MÉMOIRE SUR L'APPLICATION

DU

CALCUL DES PROBABILITÉS AUX OBSERVATIONS

ET SPÉCIALEMENT

AUX OPÉRATIONS DU NIVELLEMENT

Pierre Simon Laplace*

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In the great triangulations that one has executed for the measure of the Earth, one has observed with care the zenithal distances of the signals, either in order to reduce the angles to the horizon, or in order to determine the respective heights of the diverse stations. Terrestrial refraction has a great influence on these heights, and its variability renders them quite uncertain. I myself propose here to estimate the probability of the errors of which they are susceptible.

The theory of refraction shows us that, in a constant atmosphere, the terrestrial refraction is a fraction of the celestial arc contained between the zeniths of the observer and of the observed signal; so that, in order to obtain it, it suffices to multiply this arc by a factor which would be constant if the atmosphere were always the same, but which varies without ceasing, by reason of the continual changes of the temperature and of the density of the air. A great number of observations are able to give the mean value of this factor and the law of probability of its variations. I have concluded both from the observations of Mr. Delambre, published in the second Volume of his Work entitled: *Base du Système métrique*. By departing from these data, I have determined the probability of the errors of the height of Paris above the sea, under the hypothesis of a chain of twenty-five equilateral triangles which would join Dunkirk and Paris, this which supposes around 20000^{m} of length of each of their sides. One is able to obtain this height by diverse processes; but the one in which the law of probability of the errors is the most rapidly decreasing must be preferred as being the most advantageous. Its research is an easy corollary of the analysis that I have given besides for all these objects, and there results from it that there are odds nine against one that the error with

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respect to the height of Paris above the sea would not exceed then 8^{m}. The process that Mr. Delambre has followed, in order to conclude this height from a nearly equal number of triangles, is a little less exact than the previous; but it is principally the magnitude of the sides of several of his triangles which distribute with respect to his result the uncertainty and which do not permit to respond, with a sufficient probability, that it is not in error of 16^{m} or 18^{m}; this which forms a considerable part of it.

The equally probable errors diminish much when one brings together the stations, and it is indispensable to make it when one wishes to obtain an exact leveling. The great triangles, very proper in the measure of terrestrial degrees, are not convenient at all to the measure of heights, and it is necessary to separate these two kinds of measures. But, by multiplying the stations, the error which holds to the observation of the zenithal angles increases with their number and becomes comparable to the error which depends on the variability of terrestrial refraction. This has caused me to research the law of probability of the errors of the results when there are many sources of errors. Such are the greater part of astronomical results; because one observes the stars by means of two instruments, the meridian lunette and the circle, both susceptible of errors of which the law of probability must not be supposed the same. The analysis that I have given in the analytic theory of probabilities is applied easily in this case, whatever be the number of sources of error. It determines the most advantageous results and the laws of probability of the errors of which they are susceptible. In order to apply to the operations of leveling, it is necessary to know the law of probability of the errors due to astronomical refraction; and one has just seen that it results from the great triangulations of the meridian. It is necessary moreover to know the law of probability of the errors of the zenithal angles. We lack observations in this regard; but one will deviate little from the truth by supposing this law the same as for horizontal angles, and which is deduced from the errors observed in the sum of the three angles of each triangle of the meridian. By departing from these laws, I find that, if one divides the distance from Paris to Dunkirk into equidistant stations of an interval of 1200^{m}, there are odds one thousand against one that the error in the height of Paris above the sea will not exceed four-tenths meter. One would diminish this error by bringing together the stations; but the precision that one would obtain by this bringing together will not compensate the length of the operations that it requires.

The equations of condition that one forms in order to have the astronomical elements contain implicitly the errors of the two instruments which serve to determine the position of the stars. These errors are affected of different coefficients in each equation. Then the most advantageous system of factors by which one must multiply respectively these equations in order to obtain, by the reunion of the products, as many final equations as there are elements to determine; this system, I say, is no longer the one of the coefficients of the elements in each equation of condition. The analysis has led me to the general expression of this system of factors and thence to the results for which the same error to fear is less probable than in each other system. The same analysis gives the laws of probability of the errors of these results. These formulas contain as many constants as there are sources of errors and that depend on the laws of probability of these errors. In the case of a unique source, I have given, in my theory of probabilities, the means to eliminate the constant, by forming the sum of the squares of the remainders of each equation of condition, when one has substituted the values found for the elements. A similar process gives generally the values of these constants, whatever be their number: this which completes the application of the calculus of probabilities to the results of the observations.

I will finish with a remark that appears to me important. The small uncertainty that the observations, when they are not very multiplied, leave with respect to the values of the constants of which I have just spoken, renders a little uncertain those probabilities determined by the analysis; but it suffices nearly always to know if the probability that the errors of the results obtained are contained within narrow limits brought together extremely from unity, and, when this is not, it suffices to know to what point it is necessary to multiply the observations in order to acquire a probability such that there remains no reasonable doubt on the good quality of the results. The analytical formulas of probabilities fulfill perfectly this object, and, under this point of view, they are able to be envisioned as the necessary complement of the method of the sciences, founded on the collection of a great number of observations susceptible of errors. Thus, when one would reduce to 15^{m} the error from 18^{m} that one is able to fear in the height of Paris above the sea, concluded from the great triangles of the meridian, it would not be less true of it that this height is uncertain and that it is necessary to determine it by some more precise means. Similarly, the analytic formulas, applied to the triangles of the meridian from the base measured near to Perpignan to Formentera, give odds around seventeen hundred thousand against one that the error of the corresponding arc of the meridian, of which the length surpasses 460000^{m}, is not 60^{m} in error. This must dissipate the fears of inexactitude that the omission of a base of verification on the side of Spain was able to inspire. One would be further reassured in this regard when likewise the probability of an equal error, or greater than 60^{m}, would surpass the fraction given by the formulas and would be raised to a millionth.