; and the probability of the aforesaid event to the contrary probability will be as $4(n-1)^2(n-2)+4(n-1)^2+2(n-1)^3 : n^4 - 4(n-1)^2(n-2) - 4(n-1)^2 - 2(n-1)^3$.

21. That which has been said for two permutations can be extended to three, four etc. until the indefinite number m. There will be written therefore in column as many successive events as the number m brought, and in another lateral column will be put in order the respective numbers of the combinations, that conduct each of aforesaid events: therefore the product of these numbers will be made, and what results, will express the favorable combinations the given events to have according to the order with which are placed for the successive permutations. To the order of the events α, β, $\gamma \ldots \phi$ correspond the combinations a, b, c, $\ldots p$; we will form the columns

of these numbers thus $\begin{matrix} \alpha & a \\ \beta & b \\ \gamma & c \\ \vdots & \vdots \\ \phi & p \end{matrix}$, and the product $abc\ldots p$ will give the number of the combinations, that lead the succession of the events α, β, γ, $\ldots\phi$.

22. Where it is then demanded to have the event ϕ at the end of the entire number m of the required permutations; that is the second way to consider the problem of the two urns that we mentioned in §17. and it is precisely that of Bernoulli, it will be worthwhile to notice, that to fulfill this narrower condition, in the formation of the columns so they must associate ϕ either with themselves, or with the other events α, β, γ, $\ldots$, that ϕ always remains in the last position that corresponds to the last permutation. The events arranged thus, the lateral columns will give us the numbers of the favorable combinations to the event ϕ in the urn A after all the permutations are terminated. And if you wish to have the contrary combinations, before having found the favorable ones, that is sometimes more convenient, so you will form the columns of the events, that the event ϕ is never found in the last position. All this is very clear, nor is there need that we detain you here longer.

23. Demand now the number of the favorable combinations to have the event $n-2$ of white ones in the urn A after two permutations have been executed. The columns of the successive events written with the warning of the preceding §, and the lateral ones of the combinations attached, we have $\begin{array}{ll|ll} n-2 & (n-1)^2 & n-1 & 2.(n-1) \\ n-2 & 4(n-2) & n-2 & (n-1)^2 \end{array}$ The third column, $n-2 \quad (n-1)^2$, that was gain here when the problem $n-1 \quad 4$ was exposed in the first way, is contrary to us here, because it is not useful to us that $n-2$ succeeded in the first permutation, and the event makes harm to find itself at the end of the second one. Therefore the combinations favorable to the event $n-2$ after two permutations will be $= 4(n-1)^2(n-2) + 2(n-1)^3$, and consequently the contrary $= n^4 - 4(n-1)^2(n-2) - 2(n-1)^3 = n^4 - 6n^3 + 22n^2 - 26n + 10$.

24. You want the same event $n-2$ after three permutations. In order to know the favorable combinations that this event takes, we will write the columns of the events three by three with $n-2$ in the last position, and the corresponding lateral columns. These are

$$\begin{array}{ll|ll|ll|ll|} n & 1 & n-1 & 2(n-1) & n-1 & 2(n-1) & n-2 & (n-1)^2 \\ n-1 & n^2 & n-1 & 2(n-1) & n-2 & (n-1)^2 & n-1 & 4 \\ n-2 & (n-1)^2 & n-2 & (n-1)^2 & n-2 & 4(n-2) & n-2 & (n-1)^2 \\ \hline n-2 & (n-1)^2 & n-2 & (n-1)^2 \\ n-2 & 4(n-2) & n-3 & (n-2)^2 \\ n-2 & 4(n-2) & n-2 & 9 \end{array}$$

. That is the favorable combinations are in all $n^2(n-1)^2 + 4(n-1)^4 + 8(n-1)^3(n-2) + 16(n-1)^2(n-2)^2 = 42n^4 + 9 - 224n^3 + 446n^2 - 388n + 124$; and the contrary $= n^6 - 42n^4 + 224n^3 - 446n^2 + 388n - 124$.

25. Who will have the patience to trace the numbers of the contrary combinations to the event $n-2$ after 4, 5 etc. permutations, beginning from $2n-1$, that denotes the contrary combinations for one permutation alone, will find the following series;

perm. 1;	perm. 2	perm. 3.
$2n-1$;	$n^4-6n^3+22n^2-26n+10$;	$n^6-42n^4+224n^3-446n^2+388n-124$;

perm. 4	perm. 5
$n^8-320n^5+2360n^4-6968n^3-10192n^2$ $-7320n+2056$;	$n^{10}-2715n^6+26410n^5-107591n^4$ $+232780n^3-279700n^2+175664n-44848$;

Etc.

26. This series becomes a recurrent one of second degree, where is placed $n=2$; its terms are 3, 14, 52, 216, 848 etc. the multipliers, that produce it, 8, 2; and the general term $\frac{2^{m-1}}{3}(5.2^m+(-1)^m)$, m signifying the number of terms, that is of the permutations. Therefore the general number of the favorable combinations to have $n-2$ white balls in A, that is not one white, will make $2^{2m}-\frac{2^{m-1}}{3}(5.2^m+(-1)^m)$. For any number of permutations the case can never be probable, that some white ball not remain in A, because it would be necessary, that equalizing the general numbers of the favorable and adverse combinations, it could turn out real and positive. With such adaptation we are guided to the equation $2^{m+1}+(-1)^m=0$, which is impossible and absurd, under the supposition that m is a positive or whole or broken number, or finite or infinite. Therefore etc.

27. Generally however our series is recurrent of a degree certainly superior to the fourth, eagerly awaited the experiment that I have made, and consequently difficult to manage. Nevertheless it, indeed its first term alone, can be enough to introduce the error of Bernoulli. Let the hypothesis of $n=4$ be made, that gives $n-2=2$. Since 2 is the half of the white balls, that we have, for the Bernoullian Theory, departing from the original state of the 3 white ones and 1 black in the urn A, will make it necessary to permutation countless times, so that it becomes probable, that two white remain in A; and every finite number of permutations will render this event improbable. But the probability of an event leads to the equalization between the number of combinations, that that event takes, and the number of the contrary; and the improbability of the same event causes that the number of its favorable combinations is always smaller than the number of the adverse ones. Therefore for any finite number of permutations, the favorable combinations to have 2 white ones will be less than the contrary. Let us see now that it results from the above-mentioned series. Made, as it has been said, $n=4$ we will have for the first permutation 7 contrary combinations; for 2 permutations, 130; for 3 permutations, 1972 etc. Now the entire number of combinations of all the balls being, for one permutation $=16$, for 2 permutations $=256$, for 3 permutations $=4096$ etc, the favorable ones for one permutation will be $=16-7=9$; for 2 permutations $=256-130=126$; for 3 permutations $=4096-1972=2124$ etc. Therefore so many and so far, that they may demand infinite permutations for the probability of the two white, that on the contrary, executed only the first, is more than probable, that to me these two white ones remain in A, having for me 9 propitious combinations against 7 contrary. This advantage of greater probability, that is lacking to me in two permutations, because I have 126 combinations in favor, and 130 against, return to me in the 3 permutations, and followed me for all the series. Therefore to induce the probability for the event of two white ones, that is to give birth to the equalization between the prosperous and adverse combinations, I will want a number of permutations to you that is average either between 1 and 2, or between 2 and 3, but not already an infinite number, as

Bernoulli claims.

28. From all this it seems to me to be able to conclude legitimately, that the principle, of which *Bernoulli* serves himself, is not what ought to preside over the solution to his problem; that his other formulas also, which depend on the same principle, and correspond to the following hypotheses of more than two urns, must be considered as illegitimate, and inducing to error; in a word, that the problem of the tickets is of a very different nature than that of the fluids, and should be treated in a way much different from the Bernoullian.

29. Refused as insufficient the resolution of the Geometer of Basel, could appear to some, that I could not be dispensed from substituting the true one; and I also see, that this would be very convenient; but neither my occupations, nor the time prescribed by the illustrious Binder of the present Memoir in this edition of this first volume has permitted me to apply with the necessary comfort to an investigation, that must be of an extreme difficulty. But it would be able to estimate as a work in some satisfactory way of the obligation that the criticism to our celebrated Author requires me make, it presents to me in this Book with the solution of the problem mentioned in §17, which with little difference from Bernoulli has been imagined by me also before reading volume XIV. of the *Comm. of Petersburg*; forgotten then in my notebooks it was waiting for my reminiscence and the occasion to see the public light. This is that which I am going to do now, hoping that the method, by which I serve myself, may be useful to those who took in hand either the problem of *Bernoulli*, or others that may be similar. If this method is not as simple, as I would have wished, it will be worth to excuse the difficulty of a question, that is of the number of these problems of probability, that are called of dependent events, ordinarily much more rough and hindered than the problems of events independent, which often furnish formulas most elegant and not hoped for in those of the other class.

30. Let there be therefore two urns A, B, each of which it has n balls, the first white, second the black. The reduced state of the balls with the first operation to be that which we will name original; white ones in $A = n-1$; black $= 1$, and on the contrary in B, the favorable and contrary combinations are sought to have in the urn A a given number of white balls within a given number of permutations. In order to proceed with order, I begin from

PROBLEM I.

31. *The number of the contrary combinations is sought to have in A with a single permutation* $n-2$ *white balls.*

Since there are only three cases of white ones in A, that can take place; first that $n-1$ returns; second that one recovers n; third that $n-2$ remains; the contrary cases will be the first two; $n-1$, n. But for the Lemma, make $p = 1$ in it, to the event $n-1$ corresponds $2(n-1)$ combinations, and to the event n 1 combination. Therefore the number of combinations contrary to the event $n-2$ in a single permutation will make $2(n-1)+1$.

PROBLEM II.

32. *The number of the contrary combinations is sought to have in A $n-2$ white balls either in the one or in the other with two consecutive permutations.*

We notice the columns of the contrary events and of the respective combinations, as it has been perceived §23.

$$\begin{array}{ll|ll|ll|}
n & 1 & n-1 & 2(n-1) & n-1 & 2(n-1) \\
n-1 & n^2 & n & 1 & n-1 & 2(n-1)
\end{array}$$

Therefore the products of the numbers of the combinations made in each column, we will know quickly, to be the total number of the contrary combinations; $n^2+2(n-1)+4(n-1)^2$.

PROBLEM III.

33. *The contrary combinations are sought to have at least once in A $n-2$ white balls in the event of three consecutive permutations.*

All it reduces to noticing the double columns for the unfavorable events, without omitting any. For our problem they are the following ones;

$$\begin{array}{ll|ll|ll|ll|ll}
n & 1 & n & 1 & n-1 & 2(n-1) & n-1 & 2(n-1) & n-1 & 2(n-1) \\
n-1 & n^2 & n-1 & n^2 & n-1 & 2(n-1) & n & 1 & n-1 & 2(n-1) \\
n & 1 & n-1 & 2(n-1) & n & 1 & n-1 & n^2 & n-1 & 2(n-1)
\end{array}$$

Therefore the number of the contrary combinations is $n^2+4n^2(n-1)+4(n-1)^2+8(n-1)^3$.

PROBLEM IV.

34. *The contrary combinations are sought to have at least once in A $n-2$ white balls in the course of four consecutive permutations.*

The columns for the contrary events made as usual, they turn out thus.

$$\begin{array}{ll|ll|ll|ll}
n & 1 & n & 1 & n-1 & 2(n-1) & n & 1 \\
n-1 & n^2 & n-1 & n^2 & n & 1 & n-1 & n^2 \\
n-1 & 2(n-1) & n & 1 & n-1 & n^2 & n-1 & 2(n-1) \\
n & 1 & n-1 & n^2 & n & 1 & n-1 & 2(n-1) \\
\hline
n-1 & 2(n-1) & n-1 & 2(n-1) & n-1 & 2(n-1) & n-1 & 2(n-1) \\
n & 1 & n-1 & 2(n-1) & n-1 & 2(n-1) & n-1 & 2(n-1) \\
n-1 & n^2 & n & 1 & n-1 & 2(n-1) & n-1 & 2(n-1) \\
n-1 & 2(n-1) & n-1 & n^2 & n & 1 & n-1 & 2(n-1)
\end{array}$$

from which is found, to be the number of the contrary combinations; $n^4+4n^2(n-1)+12n^2(n-1)^2+8(n-1)^3+16(n-1)^4$.

PROBLEM V.

35. *The contrary combinations are sought to have at least once in A $n-2$ white balls in the course of five consecutive permutations.*

Here are the 13 columns together with the laterals, that present to us the solution of this Problem.

n	1	$n-1$	$2(n-1)$	n	1	n	1	n	1
$n-1$	n^2	$n-1$	$2(n-1)$	$n-1$	n^2	$n-1$	n^2	$n-1$	n^2
n	1	n	1	$n-1$	$2(n-1)$	n	1	$n-1$	$2(n-1)$
$n-1$	n^2	$n-1$	n^2	n	1	$n-1$	n^2	$n-1$	$2(n-1)$
n	1	n	1	$n-1$	n^2	$n-1$	$2(n-1)$	n	1
$n-1$	$2(n-1)$	$n-1$	$2(n-1)$	n	1	$n-1$	$2(n-1)$	$n-1$	$2(n-1)$
n	1	n	1	$n-1$	n^2	n	1	$n-1$	$2(n-1)$
$n-1$	n^2	$n-1$	n^2	$n-1$	$2(n-1)$	$n-1$	n^2	n	1
$n-1$	$2(n-1)$	n	1	$n-1$	$2(n-1)$	$n-1$	$2(n-1)$	$n-1$	n^2
n	1	$n-1$	n^2	$n-1$	$2(n-1)$	$n-1$	$2(n-1)$	$n-1$	$2(n-1)$
$n-1$	$2(n-1)$	$n-1$	$2(n-1)$	$n-1$	$2(n-1)$				
$n-1$	$2(n-1)$	$n-1$	$2(n-1)$	$n-1$	$2(n-1)$				
$n-1$	$2(n-1)$	$n-1$	$2(n-1)$	$n-1$	$2(n-1)$				
n	1	$n-1$	$2(n-1)$	$n-1$	$2(n-1)$				
$n-1$	n^2	n	1	$n-1$	$2(n-1)$				

The products of the numbers of the lateral ones furnish the contrary combinations with this formula; $n^4+6n^4(n-1)+12n^2(n-1)^2+32n^2(n-1)^3+16(n-1)^4+32(n-1)^5$.

36. We put in order the contrary combinations from the first until the fifth permutation, and this series will be born; $1+2(n-1)$; $n^2+2(n-1)+4(n-1)^2$; $n^2+4n^2(n-1)+4(n-1)^2+8(n-1)^3$; $n^4+4n^2(n-1)+12n^2(n-1)^2+8(n-1)^3+16(n-1)^4$; $n^4+6n^4(n-1)+12n^2(n-1)^2+32n^2(n-1)^3+16(n-1)^4+32(n-1)^5$; etc., which is a recurrent one of the second degree; and its multipliers are $2(n-1)$; n^2, so that the first one $2(n-1)$ is that which multiplies the antecedent term by the searched term. Meaning that the aforesaid series is continued previously, we suppose that there are two terms u, t previous to the first one $1+2(n-1)$. Hence it will be $n^2t+2(n-1)+4(n-1)^2=n^2+2(n-1)+4(n-1)^2$, that is, eliminate the quantities that are destroyed; $n^2t=n^2$; that is $t=1$. In addition we will have $n^2u+2(n-1)=1+2(n-1)$, that is $u=\frac{1}{n^2}$. So that the two terms previous to the first one are $\frac{1}{n^2}$, 1; and these will be called hereinafter the appendix of the series.

37. The passage from the multipliers to the general term becomes easy in this series. Since it is of second degree, let m be called the number of the terms or of the permutations, its general term has this form; ap^m+bq^m. Now the theory of the recurrent ones teaches there, that these two equations are worth; $p+q=2(n-1)$; $-pq=n^2$, from which is drawn $p=n-1+\sqrt{(n-1)^2+n^2}$; $q=n-1-\sqrt{(n-1)^2+n^2}$. Substituting therefore these values in the general term, it makes $a(n-1+\sqrt{(n-1)^2+n^2})^m+b(n-1-\sqrt{(n-1)^2+n^2})^m$. For the determination then of the species a, b, make successively the two hypotheses of $m=1$, and of $m=2$. With the first will have to be

$a(n-1+\sqrt{(n-1)^2+n^2})+b(n-1-\sqrt{(n-1)^2+n^2})=1+2(n-1)$; and second; $a(n-1+\sqrt{(n-1)^2+n^2})^2+b(n-1-\sqrt{(n-1)^2+n^2})^2=n^2+2(n-1)+4(n-1)^2$. These two equations enable us to know the values of the two species a, b, and so find $a=\frac{\sqrt{(n-1)^2+n^2)}+n}{2\sqrt{(n-1)^2+n^2}}$; $b=\frac{\sqrt{(n-1)^2+n^2)}-n}{2\sqrt{(n-1)^2+n^2}}$, thus that the indefinite number of the contrary combinations to have in A at least $n-2$ white balls in one of the m permutations will become

$$
\begin{aligned}
&=\frac{(\sqrt{(n-1)^2+n^2}+n)(n-1+\sqrt{(n-1)^2+n^2})^m}{2\sqrt{(n-1)^2+n^2}}\\
&+\frac{(\sqrt{(n-1)^2+n^2}-n)(n-1-\sqrt{(n-1)^2+n^2})^m}{2\sqrt{(n-1)^2+n^2}}.
\end{aligned}
$$

If this for convenience is called P, the indefinite number of the favorable combinations will be had at once, that it will make $n^{2m}-\Upsilon$, so that the equation $n^{2m}-\Upsilon=\Upsilon$ established, that is $n^{2m}=2\Upsilon$, the exact or proximate value of m will express the number of the permutations that must be demanded, so that you can wager exactly or approximately even, that at least in one the event $n-2$ will be taking place.

38. But this can still be obtained under the different numerical hypotheses of the values of n by writing one below the other the series of the favorable and contrary combinations, without resorting to the general terms. Let $n=2$; under such hypothesis the first four terms of the contrary combinations are; 3, 10, 32, 104 etc. and corresponding ones of the favorable 1, 6, 32, 152 etc. In these two series the third terms are equal. Therefore there are exactly three permutations, that take probably once the event of $n-2$ white, that is of zero white in the urn A. Let $n=3$. The two series for this hypothesis are

5,	29,	161	etc.
4,	52,	568	etc.

For a single permutation there are 5 contrary and 4 favorable combinations; for two I have 52 favorable and 29 contrary; consequently playing even, I have disadvantage if I demand a single permutation, and advantage I demand two; that is to say, that the number of permutations able to render probable event $n-2$ that is 1 of white is between one and the two.

PROBLEM VI.

39. *The contrary combinations are sought to have in A* $n-3$ *white balls in a single permutation.*

Since the original state of the urn A is to enclose $n-1$ white, one sees quickly, that in a single permutation it is not possible to pass to the state of $n-3$ white; and therefore all the combinations of the balls, that are n^2, become contrary to us, and the number of the favorable ones will be $=0$.

PROBLEM VII.

40. *The contrary combinations are sought to have at least once in A* $n-3$ *white balls either in one or in the other of two permutations.*

All the connections of the contrary events to the event of $n-2$ white in two permutations are also contrary to the event $n-3$. In addition all the connections favorable to obtain $n-2$ white in two permutations, those deducted, to those which enter the event $n-3$, are also contrary to this last event. But in one way alone $n-2$ may be associated with $n-3$, that is this: $\begin{matrix} n-2 \\ n-3 \end{matrix}$, which corresponds to the combinations $\begin{matrix} (n-1)^2 \\ (n-2)^2 \end{matrix}$.

Therefore the sum of the contrary and favorable combinations for the case of $n-2$ white in two permutations, less the product $(n-1)^2(n-2)^2$ will give the sum of the contraries to the event $n-3$. But the sum of the favorables and contraries to the event $n-2$ white in two permutations is n^4. Therefore the contraries for the case of $n-3$ are $n^4-(n-1)^2(n-2)^2$, that is $6n^3-13n^2+12n-4$.

PROBLEM VIII.

41. *The contrary combinations are sought to have at least once in A* $n-3$ *white in the course of three permutations.*

Reasoning, as we have done for the preceding Problem, we will conclude, that the number of the contrary combinations sought by us will be equal to n^6 less the products of the combinations, that correspond to the columns of the events, to those which enter $n-3$; which joined to laterals are the following;

$$\begin{array}{ll|ll|ll|ll|ll}
n-1 & 2(n-1) & n-2 & (n-1)^2 & n-2 & (n-1)^2 & n-2 & (n-1)^2 & n-2 & (n-1)^2 \\
n-2 & (n-1)^2 & n-2 & 4(n-2) & n-3 & (n-2)^2 & n-3 & (n-2)^2 & n-3 & (n-3)^2 \\
n-3 & (n-2)^2 & n-3 & (n-2)^2 & n-2 & 9 & n-3 & 6(n-3) & n-4 & (n-3)^2
\end{array}$$

Because the contrary combinations become in all, $n^6-2(n-1)^3(n-2)^2-4(n-1)^2(n-2)^3-9(n-1)^2(n-2)^2-6(n-1)^2(n-2)^2(n-3)-(n-1)^2(n-2)^2(n-3)^2$, that is $33n^4-126n^3+198n^2-144n+40$.

PROBLEM IX

42. *The contrary combinations are sought to have at least once in A* $n-3$ *white in the course of four permutations.*

There are the 20 lateral columns to those of the events, in which $n-3$ enters, which furnish us the products that must be subtracted from n^8 in order to determine the contrary combinations required by the present problem. Here they are in order:

n	1	$n-1$	$2(n-1)$	$n-1$	$2(n-1)$	$n-1$	$2(n-1)$	$n-2$	$(n-1)^2$
$n-1$	n^2	$n-1$	$2(n-1)$	$n-2$	$(n-1)^2$	$n-2$	$(n-1)^2$	$n-1$	4
$n-2$	$(n-1)^2$	$n-2$	$(n-1)^2$	$n-2$	$4(n-2)$	$n-3$	$(n-2)^2$	$n-2$	$(n-1)^2$
$n-3$	$(n-2)^2$	$n-3$	$(n-2)^2$	$n-3$	$(n-2)^2$	$n-2$	9	$n-3$	$(n-2)^2$
$n-2$	$(n-1)^2$	$n-2$	$(n-1)^2$	$n-2$	$(n-1)^2$	$n-2$	$(n-1)^2$	$n-1$	$2(n-1)$
$n-3$	$(n-2)^2$	$n-2$	$4(n-2)$	$n-2$	$4(n-2)$	$n-3$	$(n-2)^2$	$n-2$	$(n-1)^2$
$n-2$	9	$n-2$	$4(n-2)$	$n-3$	$(n-2)^2$	$n-2$	9	$n-3$	$(n-2)^2$
$n-1$	4	$n-3$	$(n-2)^2$	$n-2$	9	$n-2$	$4(n-2)$	$n-3$	$6(n-3)$
$n-2$	$(n-1)^2$	$n-2$	$(n-1)^2$	$n-2$	$(n-1)^2$	$n-2$	$(n-1)^2$	$n-1$	$2(n-1)$
$n-2$	$4(n-2)$	$n-3$	$(n-2)^2$	$n-3$	$(n-2)^2$	$n-3$	$(n-2)^2$	$n-2$	$(n-1)^2$
$n-3$	$(n-2)^2$	$n-2$	9	$n-3$	$6(n-3)$	$n-3$	$6(n-3)$	$n-3$	$(n-2)^2$
$n-3$	$6(n-3)$	$n-3$	$(n-1)^2$	$n-2$	9	$n-3$	$6(n-3)$	$n-4$	$(n-3)^2$
$n-2$	$(n-1)^2$	$n-2$	$(n-1)^2$	$n-2$	$(n-1)^2$	$n-2$	$(n-1)^2$	$n-2$	$(n-2)^2$
$n-2$	$4(n-2)$	$n-3$	$(n-2)^2$	$n-3$	$(n-2)^2$	$n-3$	$(n-2)^2$	$n-3$	$(n-2)^2$
$n-3$	$(n-2)^2$	$n-3$	$6(n-3)$	$n-4$	$(n-3)^2$	$n-4$	$(n-3)^2$	$n-4$	$(n-3)^2$
$n-4$	$(n-3)^2$	$n-4$	$(n-3)^2$	$n-3$	16	$n-4$	$8(n-4)$	$n-5$	$(n-4)^2$

It will be drawn then, after the reduction of terms, the number of the contrary combinations $= 176n^5 - 943n^4 + 2132n^3 - 2484n^2 + 1472n - 352$.

43. Going forward with the research, there will come made to us to discover, that took starting from the first of Problem VI., the four formulas found; n^2; $6n^3 - 13n^2 + 12n - 4$; $33n^4 - 126n^3 + 198n^2 - 144n + 40$; $176n^5 - 943n^4 + 2132n^3 - 2484n^2 + 1472n - 352$; are the first 4 terms of a recurrent one of third degree, whose multipliers tidily placed are; $6n - 10$; $-3n^2 + 16n - 12$; $-4n^3 + 8n^2$. This pure series also will have the appendix of the two terms $\frac{1}{n^2}$, 1 previous to the first one so that the series with the appendix will be $\frac{1}{n^2}$; 1; n^2; $6n^3 - 13n^2 + 12n - 4$. etc.

PROBLEM X

43. *The contrary combinations are sought to have at least once in A* $n-4$ *white in the course of any number of permutations.*

I consider this Problem in all its generality, no longer being able to ignore after examples of the above Problems, how I must proceed also here in order to investigate the contrary combinations that are due under the hypotheses of 1, 2, 3 etc. permutations. I have separately examined them as far as to that sign that demonstrated to me the law of the series, of which I write the first 4 terms.

$$\begin{array}{lll} & 1; & n^2 \\ & 2; & n^4 \\ \text{for permut.} & 3; & 12n^5 - 58n^4 + 144n^3 - 193n^2 + 132n - 36 \\ & 4; & 114n^6 - 888n^5 + 3159n^4 - 6216n^3 + 6952n^2 \\ & & -4128n + 1008. \end{array}$$

This series is a recurrent one of 4$^{\text{th}}$ degree, its multipliers are; $12n - 28$; $-30n^2 + 148n - 156$; $-4n^3 - 52n^2 + 216n - 144$; $15n^4 - 84n^3 + 108n^2$; and it admits the appendix of the two terms $\frac{1}{n^2}$, 1, that go ahead to the first term n^2.

PROBLEM XI

44. *The contrary combinations are sought to have at least once in A $n-5$ white in the course of any number of permutations.*

If the examination will be made of this hypothesis with the rules indicated above, a series of contrary combinations will be found, that is a recurrent one of fifth degree, of which these are in order the five multipliers. $20n-60$; $-110n^2+660n-908$; $140n^3-1460n^2+4376n-3696$; $95n^4-340n^3-1124n^2+4608n-2880$; $-56n^5+640n^4-2208n^3+2304n^2$: the first five terms; n^2; n^4; n^6; $20n^7-170n^6+800n^5-2273n^4+3980n^3-4180n^2+2400n-576$; $290n^8-3800n^7+23927n^6-89480n^5+211800n^4-320000n^3+298224n^2-155520n+34560$ etc. Here also takes place the usual appendices $\frac{1}{n^2}$, 1.

45. Returning to that which has been said from §31. as far as to now pick first, that as much is the degree of the recurrent one of the contrary combinations, so much, beginning from 2, is the number of the whites, that they wish extracted from the first urn in the course of any number of permutations; second, that having all these series in the beginning some terms that succeed in continuous geometric series, if you add the appendix, as many terms will be proportionals geometrically, as is the degree of the series of the contrary combinations, namely those are its multipliers.

So that, known that it was the general law of the multipliers for the indeterminate hypothesis of $n-z-1$ residual white balls, since the number of multipliers due to the recurrent one of this indefinite event is exactly $z+1$, and the terms of the geometric series from the first $\frac{1}{n^2}$, of the appendix as far as the last n^{2z-2} are also $z+1$, there would be no need to form some column of the events (this thing is most troublesome; and the greatest number of these columns awaited, when the number of permutations grows, always cause doubt to remain of having noticed them all), and the multipliers together with the terms of the geometric series would suffice in order to investigate the following terms of our series.

That is we must direct our meditations for this purpose; and the Problem that is important for us to solve is the following.

PROBLEM XII

46. *Given the number $n-z-1$ of white balls, that is wanted to remain at least once in urn A in the course of any number of permutations, to find the general series of the contrary combinations, that is to determine its multipliers, that are $z+1$ in number, and uniformly by the geometric part, $\frac{1}{n^2}$, 1, n^2, $n^4 \ldots n^{2z-2}$ of the general series serve to make be born all its subsequent terms.*

In order to give the solution, make of the work to set under eye the multipliers that correspond to the hypotheses of the precedence problems, so as to facilitate the investigation of the law, by which they proceed. Here it is disposed with order.

For $n-2$ residual white; 1.° multip. $2n-2$; 2.° n^2
$n-3$ r. w.; 1.° m. $6n-10$; 2.° $-3n^2+16n-12$; 3.° $-4n^3+8n^2$
$n-4$ r. w.; 1.° m. $12n-28$; 2.° $-30n^2+148n-156$;
3.° $-4n^3-52n^2+216n-144$; 4.° $15n^4-84n^3+108n^2$
$n-5$ r. w.; 1.° m. $20n-60$; 2.° $-110n^2+660n-908$;

3.° $140n^3 - 1460n^2 + 4376n - 3696$;
4.° $95n^4 - 340n^3 - 1124n^2 + 4608n - 2880$;
5.° $56n^5 + 640n^4 - 2208n^3 + 2304n^2$.

47. We give another equivalent form to these multipliers, that results from leaving the formula of the products born from the lateral columns so as they stand, without reducing them to net amount; of that an example is seen in the first five problems. The new form is this.

For $n-2$ residual white	1.°	mult.	$2(n-1)$
	2.°		n^2
$n-3$ r. w.	1.°	m.	$2(n-1)+4(n-2)$
	2.°		$-8(n-1)(n-2)+4(n-1)^2+n^2$
	3.°		$-4n^2(n-2)$
$n-4$ r. w.	1.°	m.	$2(n-1)+4(n-2)+6(n-3)$
	2.°		$-24(n-2)(n-3)-12(n-1)(n-3)+9(n-2)^2-8(n-1)$ $(n-2)+4(n-1)^2+n^2$
	3.°		$-6n^2(n-3)-24(n-1)^2(n-3)+48(n-1)(n-2)(n-3)$ $-18(n-1)(n-2)^2-4n^2(n-2)$
	4.°		$24n^2(n-2)(n-3)-9n^2(n-2)^2$
$n-5$ r. w.	1.°	m.	$2(n-1)+4(n-2)+6(n-3)+8(n-4)$
	2.°		$-48(n-3)(n-4)-32(n-2)(n-4)-16(n-1)(n-4)+16$ $(n-3)^2-24(n-2)(n-3)-12(n-1)(n-3)-8(n-1)(n-2)$ $+9(n-2)^2+4(n-1)^2+n^2$
	3.°		$192(n-2)(n-3)(n-4)+96(n-1)(n-3)(n-4)$ $-72(n-2)^2(n-4)+64(n-1)(n-2)(n-4)-32(n-1)^2(n-4)$ $-8n^2(n-4)-64(n-2)(n-3)^2-32(n-1)(n-3)^2$ $+48(n-1)(n-2)(n-3)-24(n-1)^2(n-3)-6n^2(n-3)$ $-18(n-1)(n-2)^2-4n^2(n-2)$.
	4.°		$48n^2(n-3)(n-4)+192(n-1)^2(n-3)(n-4)$ $-384(n-1)(n-2)(n-3)(n-4)+144(n-1)(n-2)^2(n-4)$ $+32n^2(n-2)(n-4)(n-4)+128(n-1)(n-2)(n-3)^2$ $-64(n-1)^2(n-3)^2-16n^2(n-3)^2+24n^2(n-2)(n-3)$ $-9n^2(n-2)^2$
	5.°		$-192n^2(n-2)(n-3)(n-4)+72n^2(n-2)^2(n-4)$ $+64n^2(n-2)(n-3)^2$.

48. The aforesaid formula by putting to greater convenience $n-1=a$; $2(n-2)=b$; $3(n-3)=c$; $4(n-4)=d$, etc.; $n^2=p^2$; $4(n-1)^2=q^2$; $9(n-2)^2=r^2$; $16(n-3)^2=t^2$, etc. is transformed into these others.

For $n-2$ residual white;	1.°	mult.	$2a$
	2.°		p^2
$n-3$ r. w.;	1.°	m.	$2a+2b$
	2.°		$-2b(2a)+p^2+q^2$
	3.°		$-2bp^2$
$n-4$ r. w.;	1.°	m.	$2a+2b+2c$
	2.°		$-2c(2a+2b)-4ab+p^2+q^2+r^2$
	3.°		$2c(4ab-p^2-q^2)-2bp^2-2ar^2$
	3.°		$2c(2bp^2)-p^2r^2$
$n-5$ r. w.;	1.°	m.	$2a+2b+2c+2d$
	2.°		$-2d(2a+2b+2c)-4bc-4ac-4ab+p^2+a^2+r^2+t^2$
	3.°		$2d(4bc+4ac+4ab-p^2-q^2-r^2)+8abc-2cp^2$ $-2cq^2-2bp^2-2ar^2-2at^2-2bt^2$
	4.°		$2d(-8abc+2cp^2+2cq^2+2bp^2+2ar^2)+4bcp^2$ $-p^2r^2-p^2t^2-q^2t^2+4abt^2$
	5.°		$2d(-4bcp^2+p^2r^2)+2bp^2t^2$.

49. The law, by which the first multipliers proceed, is for itself most clear, having each of them to comprehend as many terms of the series $2a$, $2b$, $2c$ etc, that is of the series $2(n-1+2(n-2)+3(n-3)$ etc.), as is the degree of the recurrent one that belongs to the given hypothesis diminished by a unit. Wanting therefore the first multiplier for the event of $n-6$ r. w., there will come to us to be furnished from the formula $2(n-1+2(n-2)+3(n-3)+4(n-4)+5(n-5))$. Thus for w.r $n-7$ it will make $=2(n-1+2(n-2)+3(n-3)+4(n-4)+5(n-5)+6(n-6))$; and generally for the hypothesis of w.r $n-z-1$ we will have the first multiplier $=2(n-1+2(n-2)+3(n-3)\dots z(n-z))$.

50. Regarding the second multiplier, we will make two parts from each of them: the first will come composed from the aggregate of all those terms, to which the last term of the first multiplier corresponds; the second part formed from all the remaining terms. Does one want for example $n-3$ r. w.? The first part of the second multiplier noted in §48. will make $-2b(2a)$; the other p^2+q^2. Thus for the hypothesis of $n-4$ white, this first part will make $-2c(2a+2b)$, and the second one $-4ab+p^2+q^2+r^2$. A slight examination then of these first parts for the 4 hypotheses of §48. will demonstrate to us, that they are equal to the negative product of the last term of the first multiplier multiplied by the first multiplier of the hypothesis immediately preceding. So that, the first general multiplier $=2(n-1+2(n-2)+3(n-3)\dots z(n-z))$ is called $=\alpha$, and that of the next antecedent hypothesis, that is $=2(0+n-1+2(n-2)\dots(z-1)(n-z+1))=\alpha'$, this first part will be able to be expressed in general with the following formula; $-2z(n-z)(\alpha')$.

51. For the second parts I stopped, that the series of squares p^2, q^2, r^2, t^2 etc, that is n^2, $4(n-1)^2$, $9(n-2)^2$, $16(n-3)^2$, $25(n-4)^2$ etc, has for general term $z^2(n-z+1)^2$; in addition, that each of these second parts is composed of every second multiplier, that agrees with the hypothesis of the near preceding event and the square, that happens immediately in the series of the squared p^2+q^2 etc., that enter in the formation of the same previous second multiplier. Thus in §48. for the event $n-3$, our second part is p^2+q^2, that is each second multiplier for the event $n-2$ with the square q^2 more, that

in the series of squares hold back immediately to p^2. For the event $n-4$, the second part is $-4ab+p^2+q^2+r^2$, the three first terms of which are precisely the second multiplier $-2b(2a)+p^2q^2$ for the hypothesis $n-3$, and the last one r^2 is the square that follows in order q^2; and the same is said for the other events. So that calling generally γ the second multiplier, that is due to the event $n-z-1$, the γ' the second multiplier, that belongs to the preceding event $n-z$, will make this second part $=\gamma'+z^2(n-z+1)^2$; and each the second multiplier $\gamma=-2z(n-z)\alpha'+\gamma'+z^2(n-z+1)^2$.

52. I come to the third multipliers, for which I establish the following rules extracted from the examination of the four hypotheses arranged in §48, and from others more on which I have instituted my calculations. Let α be the first general multiplier, that corresponds to the hypothesis of the event $n-z-1$; β the second, or γ be the third that we demand. Beyond this α' represents the first multiplier, that belongs to the hypothesis of the immediately preceding event $n-z$, the first multiplier that concerns the antepenultimate event $n-z+1$. Thus β' expresses the second multiplier due to the penultimate event $n-z$; and for the same event $n-z$ let γ be the third multiplier. I say that it will make $\gamma=-2z(n-z)\beta'+\gamma'-z^2(n-z+1)^2(\alpha'')$. We take in hand the hypothesis of the second event $n-3$, that has for third multiplier $-2bp^2$. Since under such supposition the first multiplier is $2a+2b$; the second, $-4ab+p^2+q^2$, it will make $\alpha=2a+2b$; $\alpha'=2a$; $\alpha''=0$. Similarly $\beta=-4ab+p^2+q^2$; $\beta'=p^2$, and $\gamma'=0$, because exactly under the preceding hypothesis $n-2$ we have no third multiplier. Therefore substituting these values in the rule above, we will have $\gamma=-2(n-2).2p^2+0-4(n-1)^2(0)$, that is, replacing b instead of $2(n-2)$; $\gamma=-2bp^2$. For the event $n-4$ there becomes $\alpha=2a+2b+2c$; $\beta=-4ab-4ac-4bc+p^2+q^2+r^2$, so that $\alpha'=2a+2b$; $\alpha''=2a$; $\beta'=-4ab+p^2+q^2$, and $\gamma'=-2bp^2$, such being exactly the third multiplier for the precedence hypothesis of the event $n-3$. Therefore it will make $\gamma=-3(n-3)(-8ab+2p^2+2q^2)-2bp^2-9(n-2)^2(2a)$, that is (placing c in place of $3(n-3)$, and r^2 in place of $9(n-2)^2$) $\gamma=8abc-2cp^2-2cq^2-2bp^2-2ar^2$, as has been found.

53. This short and elegant rule has the advantage of the maximum generality; and where α, β represent the multipliers that in order precede γ, serve not only for the investigation of the third, but also of the fourth, fifth etc. multipliers. The fourth multiplier γ is wanted, that corresponds to the event $n-4$. Having α, β to be the two antecedent multipliers, it will make for §48. $\alpha=-4bc-4ac-4ab+p^2+q^2+r^2$; $\beta=8abc-2cp^2-2cq^2-2bp^2--2ar^2$; $\alpha'=-4ab+p^2+q^2$; $\alpha''=p^2$; $\beta'=-2bp^2$; $\gamma'=0$. Therefore $\gamma=-3(n-3_(-4bp^2)-9(n-2)^2p^2$, that is $\gamma=4bcp^2-p^2r^2$. The fifth multiplier γ is demanded, that belongs to the event $n-5$. In such case it will make $\alpha=8bcd+8acd+8abd-2dp^2-2dq^2-2dr^2+8abc-2cp^2-2cq^2-2bp^2-2ar^2-2bt^2-2at^2$; $\beta=-16abcd+4cdp^2+4cdq^2+4bdp^2+4adr^2+4bcp^2-p^2r^2-p^2t^2-q^2t^2+4abt^2$; $\alpha'=8abc-2cp^2-2cq^2-2bp^2-2ar^2$; $\alpha''=-2bp^2$; $\beta'=4bcp^2-p^2r^2$; and $\gamma'=0$. So that by virtue of the rule $\gamma=-4(n-4)(8bcp^2-2p^2r^2)-16(n-3)^2(-2bp^2)$, that is $r=-8bcdp^2+2dp^2r^2+2bp^2t^2$, as must be.

54. We adapt the theory to an example, and let us show in practice, how given the total number of white balls, and asked to a certain event of white that must remain in the urn, the multipliers of the recurrent one of the contrary combinations can by means of our rule be traced with sufficient rapidity, and therefore the number of the permutations be determined, that are necessary to render the given event probable. Let the

number of the white balls be $n = 8$, and the multipliers of the recurrent one are sought, which are due to the event in the urn with 4 residual white. In this case it is agreed the three events $n-2 = 6$, $n-3 = 5$, $n-4 = 4$ to exhaust. Beginning from the first $n-2$, and recalling to memory, that $a = n-1$, $b = 2(n-2)$, $c = 3(n-3)$, that is $a = 7$, $b = 12$, $c = 15$; and that $2a$ is the first multiplier for the event $n-2 = 6$; $2a+2b$ for second event $n-3 = 5$; $2a+2b+2c$ for the third $n-4 = 4$, we will soon assign to each of these events the first respective multiplier, all three will be in order, like in the annexed figure, 7.2, 19.2, 17.2^2.

6 r. w.	1.°	mult.	7.2	5 r. w.	1.°	m.	19.2	4 r. w.	1.°	m.	17.2^2
	2.°		2^6		2.°		-19.2^2		2.°		-223.2^2
					3.°		-3.2^9		3.°		-237.2^4
									4.°		99.2^8

Resume then the rules $\gamma = -2z(n-z)$. $\alpha' + \gamma' + z^2(n-z+1)^2\gamma = -2z(n-z).\beta' + \gamma' - z^2(n-z+1)^2.\alpha''$, the first of which serves in order to recover the second multipliers, and the second in order to recover the others, we will speak thus. For the first event of 6 r. w. we will have $z = 1$, $\alpha = 7.2$, $\alpha' = 0$, $\gamma' = 0$. Therefore the second multiplier γ will be reduced to the formula $z^2(n-z+1)^2$, that is with the values of $n = 8$, $z = 1$; $\gamma = 2^6$. For the second event of 5 r. w. it becomes $z = 2$, $\alpha = 19.2$, $\alpha' = 7.2$, $\gamma' = 2^6$; and however the second multiplier γ is made $= -4.6.7.2 + 2^6 + 4.49 = -19.2^2$. For the third event then of 4 r. w. becomes $z = 3$, $\alpha = 17.2^2$; $\alpha' = 19.2$; $\gamma' = -19.2^2$; so that the second multiplier $\gamma = -6.5.19.2 - 19.2^2 + 9.36 = -223.2^2$. Next I pass to the third multipliers, and let the principle of the second hypothesis be given, since the first is lacking. Since α, β must be the two multipliers immediately preceding the third that we seek, it will make $\alpha = 19.2$, $\beta = -19.2^2$; consequently $\alpha' = 7.2$, and (we had no hypothesis superior to the first of 6 r. w.) $\alpha'' = 0$. Moreover $\beta' = 2^6$; $\gamma' = 0$. Hence, these values introduced in the second of the aforesaid rules, the third multiplier $\gamma = -4.6.2^6 = -3.2^9$ will arise. To the third hypothesis of 4 r. w. correspond these values; $z = 3$, $\alpha = 17.2^2$; $\beta = -223.2^2$; $\alpha' = 19.2$; $\alpha'' = 7.2$; $\beta' = -19.2^2$; $\gamma' = -3.2^9$, that give birth to the third multiplier $\gamma = 6.5.19.2^2 - 3.2^9 - 9.36.7.2$, that is $\gamma = -237.2^4$. Nothing remains presently other than the fourth multiplier of the third hypothesis, because it is lacking in the other two, it will make, for this last investigation, $\alpha = -223.2^2$; $\alpha' = -19.2^2$; $\alpha'' = 2^6$; $\beta = 237.2^4$; $\beta' = -3.2^9$; $\gamma' = 0$, and $z = 3$. Therefore the fourth multiplier $\gamma = 6.5.3.2^9 - 9.36.2^6 = 99.2^8$.

55. The four multipliers for the event of 4 r. w. known, there remains that the terms of the contrary series are found, of which with the aid of the appendix the first four terms $\frac{1}{2^6}$, 1, 2^6, 2^{12} are noted by us, that they observe the geometric proportion. We put under them the four multipliers, as in the present figure;

$\frac{1}{2^6}$,	1,	2^6,	2^{12},	54511.2^2,	684751.2^4,	38988839.2^4,	498813699.2^6
		0,	0,	11025.2^2,	363825.2^4,	28120025.2^4,	574928125.2^6.
99.2^8,	-237.2^4,	-223.2^2,	17.2^2				

And then let us just make the usual multiplication the rule of the recurrent ones. They will give us the fifth term $= 54511.2^2$, the sixth $= 684751.2^4$, the seventh $= 38988839.2^4$, the eighth $= 498813699.2^6$ etc. In order not to go forward without purpose, it will be good to have present that which we have perceived in §15. on the total number of combinations of all the balls in the diverse permutations to be able, to every term of the recurrent one of the contrary combinations, write under from hand to hand the analogous one in the series of the favorable ones. Thus in our case one will see, that to the terms of the contrary series, beginning from 2^6 that it is really the first one as far as the term 498813699.2^6 correspond to the favorable terms 0; 0; 11025.2^2; 363825.2^4; 28120025.2^4; 574928125.2^6. Now in the antecedent terms to the sixth adverse combinations are always in greater number than the propitious, but that in the sixth exceeds the first, and this excess, as if experience can be made, is always increasing more in the following terms. Therefore it would be a useless thing to go much further in the series, having already so much that is enough in order to conclude, that the number of the permutations necessary to render probable the event of reduction in the first urn to the half of all the white balls, is a middle number between 5 and 6, not counting already, as we have always done, the first operation, that of 8 white which were to be made to remain seven. I will come closer to the just, if I demand six permutations rather than five, because the contrary combinations, and the useful ones in five permutations stand between them :: 38988839 : 28120025, that is :: 1.38 : 1 nearly; whereby in six permutations the proportion of the contraries to the useful ones is that of 498813699 : 574928125, that is of 1 : 1.15 thereabouts; and the two numbers 1, 1.15 are approaching more to equality from that which the others 1.38, 1 make.

56. I cannot disguise the trouble, that the method brings to calculating that I have presented for the finding of the multipliers, because to determine any multiplier with respect to some hypothesis, it is necessary that the analogous multiplier of the preceding hypothesis be known; this supposes noting the analogous of the hypothesis, that it goes forward to this last one, and so on away until with retrograde step it is arrived to the first hypothesis of the event $n-2$. So that if, for example, given the two multipliers for the first event, one wants to determine the seven multipliers, that are demanded from the hypothesis of the event $n-7$, will make of necessity by the aid of the first two to find the three multipliers for the event $n-3$, then the four for the event $n-4$ etc. as far as the seven multipliers for the last one that we demand. But this trouble remains much diminished, where a numerical value is given to the species n, as it is the habit to do on the occasion that similar questions come in practice; and in such case, when the number of all the white balls, and those of the white ones, that you want removed from the urn are not very large, with a few numerical operations give birth to the multipliers of all the hypotheses that precede this last, and the relevant multipliers are assigned to these same.

Also the use of logarithms, especially in the multiplications that must be made in order to find the terms of the series of the combinations, will be able to contribute to facilitate the matter more ways; and considering everything together, it will have to be concluded for the venture, that in a problem of no small difficulty, as ours, the method exposed is effectively one of more simplicity and of less difficulty.

57. In order to satisfy nevertheless who may love a greater generality also at the cost of the quickness, and for the determination of the multipliers that a given event requires would not want to have need of the flow for the hypotheses of all the previous events, there remains to us finally to find the way to generalize these same multipliers, and to enclose the multipliers in such formulas only that belong to all possible events. This is that which we will execute in the subsequent paragraphs; but the solution premissed on the present.

PROBLEM XIII.

58. *Let the general formula (M) be* $\phi = az^m + bz^{m-1} + cz^{m-2} + dz^{m-3} + ez^{m-4} + fz^{m-5} + gz^{m-6}$ *etc., then another formula (N)* $\gamma = pz^{m+1} + qz^m + rz^{m-1} + fz^{m-2} + tz^{m-3} + uz^{m-4} + xz^{m-5}$ *etc. Decided that z decreases by the positive unit, so that, making use of finite differences in the manner of writing, let* $z - z' = \delta z = 1$*; and suppose that there is* $\gamma - \gamma' = \delta\gamma = \phi$, γ *is sought, namely the indeterminate values p, q, r, s, etc. are demanded given for m, and for the other known a, b, c, d etc.*

The solution of this problem is easy, if we make use of the well-known theorem comprehended in this equation; $\delta\gamma = d\gamma - \frac{d^2\gamma}{2} + \frac{d^3\gamma}{2.3} - \frac{d^4\gamma}{2.3.4} + \frac{d^5\gamma}{2.3.4.5}$ etc. in which the d represent ordinary differentiations, each of which carried, admit to change the difference dz that each term has in dz, that is in 1. Since, by virtue of this theorem, we will have

$$\begin{aligned}
d\gamma &= (m+1).pz^m + mqz^{m-r} + (m+1)rz^{m-1} + (m-2)sz^{m-3} \\
&\quad + (m-3)tz^{m-4} + (m-4)uz^{m-5} \text{ etc.} \\
d^2\gamma &= m(m+1)pz^{m-1} + (m-1)mqz^{m-r} + (m-2)(m-1)rz^{m-3} \\
&\quad + (m-3)(m-2)sz^{m-4} + (m-4)(m-3)tz^{m-5} \text{ etc.} \\
d^3\gamma &= (m-1)(m)(m+1).pz^{m-2} + (m-2)(m-1)(m)qz^{m-3} \\
&\quad + (m-3)(m-2)(m-1)rz^{m-4} + (m-4)(m-3)(m-2)sz^{m-5} \text{ etc.} \\
d^4\gamma &= (m-2)(m-1)(m)(m+1)pz^{m-3} + (m-3)(m-2)(m-1)(m)qz^{m-4} \\
&\quad + (m-4)(m-3)(m-2)(m-1)rz^{m-5} \text{ etc.} \\
d^5\gamma &= (m-3)(m-2)(m-1)(m)(m+1)pz^{m-4} \\
&\quad + (m-4)(m-3)(m-2)(m-1)(m)qz^{m-5} \text{ etc.} \\
d^6\gamma &= (m-4)(m-3)(m-2)(m-1)(m)(m+1)pz^{m-5} \text{ etc.,}
\end{aligned}$$

and therefore etc.

$$
\begin{aligned}
\delta\gamma = {} & (m+1)pz^m + \left(-\frac{m(m+1)}{2}p + mq\right) z^{m-1} \\
& + \left(\frac{(m-1)(m)(m+1)}{2.3}p - \frac{(m-1)(m)}{2}q + (m-1)r\right) z^{m-2} \\
& + \left(-\frac{(m-2)(m-1)m(m+1)}{2.3.4}p + \frac{(m-2)(m-1)(m)q}{2.3}\right. \\
& \left. - \frac{(m-2)(m-1)r}{2} + (m-2)s\right) z^{m-3} + \left(\frac{(m-3)(m-2)(m-1)(m)(m+1)p}{2.3.4.5}\right. \\
& - \frac{(m-3)(m-2)(m-1)(m)q}{2.3.4} + \frac{(m-3)(m-2)(m-1)r}{2.3} \\
& \left. - \frac{(m-3)(m-2)s}{2} + (m-3)t\right) z^{m-4} \\
& + \left(-\frac{(m-4)(m-3)(m-2)(m-1)(m)(m+1)p}{2.3.4.5.6}\right. \\
& + \frac{(m-4)(m-3)(m-2)(m-1)(m)q}{2.3.4.5} - \frac{(m-4)(m-3)(m-2)(m-1)r}{2.3.4} \\
& \left. + \frac{(m-4)(m-3)(m-2)s}{2.3} - \frac{(m-4)(m-3)t}{2} + (m-4)u\right) z^{m-5} \text{ etc.}
\end{aligned} \tag{O}
$$

Nothing therefore remains to do but to compare the terms of this equation $\phi =$ etc., and we will find the values of the aforesaid indeterminate ones p, q, etc. With the comparisons then of the subsequent first terms offer us the equation $(m+1)p = a$, that gives $p = \frac{a}{m+1}$. From the comparison of the second there results $-\frac{(m)(m+1)}{2}p + mq = b$, or, substituting the value of p, and the necessary operations made $q = \frac{b}{m} + \frac{a}{2}$. With the comparisons of the terms then there will come given to us the other indeterminate ones, which I arrange in order, beginning from the first;
$p = \frac{1}{m+1}$; $q = \frac{b}{m} + \frac{a}{2}$; $r = \frac{c}{m-1} + \frac{b}{2} + \frac{ma}{2.2.3}$; $s = \frac{d}{m-2} + \frac{c}{2} + \frac{(m-1)b}{2.2.3} + a.0$; $t = \frac{e}{m-2} + \frac{d}{2} + \frac{(m-2)c}{2.2.3} + b.0 - \frac{(m-2)(m-1)(m)a}{2.3.2.3.4.5}$; $u = \frac{f}{m-4} + \frac{e}{2} + \frac{(m-3)d}{2.2.3} + c.0 - \frac{(m-3)(m-2)(m-1)}{2.3.2.3.4.5}.b + a.0$; $x = \frac{g}{m-5} + \frac{f}{2} + \frac{(m-4)e}{2.2.3} + d.0 - \frac{(m-4)(m-3)(m-2)c}{2.3.2.3.4.5} + b.0 + \frac{(m-4)(m-3)(m-2)(m-1)(m)a}{2.3.2.3.4.5.6.7}$ etc.

The law of the first two terms in all the values of these indeterminate ones is clear. Beginning then the series of the remaining from the third terms, we see, that the terms in the equal places are null, that those of the uneven places progress with alternate signs, and that the number of the factors, in which m enters for the terms of the uneven places follow the law of the uneven numbers 1,…5, 7 etc. As for the law of the numerical coefficients, one should not permit oneself to be deceived by those few values of the species p, q, r etc. that we have determined, with which it could seem, that the numerators of the terms may have no other numerical coefficient than the unit. If it is advanced as far as the ninth term of the generating equations, and so it is $\phi = az^m \cdots + hz^{m-7} + iz^{m-8}$ etc., $\gamma = pz^{m+1} \cdots + yz^{m-6} + \pi z^{m-7}$, one finds $\pi = \frac{i}{m-7} + \frac{b}{2} + \frac{(m-6)g}{2.2.3} + f.0 - \frac{(m-6)(m-3)(m-4).e}{2.3.2.3.4.5.} + d.0 + \frac{(m-6)(m-5)(m-4)(m-3)(m-2).c}{2.3.2.3.4.5.6.7} + b.0 - \frac{3(m-6)(m-5)(m-4)(m-3)(m-2)(m-1)(m)a}{2.5.2.3.4.5.6.7.8.9}$; where one sees that 3 appears in the numerator of the last term. The true rule, by which these coefficients are determined, is the following one, as each can check. I put for convenience $\frac{1}{2.2.3} = \omega$, and it will make $\frac{1}{2.3.2.3.4.5} = \frac{-3}{2.2.3.4.5} + \frac{\omega}{2.3}$. This homogeneous one of comparison is made $= \omega'$, and there results $\frac{1}{2.3.2.3.4.5.6.7} = \frac{5}{2.2.3.4.5.6.7} - \frac{\omega}{2.3.4.5} + \frac{\omega'}{2.3}$, which is $= \omega''$; and we will have $\frac{3}{2.5.2.3.4.5.6.7/8/9} = \frac{-7}{2.2.3.4.5.6.7.8.9} + \frac{\omega}{2.3.4.5.6.7} - \frac{\omega'}{2.3.44.5} + \frac{\omega''}{2.3}$ etc. with the law, that is manifest.

PROBLEM XIV.

59. *Given the general event of* $n-z-1$ *white, to determine the first general multiplier of the contrary recurrent series of degree* $z+1$, *that corresponds to the given event.*

We have seen in §49, that the first general multiplier is equal to the sum of the series $2(n-2)+4(n-2)+6(n-3)+\cdots+2(z-1)(n-z+1)+2z(n-z)$, that I suppose $=\gamma$. Now since this series diminished by the last term $2z(n-z)$ is that which becomes γ, if in it in place of z is put $z-1$, it will make $2(n-1)+4(n-2)+6(n-3)\cdots 2(z-1)(n-z+1)=\gamma'$, for preceding Problem, and therefore $\gamma-\gamma'$, that is $\delta\gamma = 2z(n-z) = -2z^2+2nz$. Let the comparison of this equation with the rule (O) of §58 be instituted, and we will have $m=2$; $(m+1)p=-2$, that is $p=-\frac{2}{3}$ with the comparison of the first terms. Those of the second furnish us the equation $-3p+2q=2n$, from which with the substitution of the value of p is drawn $q=n-1$. Because then the constant term is lacking in our formula, it will make with the comparison of the terms $p-q+r=0$, that is $r=-p+q=n-1+\frac{2}{3}=\frac{3n-1}{3}$. Let now these values be introduced into the general formula (N) of §58, and the sum of the series will be $\gamma=\frac{-2z^3}{3}+(n-1)z^3+\frac{(3n-1)z}{3}+s$. The species s will come determined from the condition, that, when $z=1$, the sum of our series is equal to the first term $2(n-1)$; so that is born $-\frac{2}{3}+n-1+\frac{3n-1}{3}+s=2n-2$, that is $-\frac{2+3n-3+3n-1}{3}+s=2n-2$ that is $\frac{6n-6}{3}+s=2n-2$, that gives $s=0$. Since $\delta\gamma=\phi$, one can also make the comparison of our general term $-2z^2+2nz$ with the formula (M) of §58, and derive thence the values of the symbols m, a, b, etc. Here is that which would result; $m=2$, $a=-2$, $b=2n$, $c=0$. Therefore, for the general determination of the species p, q etc. $p=-\frac{2}{3}$, $q=n-1$, $r=\frac{3n-1}{3}$, as is already found. From this the first multiplier of the contrary recurrent is deduced for the general event of $n-z+1$ white, it will make $=\frac{-2z^3+3nz^2-3z^2+3nz-z}{3}=(z+1)(zn-\frac{2z^2+z}{3})$.

PROBLEM XV.

60. *Given the general event of* $n-z-1$ *white, to determine the second multiplier of the contrary series of degree* $z+1$, *that corresponds to the given event.*

Let θ be any function of z; θ' that which θ becomes, if in it is put $z-1$ in place of z; θ'' that which θ' becomes, if in θ is put another time $z-1$ in place of z; etc. The theory of finite differences furnishes us as many series as there are θ', θ'' etc., by means of which the given come to the same θ', θ'' etc for first symbol θ; and is had.

$$\theta' = \theta - d\theta + \frac{d^2\theta}{2} - \frac{d^3\theta}{2.3} + \frac{d^4\theta}{2.3.4} - \frac{d^5\theta}{2.3.4.5} + \frac{d^6\theta}{2.3.4.5.6} \text{ etc.} \qquad \text{(P)}$$

$$\theta'' = \theta - 2d\theta + \frac{2^2 d^2\theta}{2} - \frac{2^3 d^3\theta}{2.3} + \frac{2^4 d^4\theta}{2.3.4} - \frac{2^5 d^5\theta}{2.3.4.5} \text{ etc.} \tag{Q}$$

where d signifies the ordinary differentiations. Of the two series noticed, the first is useful to us in order to find our second multiplier; both in order to find the following. Recall to such end the formula of §51. that is $\gamma = -2z(n-z)\alpha' + \gamma' + z^2(n-z+1)^2$, in which γ represents the second multiplier, α the first, and α' that which α becomes if in it in place of z is put $z-1$. It will be $\gamma - \gamma' = \delta\gamma = -2z(n-z)\alpha' + z^2(n-z+1)^2$.

Now since $\alpha = \frac{-2z^3+3nz^2-3z^2+3nz-z}{3}$, made $\theta = \alpha$, and 1 substituted in place of dz, we have by rule(P); $\alpha' = \frac{-2z^3+3nz^2-3z^2+3nz-z}{3} + \frac{6z^2-6nz+6z-3n+1}{3} + \frac{-6z+3n-3}{3} + \frac{2}{3}$; that is $\alpha' = \frac{-2z^3+(3n+3)z^2-(3n+1)z}{3}$; and therefore $\delta\gamma = 2z(n-z)\frac{(2z^3-(3n-3)z^2+(3n+1)z}{3} + z^2(n-z+1)^2 = \frac{-4z^5+(10n+9)z^4-(6n^2-18n-)z^3+(9n^2+8n+3)z^2}{3}$.

Compare this formula with the general (M), and there will result $m=5$, $a = -\frac{4}{3}$, $b = \frac{10n+9}{3}$, $c = \frac{-6n^2-18n-8}{3}$, $d = \frac{9n^2+8n+3}{3}$, $e=0$, $f=0$, because the term is lacking, in which z is in the first dimension, and the constant term. Substituting then these values of m, a, b etc. in the general determinations of the symbols p, q etc, it has $p = -\frac{2}{3}$, $q = \frac{10n-1}{3/5}$, $r = \frac{-9n^2+3n+5}{2.3^2}$, $s = -n$, $t = \frac{18n^2-3n-1}{2.3^2}$, $u = \frac{15n^2+10n+2}{2.3.5}$; so that

$$\gamma = \frac{-20z^6 + \overline{60n-6}z^5 - \overline{45n^2+15n+25}z^4 - 90nz^3 + \overline{90n^2-15n-5}z^2 + \overline{45n^2+30n+6}z.}{2.3.3.5}$$

Do not add some constant term, because the finite difference $\delta\gamma$ being an aggregate of terms multiplied by z^2, and canceling it when $z=0$, under the same hypothesis also the integral γ must be zero. So that it will make known the second general multiplier, that is the responsibility of our series, and we will express it in this other equivalent form;

$$\frac{z(z+1)}{2}\left(-(z^2+z+1)n^2 + \frac{(4z^3-3z^2-3z+2)}{3}n + \frac{-20z^4+14z^3+11z^2-11z+6}{3.3.5}\right).$$

PROBLEM XVI

61. *Given the general event of $n-z-1$ white, to determine the third multiplier of the contrary series of degree $z+1$, that corresponds to the given event.*

For §51 we have the third multiplier $\gamma = -2z(n-z)\beta' + \gamma' - z^2(n-z+1)^2.\alpha''$, the symbols α, β denoting the first and the second multiplier already found: and therefore $\gamma - \gamma' = \delta\gamma = -2z(n-z)\beta' - z^2(n-z+1)^2\alpha''$. From the value of $\beta =$

$$\frac{-20z^6 + \overline{60n-6}.z^5 - \overline{45n^2+15n+25}.z^4 - 90nz^3 + \overline{90n^2-15n-5}.z^2 + \overline{45n^2+30n+6} \times z}{2.3.3.5}$$

it is deduced by means of the rule (P) $\beta' =$

$$\frac{-20z^6 + \overline{60n+114}.z^5 - \overline{45n^2-285n-245}.z^4 + \overline{180n^2+450n+240}.z^3 - \overline{180n^2+255n-95}.z^2 + \overline{45n^2+30n+6}.z}{2.3.3.5}$$

and from the value of $\alpha = \frac{-2z^3+(3n-3)z^2+(3n-1)z}{3}$ with second rule (E) is drawn

$$\alpha'' = \frac{-2z^3+(3n+9)z^2+(-9n-13)z+6n+6}{3}.$$

With the introduction then of these values in the formula above $\delta\gamma = -2z(n-z)\beta' - z^2(n-z-1)^2\alpha''$ there will be obtained after the necessary reductions

$$\delta\gamma = \frac{1}{3.3.5}(-20z^8 + (80n+144)z^7 + (-105n^2 - 504n - 440)z^6 \\ + (45n^3 + 585n^2 + 1250n + 735)z^5 + (-225n^3 - 1125n^2 - 1560n - 710)z^4 \\ + (315n^3 + 945n^2 + 1010n + 381)z^3 + (-135n^{-}300n^2 - 276n - 90)z^2)$$

This formula must be compared with the rule (M) for the determination of the species m, a, b etc., so that the other symbols p, q, r etc. also become known and its integral is finally had that will make the sought third multiplier. But if by reason of the very complex coefficients of the power of z it is found more convenient to integrate by parts the aforesaid differential formula, it ought to be transformed in such case into this other equivalent

$$\delta\gamma = (z^5 - 5z^4 + 7z^3 - 3z^2)n^3 + \frac{(-7z^6 + 39z^5 - 75z^4 + 63z^3 - 20z^2)n^2}{3} \\ + \frac{(80z^7 - 504z^6 + 1250z^5 - 1560z^4 + 1010z^3 - 276z^2)n}{3.3.5} \\ + \frac{-20z^8 + 144z^7 - 440z^6 + 735z^5 - 710z^4 + 381z^3 - 90z^2}{3.3.5},$$

and to consider that each factor of the power of n is a formula of finite differences. Each of these factors integrated therefore with the aid of the rule (M), without adding constant, that has no place, the third multiplier demanded will be found from the Problem

$$= \frac{(z-1)(z(z+1)}{2.3}\Bigg((z^3 - 3z^2 - z + 2)n^3 + \frac{(-6z^4 - 18z^3 - 15z + 6)n^2}{3} \\ + \frac{(60z^5 - 192z^4 + 78z^3 + 174z^2 - 102z + 36)n}{3.3.5} \\ + \frac{-280z^6 + 1008z^5 - 808z^4 - 693z^3 + 1061z^2 - 126z + 144}{3.3.3.5.7}\Bigg).$$

PROBLEM XVII.

62. *Given the general event of* $n-z-1$ *white, to determine the fourth multiplier of the contrary series of degree* $z+1$*, that corresponds to the given event.*

γ representing second as usual this fourth multiplier, will make α the second multiplier, and β the third, that are already noticed. Therefore there passes with the rule (Q) from the value of α to that of α'', and from the value of β to that of β', and substitute these new values into the general formula $\delta\gamma = -2z(n-z)\beta' - z^2(n-z+1)^2\alpha''$. There will be $\delta\gamma$ given for z and for n; and the integration of this formula either all at once, or by parts, executed in the manner of the preceding Problems will make us recognize the fourth multiplier, that we demand, and will make it

$$= \tfrac{(z-2)(n-1)z(z+1)}{2.3.4}\big\{-z^4+6z^3-5z^2-8z+3)n^4$$
$$+(8z^5-50z^4+64z^3+36z^2-70z+12)\tfrac{n^3}{3}$$
$$+(-120z^6+804z^5-1437z^4+78z^3+1503z-792z+108)\tfrac{n^2}{3.3.5}$$
$$+(1120z^7-8232z^6+19108z^5-10212z^4-15476z^3+18228z^2-3816z+864)\tfrac{n}{3.3.3.5.5.7}$$
$$-\tfrac{1}{3.3.3.3.5.5.7}(2800z^8-22960z^7+62836z^6-64288z^5-30305z^4+83310z^3-37251z^2+198z-3240)\big\}.$$

Let the same method be kept for the 5.$^{\text{th}}$, 6.$^{\text{th}}$ etc. multipliers, and each of them will be in our hand, without there be need to have them by the hypothesis ascend from the event $n-z-1$ to all the preceding as far as the last one of the event $n-2$.

63. We will make the reflection, that the third multiplier has the factor $z-1$, and becomes zero, when $z=1$; which must to be necessarily, because under such hypothesis the recurrent series of the combinations is precisely of second degree, and which does not admit consequently two multipliers only. Thus the fourth multiplier receives the factor $z-2$, and therefore it is null under the hypothesis of $z=2$, because in such case the recurrent one, that corresponds to it, is only of third degree, and ought really be lacking this fourth multiplier. So that, a similar discourse adapted to the subsequent multipliers, will be assembled, that the number of the multipliers is called m, the last denominator of the number m will have the factor $z-m+2$. It is observed further, that the factors of our general multipliers constitute an arithmetical series, and that are as many, as is the maximum exponent of n for each of them, that is as many is the maximum exponent of z in the formula, that multiplies the maximum power of n. Moreover, that the only factor of the first multiplier is divided by 1, the two of the second by 1.2, the three of the third by 1.2.3; the four of the fourth by 1.2.3.4; consequently the number m factors of the multiplier named by m will be divided by $1.2.3.4\ldots(m-1)m$: Finally that the maximum exponent of z increases successively by one unit in the multiplying formula of the power of n decreasing successively by one unit in every multiplier; and similarly the maximum exponent of z increases by one unit in the formula that multiplies the maximum power of n, when one passes from a multiplier to its immediate successor.

64. These things put, in order to avoid making so many divisions in the multipliers found with the above-mentioned method, how many are their factors, so that to reduce them to a more comfortable form, the appropriate solution may fall from the present

PROBLEM XVIII.

65. *To find for every multiplier the multiplying formula of the power of n, that united constitute the quotient of the entire multiplier divided by its respective factors.*

To this end I establish any three general multipliers, that they succeed one after the other. The antepenultimate is

$$A = \frac{(z+1)z(z-1)\cdots(z-m+4)}{1.2.3\ldots(m-2)}(an^{m-2}+bn^{m-3}+cn^{m-4}+en^{m-5}+fn^{m-6}\cdots \text{ etc.}).$$

The penultimate one

$$B = \frac{(z+1)z(z-1)\cdots(z-m+3)}{1.2.3\ldots(m-1)}(\alpha n^{m-1}+\beta n^{m-2}+\gamma n^{m-3}+\varepsilon n^{m-4}+\phi n^{m-5}\cdots \text{ etc.}).$$

The last one, that is sought

$$C=\frac{(z+1)z(z-1)\cdots(z-m+2)}{1.2.3\dots m}(sn^m+tn^{m-1}+un^{m-2}+xn^{m-3}+yn^{m-4}\cdots\text{ etc.}).$$

The species a, b, c etc., α, β, γ etc., s, t, u etc. are functions of z. Let further A', B', C', a', b', c', etc., α', β' etc., s', t' etc. be that which A, B, C, a etc. become, if $z-1$ is put instead of z in these; and A'', B'', C'', a'' etc. that which A', B', C', a' etc. become if $z-1$ is put instead of z in them another time. There will be

$$A'=\frac{z(z-1)\cdots(z-m+3)}{1.2.3\dots(m-2)}(a'n^{m-2}+b'n^{m-3}+c'n^{m-4}+e'n^{m-5}+f'n^{m-6}\cdots\text{ etc.}$$
$$A''=\frac{(z-1)(z-2)\cdots(z-m+2)}{1.2.3\dots(m-2)}(a''n^{m-2}+b''n^{m-3}+c''n^{m-4}+e''n^{m-5}+f''n^{m-6}\cdots\text{ etc.}$$
$$B'=\frac{z(z-1)\cdots(-m+2)}{1.2.3\dots(m-1)}(\alpha'n^{m-1}+\beta'n^{m-2}+\gamma'n^{m-3}+\varepsilon'n^{m-4}+\phi'n^{m-5}\cdots\text{ etc.}$$
$$C'=\frac{z(z-1)\cdots(z-m+1)}{1.2.3\dots m}(s'n^m+t'n^{m-1}+u'n^{m-2}+x'n^{m-3}+y'n^{m-4}\cdots\text{ etc.}$$

Therefore $C-C'=\delta C=$

$$\begin{aligned}\frac{z(z-1)\cdots(z-m+2)}{1.2.3\dots m}\Big(\big((z+1)(s-s')+ms'\big)\times n^m+\big((z+1)(t-t')+mt'\big)\big)\times\\ n^{m-1}+\big((z+1)(u-u')+mu'\big)\times n^{m-2}+\big((z+1)(x-x')+mx'\big)\times n^{m-3}\\ +\big((z+1)(y-y')+my'\big)\times n^{m-4}\text{ etc.}\Big).\end{aligned}$$

But, by the determining formula of our multipliers we have also $\delta C=-2z(n-z)B'-z^2(n-z+1)^2A''$, that is, after the substitutions of the values of B', A'', and the appropriate reductions;

$$\begin{aligned}\delta C=\frac{z(z-1)\cdots(z-m+2)}{1.2.3\dots m}&\Big(-\big(m(m-1)za''+2mz\alpha'\big)\times n^m\\ &+\big(2m(m-1)(z-1)za''-m(m-1)zb''+2mz^2\alpha'-2mz\beta'\big)\times n^{m-1}\\ &-\big(m(m-1)(z-1)^2za''+2m(m-1)(z-1)zb''-m(m-1)zc''+2mz^2\beta'-2mz\gamma'\big)\times n^{m-2}\\ &-\big(m(m-1)(z-1)^2zb''+2m(m-1)(z-1)zc''-m(m-1)ze''+2mz^2\gamma'-2mz\varepsilon'\big)\times n^{m-3}\text{ etc.}\Big)\end{aligned}$$

with order for the subsequent terms similar to that one, that begins with the third term. Put therefore δs, δt, δu etc. in place of $s-s'$, $t-t'$, $u-u'$ etc., and moreover $s-\delta s$, $t-\delta t$, $u-\delta u$, in place of s', t', u'; and equate the two values of δC, with removing the equal factors results the equation;

$$
\begin{aligned}
&((z-(m-1))\delta s+ms)\times n^m+((z-(m-1))\delta t+mt)\times n^{m-1}\\
&+((z-(m-1))\times\delta u+mu)\times n^{m-2}+((z-(m-1))\times\delta x+mx)\times n^{m-3}\text{ etc.}\\
&=\left(-m(m-1)za'-2mz\alpha'\right)\times n^m\\
&+\left(2m(m-1)(z-1(za''-m)m-1)zb''+2mz^2\alpha'-2mz\beta'\right)\times n^{m-1}\\
&+\left(-m(m-1)(z-1)^2za''+2m(m-1)(z-1)zb''-m(m-1)zc''\right.\\
&\left.+2mz^2\beta'-2mz\gamma'\right)\times n^{m-2}+\left(-m(m-1)(z-1)^2zb''\right.\\
&\left.+2m(m-1)(z-1)zc''-m(m-1)ze''+2mz^2\gamma'-2mz\varepsilon'\right)\times n^{m-3}\text{ etc.}
\end{aligned}
$$

where the formulas, that multiply the power of n in the first member must be identical with the formulas, that multiply the analogous power of n in the homogenous one of comparison. Now, as m is the known quantity, and the species a, b, c etc. α, β, γ etc. are also known, because they belong to the two known multipliers which immediately precede the multiplier C; and therefore, by rules (P) (Q) the others, that derive from it, a'', b'', c'' etc. α', β', γ' etc., will not remain to make other, that give for these the unknown symbols s, t, u etc., so that to exhibit the sought quotient of the multiplier C, and consequently the same multiplier.

66. To obtain this, I establish etc in general,

$$S=Fz^r+Gz^{r-1}+Hr^{r-2}+Iz^{r-3}+Lr^{r-4}\text{ etc.}$$

It will be by the rule (O) of §58.

$$
\begin{aligned}
\delta S=rFz^{r-1}&+\left(\frac{-r(r-1)F}{2}+(r-1)G\right)\times z^{r-2}\\
&+\left(\frac{r(r-1)(r-2)F}{2.3}-\frac{(r-1)(r-2)}{2}G+(r-2)H\right)\times z^{r-3}\cdots\text{ etc.,}
\end{aligned}
$$

from that is gained, being

$$
\begin{aligned}
&(z-(m-1))\delta S+mS=((r+m-1)F+F)\times z^r\\
&+\left(\frac{-r(r+2m-3)}{2}F+(r+m-2)G+G\right)z^{r-1}\\
&+\left(\frac{r(r-1)(r+3m-5)}{2.3}F-\frac{(r-1)(r+2m-4)}{2}G+(r+m-3)H+H\right)z^{r-2}\\
&+\left(\frac{-r(r-1)(r-2)(r+4m-7)}{2.3.4}F+\frac{(r-1)(r-2)(r+3m-6)}{2.3}G\right.\\
&\left.-\frac{(r-2)(r+2m-5)}{2}H+(r+m-4)I+I\right)z^{r-3}\text{ etc.}
\end{aligned}
$$

The law of this series is itself manifest, and when I may wish gain, will be able to produce as far as that number of terms that the nature of the formula will demand, and the number of the terms, from which they come constituted.

67. One demands now for the multiplier C named from m the formula that multiplies n^m, will make $r=m$, $S=s$, however $s=Fz^m+Gz^{m-1}+Hz^{m-2}+Iz^{m-3}$ etc., and one has $(z-(m-1))\delta s+ms$, that is

$$\begin{aligned}
&((2m-1)F+F)\times z^m+\left(\frac{-m(3m-3)}{2}F+(2m-2)G+G\right)\times z^{m-1}\\
&+\left(\frac{m(m-1)(4m-5)}{2.3}F-\frac{(m-1)(3m-4)}{2}G+(2m-3)H+H\right)\times z^{m-2}\\
&+\left(\frac{-m(m-1)(m-2)(5m-7)}{2.3.4}F+\frac{(m-1)(m-2)(4m-6)}{2.3}G\right.\\
&\left.-\frac{(m-2)(3m-5)}{2}H+(2m-4)I+I\right)\times z^{m-3}\text{ etc.}\\
&=m(m-1)za''-2mz\alpha'.
\end{aligned}$$

The functions a'', α' being known in this second member, that belongs to the preceding multipliers, if their given values are substituted for z, there will be born in the homogeneous formula, in which it will make m the maximum exponent of z; and comparing the terms that result with the analogous of the first member, will determine the values of F, G, H etc. so that the formula will remain known, that in the multiplier C named from m multiplies n^m.

68. This operation accomplished, it must proceed to find the formula t, that in C multiplies n^{m-1}. Let us recall, we having said in §63, that the maximum exponent of z in t becomes $m+1$, and we will see that it must have for $r=m+1$. Make then $S=t$, it is necessary that this equation be verified: $(z-(m-1))\delta t+mt$, that is

$$\begin{aligned}
&(2mF+F)\times z^{m+1}+\left(\frac{-(m+1)(3m-2)}{2}F+(2m-1)G+G\right)\times z^m\\
&+\left(\frac{(m+1)(m(4m-4)}{2.3}F-\frac{m(3m-3)}{1}G+(2m-2)H+H\right)\times z^{m-1}\\
&\left(\frac{-m(m+1)(m(m-1)(5m-6)}{2.3.4}F+\frac{m(m-1)(4m-5)}{2.3}G\right.\\
&\left.-\frac{(m-1)(3m-4)}{2}H+(2m-3)I+I\right)z^{m-2}\text{ etc.}\\
&=2m(m-1)(z-1)za''-m(m-1)zb''+2mz^2\alpha'-2mz\beta'.
\end{aligned}$$

Here also the functions a'', b'', α', β' being known, with the substitution of their values into this last member, we will establish a formula, in which the maximum exponent of z arrives to $m+1$; and the comparison of the terms with those which they correspond in first part of the equation will give the values of F, G, H etc. and will render the formula t known, that in C multiplies n^{m-1}.

69. In the third term of the multiplier C, in which n^{m-2} starts, corresponds $r=m+2$, $S=u=Fz^{m+2}+Gz^{m+1}+Hz^m+Iz^{m-1}$ etc., so that $(z-(m-1))du+mu$, that

is

$$
\begin{aligned}
&((2m+1)F+F)\times z^{m+2}+\left(\frac{-(m+2)(3m-1)}{2}F+2mG+G\right)\times z^{m+1}\\
&+\frac{(m+2)(m+1)(4m-3)}{2.3}F+\left(\frac{-(m+1)(3m-2)}{2}G+(2m-1)H+H\right)\times z^{m}\\
&+\left(\frac{-(m+2)(m+1)m(5m-5)}{2.3.4}F+\frac{(m+1)m(4m-4)}{2.3}G\right.\\
&\left.-\frac{m(3m-3)}{2}H+(2m-2)I+I\right)\times z^{m-1}\text{ etc.}\\
&=-m(m-1)z(z-1)^2a''+2m(m-1)z(z-1)b''-m(m-1)zc''+2mz^2\beta'-2mz\gamma'.
\end{aligned}
$$

The substitution of the known values of the functions a'', b'', c'', β', γ', and the comparison of the analogous terms in the two members of the equation make known the intermediates F, G, H etc., and consequently the third term of the multiplier C.

70. Then the aforesaid series modified with the value that r receives in the fourth term of C, must make the equality between them, and $-m(m-1)z(z-1)^2b''+2m(m-1)z(z-1)c''-m(m-1)ze''+2mz^2\gamma'-2mz\varepsilon'$; and indeed the form of this as in the homogeneous antecedent of comparison will make that which maintains all the following. For that there remains to find the way to exhibit the entire quotient, and consequently the entire multiplier C that was sought. This method serves for all how many multipliers of the general recurrent, minus the first; and it must be noted, that when the second multiplier is sought, there becomes $a''=-1$, and the other symbols b'', c'' etc. are zero.

71. For the complete solution of the Problem, that we have proposed in §30, there yet remains to consider two events of white balls in urn A, which make a class apart, and have their recurrent of their contrary combinations of the general series, that is found to belong to the event of $n-z-1$ white. One of these is $n-1$, the other n; that is I demand how many permutations are necessary to me, because, the urn A reduced to the state of $n-1$ white which I have called original, it may succeed probably, that in some one of them I have the event of white in A equal to that of the original state; and how many you want, because it is probable, that in some of them I replace all the white balls in the first urn.

72. I have said, that these two cases fall outside of the general rule of the other events, and I prove it thus. If the general event of $n-z-1$ white embraced also these two cases, the hypothesis of $z=0$ could take place, and of $z=-1$, that gives $z+1=0$. With the first the formula $n-z-1$ would become exactly $n-1$, and with the second it would make $n-z-1=n$. But each of the general multipliers of the contrary series to the event $n-z-1$ having the two factors z, $z+1$, and both of the two hypotheses would reduce them all to zero; which thing would cancel the number of combinations contrary to the aforesaid events for any course of permutations. Therefore, either we are sure in the permuting in having always those events, because all the combinations are propitious to us, or they are removed to the law of the others. The absurdity is evident, that the two aforesaid events are had constantly, also by reflex alone, that being between them different, one necessarily excludes the other; and therefore there remains to be said, that they demand other regulating formulas, and furnish us argument for two

new Problems. Let be therefore

PROBLEM XIX.

73. *Departing from the original state of* $n-1$ *white in the first urn, the number of contrary combinations is sought to have at least once the same state* $n-1$ *in the course of any number of permutations.*

Let us form the columns of the contrary events and the respective combinations in the manner practiced in the first until the ninth Problem; and we will have for 1 permutation; $n \quad 1 \mid n-2 \quad (n-1)^2$: therefore the contrary combinations; $1+(n-1)^2$:

$$\text{with 2 perm.} \quad \begin{array}{ll|ll|l} n-2 & (n-1)^2 & n-2 & (n-1)^2 & \text{contr. combin.} \\ n-2 & 4(n-2) & n-3 & (n-2)^2 & (n-1)^2(n-2)(n+2) \end{array} :$$

$$\text{with 3 perm.} \quad \begin{array}{ll|ll|ll} n-2 & (n-1)^2 & n-2 & (n-1)^2 & n-2 & (n-1)^2 \\ n-2 & 4(n-2) & n-2 & 4(n-2) & n-3 & (n-2)^2 \\ n-2 & 4(n-2) & n-3 & (n-2)^2 & n-2 & 9 \end{array}$$

$$\text{etc.} \quad \begin{array}{ll|ll|c} n-2 & (n-1)^2 & n-2 & (n-1)^2 & \text{contr. comb.} \\ n-3 & (n-2)^2 & n-3 & (n-2)^2 & (n-1)^2(n-2)^2(n^2+4n+8) \\ n-3 & 6(n-3) & n-4 & (n-3)^2 & \text{etc.} \end{array}$$

With the patience to go ahead in the search of the terms of the recurrent one, one finds, that they go with this order; $(n-1)^2+1$; $(n-1)^2(n-2)(n+2)$; $(n-1)^2(n-2)^2(n^2+4n+8)$; $(n-1)^2(n-2)^2(n^4+4n^3+8n^2-100)$; $(n-1)^2(n-2)^2(n^6+4n^5+8n^4-100n^2-760n+1736)$; $(n-1)^2(n-2)^2(n^8+4n^7+8n^6-100n^4-760n^3-4940n^2+32176n-39520)$ etc., and the examination of this series will make known a recurrent one of 4.th degree, that has the multipliers
1.° $(n-2)(-56n^4+528n^3-1152n^2)$; 2.° $79n^4-460n^3+156n^2+2208n-2304$; 3.° $-18n^3-21n^2+516n-796$; 4.° $n^2+18n-58$. It is perceived however, that the first term of this series is not already $(n-1)^2+1$, as the number of the combinations bears, but only $(n-1)^2$. That unit more derived from the possibility of the event of n white in one permutation alone, whereby it cannot ever take place in any of the subsequent, because, when it happens, the other $n-1$ is taken back to it necessarily: and here the contraries are looking for the same event $n-1$. Wishing therefore to establish the general term of our series, that, exactly the known rule of the recurrent ones must have this form; $ap^v+bq^v+cr^v+ds^v$, to have the true number of contrary combinations under the hypothesis of $v=1$, that is for the case of a single permutation, we will add the unit to the result exhibited from the formula; or also, in order to have always present the necessity of this addition, we will thus write the general term of the recurrent one: $o^{v-1}+ap^v+bq^v+cr^v+ds^v$. By such manner under the supposition that the number v of the permutations is greater than the unit, with the first o^{v-1} term will add nothing to the value of the subsequent, and only under the hypothesis of $v=1$ we have $o^{v-1}=o^0=1$.

LAST PROBLEM.

74. *Departing from the original state of the two urns, the number of the contrary combinations is sought to replace at least once in A all n white in the course of any number of permutations,*

Erect the columns relevant to each permutation, there will be had
for 1 permutation: $n-1 \quad 2(n-1) \;\big|\; n-2 \quad (n-1)^2 \;\big|$; so that the contrary combination; $(n-1)(n+1)$;

$$\begin{array}{lll|ll|ll|ll}
\text{for 2 perm.} & n-1 & 2(n-1) & n-1 & 2(n-1) & n-2 & (n-1)^2 & n-2 & (n-1)^2 \\
 & n-1 & 2(n-1) & n-2 & (n-1)^2 & n-1 & 4 & n-2 & 4(n-2) \\
 & n-2 & (n-1)^2 & & & & & & \\
\text{etc.} & n-3 & (n-2)^2; & \multicolumn{6}{l}{\text{contr. comb. } (n-1)^2(n^2+2n+2).\text{ etc.}}
\end{array}$$

No other columns known, because in the successive permutations their number increases by much; and one already knows how they must be formed. The contrary series, that from them is demanded, is the following one;
$(n-1)(n+1)$; $(n-1)^2(n^2+2n+2)$; $(n-1)^2(n^4+2n^3+2n^2-8)$; $(n-1)^2(n^6+2n^5+2n^4-8n^2-40n+56)$; $(n-1)^2(n^8+2n^7+2n^6-8n^4-40n^3-188n^2+752n-608)$ etc., that also is recurrent of 4.$^{\text{th}}$ degree generated from the multipliers:
1.° $(n-1)(-6n^4+72n^3-144n^2)$; 2.° $31n^4-142n^3+78n^2+216n-144$; 3.° $-12n^3+148n-156$; 4.° $n^2+12n-28$.

75. We adorn this last Problem with some examples. Let the number of white be $n=1$. Under such hypothesis all the terms of the recurrent contrary are canceled; which gives a certainty to us of having 1 white at least once whatever be the number of the permutations that we demand. In fact, the original state of the urns being, that in A is only one black, and in B only one white, with the first permutation it is manifest, that in A the white is transfered, passing the black one into the second urn. This happening infallibly in the first permutation, then in the third, then in the fifth etc, it is clear, that whatever number of permutations requested, in one of them we have always certain the event of only white; and therefore every combination will be favorable to us, and we will not have any contrary.

76. Let us make transition to the other hypothesis of white $n=2$: and for the two series, contrary, and favorable will be born to us these terms;

$$\begin{array}{rrrrl} 3, & 10, & 32, & 104 & \text{etc.} \\ 1, & 6, & 32, & 152 & \text{etc.} \end{array}$$

that they are the same with those which we have found in S 38. for the probability of the event of $n-2$ white, that is of no white under the same supposition of white $n=2$. Therefore, where all the white are two, it is the same to seek, how many the combinations be, that lead to the event of all the black ones, and how many are those which favor the return of all the white into the first urn: which thing will not seem a point strange to who makes the reflection, that in the original state finding a white in each urn, must be equally difficult to raise white from the one, black from the other, and to conduct the event of §38., that raises black from the first, white from the second one, and to replace in the first all the white balls.

77. But how to reconcile under the two hypotheses the identity of the number of the combinations with the nature of the two series, being that of §38. only of the second degree, and arriving to the fourth that of the present Problem? This is done easily. Since to all it is known that, the four multipliers of a recurrent one of fourth degree are called p, q, r, s, the equation $x^4-sx^3-rx^2-qx-p=0$ comprehends the four roots that I name P, Q, R, S, which enter into the formation of the general terms $aP^v+bQ^v+cR^v+dS^v$, that correspond to the series. I shall now put, that under one given hypothesis the aforesaid equation of fourth degree is divisible into two of second, that is there is born $(x^2+ex+f)\times(x^2+gx+h)=0$; and P, Q are the roots

of the first trinomial, R, S those of the other. It is evident, that all four roots of square-quadratic, that originally were, become quadratic; and that they have no dependency on the values of the coefficients a, b, c, d, which come determined from the comparisons of the modified general term to the four hypotheses of $v = 1$, $v = 2$, $v = 3$, $v = 4$, with the first four terms of the series. If therefore it will happen, that such comparisons make to vanish the first two coefficients a, b, we can consider the general formula reduced to the two terms $cR^v + dS^v$, that of itself only leads to a recurrent of second degree; and we can still take account of the two terms $aP^v + bQ^v$, which are destroyed not because they are zero, but because they cancel a and b. Therefore it appears, how the use of the two multipliers must be indifferent for the formation of the series, that are the determined states in §38, or to make use of the four, that it gives to us our Problem, and because the two series are identified between them perfectly.

78. Put finally $n = 3$, the two series[1] contrary and favorable proceed in this manner:

contr.	8,	68,	580,	4964,	42484,	363668,	3112996,	26647556,	228105364,
fav.	1,	13,	149,	1597,	16565,	167773,	1699973,	16399165,	159315125,

contr.	1952603060,	16714460740,	143077320356,	1224754999348	etc.
fav.	1544181341,	14666598869,	139352216125,	1317110828981	etc.,

in which it is observed, that as far as the twelfth term inclusively the contrary combinations exceed the favorable ones, commencing only at the thirteenth to be those in greater number than the first. From that I infer, that, two white and one black being in A, one white and two black in B, in order to play soon even on the probability of the event of all three white in the first urn, I must request either 12, or 13 permutations.

[1]The first entries in the third and fourth rows apparently have printing errors and should be 1952603087 and 15334181341 respectively.