

# MÉMOIRE SUR LES SUITES

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*Mémoires de l'Académie royale des Sciences de Paris*, year 1779; 1782.  
*Oeuvres complètes X*, pp. 1–89.

## I.

The theory of series is one of the most important objects of Analysis: all problems which reduce to some approximations, and consequently nearly all the applications of Mathematics to Nature, depend on this theory; thus we see that it has principally fixed the attention of the geometers; they have found a great number of beautiful theorems and ingenious methods, either in order to expand function into series, or in order to sum series exactly or for approximation; but they have attained them only by some indirect and particular ways, and we can not doubt that, in this branch of Analysis, as in all others, there is a general and simple manner to view it, from which the already known truths derive, and which lead to many new truths. The research of a similar method is the object of this Memoir; that to which I am come is founded on the consideration of that which I name *generating functions*: this is a new kind of calculus which we can name *calculus of generating functions*, and which has appeared to me to merit being cultivated by the geometers. I exhibit first some very simple results on these functions and I deduce from them a method to interpolate series, not only when the consecutive differences of the terms are convergent, that which is the sole case which we have considered until now, but yet when the proposed series converges towards a recurrent series, the final ratio of its terms being given by a linear equation in finite differences of which the coefficients are constants. Integration of this kind of equation is a corollary of this analysis. In passing next from the finite to the infinitely small, I give a general formula to interpolate the series of which the final ratio of the terms is represented by a linear equation in infinitely small differences, of which the coefficients are constants; whence I conclude the integration of these equations. By applying the same method to the transformation of series, there results from it a quite simple way to transform them into some others of which the terms follow a given law; finally the relationship of the generating functions to the corresponding variables leads me immediately to the singular analogy of the positive powers with the differences and of the negative powers with the integrals, an analogy observed first by Leibnitz, and since brought to greater light by Mr de la Grange (*Mémoires de Berlin*, 1772); all the theorems of which the

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second of these two great geometers is attained in the Memoirs cited after this analogy, and many others again, are deduced with the greatest ease in this report.

By considering in the same manner series in two variables, I exhibit a general method to interpolate them, not only in the case where the consecutive differences of the terms of the series are convergent, but again when the series converges towards a *récurro-récurrente* series, the final ratio of its terms being given by a linear equation in partial finite differences of which the coefficients are constants; whence result the integration of this kind of equations. This material is of the greatest importance in the analysis of chances; I believe to be the first who has considered it [*see* Books VI and VII of the *Savants étrangers*]. Mr. de la Grange has since treated it by a very good and very learned analysis in the *Mémoires de Berlin* for the year 1775; I dare to hope that the new manner in which I envision it in this Memoir will not offend the geometers. It follows from my researches that the integration of any linear equation in partial finite differences, of which the coefficients are constants, can be restored to that of a linear equation in infinitely small differences, by means of *definite integrals* taken with respect to a new variable; I name *definite integral* an integral taken from one determined value of the variable to another determined value. This remark, more curious than useful in the theory of finite differences, becomes very useful when we transport it to the equations linear in the infinitely small partial differences: it gives a means of integrating them in an infinity of cases which withstand all the known methods, and, without it, it had been nearly impossible to foresee the forms of which the integrals are then susceptible. But, in order to render that which I just said more sensible, it will not be useless to recall in a few words that which we have discovered on linear equations in infinitely small partial differences of the second order. The integral of these equations contain, as we know, two arbitrary functions; we have, moreover, remarked that these functions can be, in the integral, affected with the differential sign  $d$ ; and it is, if I do not deceive myself, to Messrs. Euler and de la Grange that we owe this important remark to which they have been led by the theory of sound, in the case where the air is considered with its three dimensions.

These two great geometers have next extended their methods to some equations more complicated than those of this problem; but there remains to find a method by means of which we could generally, either integrate any linear equation of the second order, or be assured that its integral is impossible in finite terms, by having regard only to the sole variables that they contain: this is the object of a Memoir<sup>1</sup> that I have inserted in the Volume of the Academy for the year 1773. In this Memoir, I have demonstrated: 1° that the arbitrary functions can exist in the integral only under a linear form; 2° that if the integral is possible in finite terms, by considering only the sole variables of the equation, one of the two arbitrary functions is necessarily delivered with the integral sign  $\int$ . I have given next a general method to have in this case the complete integral of the differential equation, by supposing even that this equation contains a term independent of the principal variable, and which is any function whatever of two other variables; whence it follows that, when a proposed equation withstands this method, we can be assured that its complete integral is impossible in finite terms, by having regard only to the sole variables of the equation. Now, the remark of which I have

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<sup>1</sup>“Recherches sur le Calcul intégral aux différences partielles,” *Oeuvres de Laplace*, T. IX, p. 5.

spoken above has made me see that, in this case, the integral is possible in finite terms, by means of *definite* integrals taken with respect to a new variable which it is necessary necessarily then to introduce into the calculation. We will see after this that these forms of integrals are of the same use in the solution of the problems as the known forms; I give in order to obtain them a method which extends to a great number of cases, and especially to many important physical questions, such as the movement of the vibrating strings in a medium resistant as the speed, the propagation of the sound in a plane, etc., of which we have been able to find yet only some particular solutions.

By transporting to the infinitely small differences the remarks that I make on a particular equation in partial finite differences, I succeed in assuring myself by an uncontested manner that, in the problem of the vibrating strings, we can admit some discontinuous functions, provided that none of the angles formed by two contiguous sides of the initial figure of the string is finite; whence it appears to me that these functions can be generally employed in all the problems which are related to partial differences, provided that they can subsist with the differential equations and with the conditions of the problem; thus, the only condition which is necessary in the determination of the arbitrary functions of a proposed equation in partial differences of order  $n$  is that it have no jump point between two consecutive values of a difference of these functions, smaller than the  $n^{\text{th}}$  difference, and, consequently, that, in the curves by means of which we represent these arbitrary functions, there is no jump point between two consecutive tangents, if, as in the problem of the vibrating strings, the differential equation is of the second order, or that it have no jump point between to consecutive osculating radii, if the equation is of the third order, etc., that which is conformed to that which as Mr. le marquis de Condorcet has found, by another method, in the *Mémoires de l'Académie* for the year 1771, pages 70 and 71. But it is essential to observe that, if the integral contains the differences of the arbitrary functions, we must consider the most elevated differences as the true arbitrary functions of the integral, and to apply the preceding rule only to these differences. This manner of illuminating the delicate points of the theory of infinitely small differences by that of the finite differences is, if I do not deceive myself, the most proper to realize this object, and it seems to me that, after the theory that I exhibit, there must remain no doubt on the use of discontinuous functions in the integral Calculus with the partial differences. Finally, I end this Memoir with the consideration of equations linear in the partial differences, in finite parts and in infinitely small parts, and by some theorems on the reduction into series of the functions in two variables. All these researches being only the expansion of a very simple consideration on the nature of generating functions, I dare flatter myself that the analysis of which I have made use could merit, by its generality, the attention of the Geometers.

## II.

### *On the series in one variable.*

Let  $y_x$  be any function whatever of  $x$ ; if we form the infinite series

$$y_0 + y_1t + y_2t^2 + y_3t^3 + \cdots + y_x t^x + y_{x+1}t^{x+1} + \cdots + y_\infty t^\infty,$$

and if we name  $u$  the sum of this series, or, what returns to the same, the function of

which the expansion forms this series, this function will be that which I name *generating function* of the variable  $y_x$ .

A generating function of any variable  $y_x$  is thus generally a function of  $t$ , which, expanded according to the powers of  $t$ , has this variable  $y_x$  for the coefficient of  $t^x$ ; and, reciprocally, the corresponding variable of a generating function is the coefficient of  $t^x$  in the expansion of this function according to the powers of  $t$ . It follows from these definitions that,  $u$  being the generating function of  $y_x$ , that of  $y_{x-r}$  will be  $ut^r$ ; because it is clear that the coefficient of  $t^x$  in  $ut^r$  is equal to the one of  $t^{x-r}$  in  $u$ , and consequently equal to  $y_{x-r}$ .

The coefficient of  $t^x$  in  $u\left(\frac{1}{t} - 1\right)$  is evidently equal to  $y_{x+1} - y_x$ , or to  $\Delta y_x$ ,  $\Delta$  being the characteristic of finite differences; we will have therefore the generating function of the finite difference of one quantity by multiplying by  $\frac{1}{t} - 1$  the generating function of the quantity itself; the generating function of  $\Delta^2 y_x$  is thus  $u\left(\frac{1}{t} - 1\right)^2$ , and, generally, that of  $\Delta^i y_x$  is  $u\left(\frac{1}{t} - 1\right)^i$ ; whence we can conclude that the generating function of  $\Delta^i y_{x-r}$  is  $ut^r\left(\frac{1}{t} - 1\right)^i$ .

Similarly, the coefficient of  $y^x$  in

$$u\left(a + \frac{b}{t} + \frac{c}{t^2} + \frac{e}{t^3} + \cdots + \frac{q}{t^n}\right)$$

is

$$ay_x + by_{x+1} + cy_{x+2} + ey_{x+3} + \cdots + qy_{x+n};$$

by naming therefore  $\nabla y_x$  this quantity, its generating function will be

$$u\left(a + \frac{b}{t} + \frac{c}{t^2} + \frac{e}{t^3} + \cdots + \frac{q}{t^n}\right)$$

If we name  $\nabla^2 y_x$  the quantity

$$a\nabla y_x + b\nabla y_{x+1} + c\nabla y_{x+2} + e\nabla y_{x+3} + \cdots + q\nabla y_{x+n};$$

$\nabla^3 y_x$  the quantity

$$a\nabla^2 y_x + b\nabla^2 y_{x+1} + c\nabla^2 y_{x+2} + \cdots + q\nabla^2 y_{x+n};$$

and thus in sequence, their corresponding generating functions will be

$$\begin{aligned} & u\left(a + \frac{b}{t} + \frac{c}{t^2} + \cdots + \frac{q}{t^n}\right)^2, \\ & u\left(a + \frac{b}{t} + \frac{c}{t^2} + \cdots + \frac{q}{t^n}\right)^3, \\ & \dots \end{aligned}$$

and, generally, the generating function of  $\Delta^i y_x$  will be

$$u\left(a + \frac{b}{t} + \frac{c}{t^2} + \frac{e}{t^3} + \cdots + \frac{q}{t^n}\right)^i;$$

hence the generating function of  $\Delta^i \nabla^s y_{x-r}$  will be

$$ut^r \left( a + \frac{b}{t} + \frac{c}{t^2} + \cdots + \frac{q}{t^n} \right)^s \left( \frac{1}{t} - 1 \right)^i.$$

We can generalize again the preceding theorems, by supposing that  $\nabla y_x$  represents any linear function of  $y_x, y_{x+1}, y_{x+2}, \dots$ ; that  $\nabla^2 y_x$  represents a new function in which  $\nabla y_x$  enters in the same manner as  $y_x$  in  $\nabla y_x$ ; that  $\nabla^3 y_x$  represents a function of  $\nabla^2 y_x$  similar to that of  $\nabla y_x$  in  $y_x$ , and thus in sequence; because,  $u$  being the generating function of  $y_x$ , if we name  $us$  that of  $\nabla y_x$ ,  $us^2$ ,  $us^3$ ,  $\dots$  will be the generating functions of  $\nabla^2 y_x$ ,  $\nabla^3 y_x$ ,  $\dots$ . By multiplying therefore the function  $u$  by the successive powers of  $s$ , we will have the generating functions of the products of  $y_x$  by the corresponding powers of  $\nabla$ ,  $\nabla$  being at no point a quantity, but a characteristic; and this will be again true by supposing these powers fractional and even incommensurates.

$s$  being any function whatever of  $\frac{1}{t}$ , if we expand  $s^i$  according to the powers of  $\frac{1}{t}$ , and if we designate by  $\frac{K}{t^m}$  any term of this expansion, the coefficient of  $t^x$  in  $\frac{Ku}{t^m}$  will be  $Ky_{x+m}$ ; we will have therefore the coefficient of  $t^x$  in  $us^i$ , or, what comes to the same, we will have  $\nabla^i y_x$ :  $1^\circ$  by substituting, into  $s$ ,  $y_x$  in place of  $\frac{1}{t}$ ;  $2^\circ$  by expanding that which  $s^i$  then becomes, according to the powers of  $y_x$ , and by adding to  $x$ , in each term, the exponent of the power of  $y_x$ , that is by writing  $y_x$  in place of  $(y_x)^0$ ,  $y_{x+1}$  in place of  $(y_x)^1$ ,  $y_{x+2}$  in place of  $(y_x)^2$ , and thus in sequence.

If, instead of expanding  $s^i$  according to the powers of  $\frac{1}{t}$ , we expand it according to the powers of  $\frac{1}{t} - 1$ , and if we designate by  $K \left( \frac{1}{t} - 1 \right)^m$  any term of this expansion, the coefficient of  $t^x$  in  $Ku \left( \frac{1}{t} - 1 \right)^m$  will be  $K \Delta^m y_x$ . We will have therefore  $\nabla^i y_x$ :  $1^\circ$  by substituting into  $s$ ,  $\Delta y_x$  in place of  $\frac{1}{t} - 1$ , or, what comes to the same,  $1 + \Delta y_x$  in place of  $\frac{1}{t}$ ;  $2^\circ$  by expanding that which  $s^i$  then becomes according to the powers of  $\Delta y_x$ , and by applying to the characteristic  $\Delta$  the exponents of the powers of  $\Delta y_x$ , that is by writing  $\Delta^0 y_x$  or  $y_x$  in place of  $(\Delta y_x)^0$ ,  $\Delta^2 y_x$  in place of  $(\Delta y_x)^2$ , and thus in sequence.

In general, if we consider  $s$  as a function of  $r$ ,  $r$  being a function of  $\frac{1}{t}$ , such that the coefficient of  $t^x$  in  $ur$  is  $\square y_x$ , we will have  $\nabla^i y_x$  by substituting, into  $s$ ,  $\square y_x$  in place of  $r$ ; by developing next that which  $s^i$  then becomes according to the powers of  $\square y_x$ , and by applying to the characteristic  $\square$  the exponents of the powers of  $\square y_x$ , that is, by writing  $\square^0 y_x$ , or  $y_x$  in place of  $(\square y_x)^0$ ,  $\square^2 y_x$  in place of  $(\square y_x)^2$ , and thus of the rest. We will have therefore the values of  $\nabla y_x$ ,  $\nabla^2 y_x$ ,  $\dots$  by some simple expansions of algebraic functions.

Let  $z$  be the generating function of  $\Sigma^i y_x$ ,  $\Sigma$  being the characteristic of finite integrals; we will have, by that which precedes,  $z \left( \frac{1}{t} - 1 \right)^i$  for the generating function of  $y_x$ ; but this function must, by having regard only for the positive or null powers of  $t$ , be reduced to  $u$ . We will have therefore

$$z \left( \frac{1}{t} - 1 \right)^i = u + \frac{A}{t} + \frac{B}{t^2} + \frac{C}{t^3} + \cdots + \frac{F}{t^i},$$

whence we deduce

$$z = \frac{ut^i + At^{i-1} + Bt^{i-2} + Ct^{i-3} + \cdots + F}{(1-t)^i};$$

$A, B, C, \dots, F$  being the  $i$  arbitrary constants which the  $i$  successive integrations of  $y_x$  introduce. By setting aside these constants, the generating function of  $\Sigma^i y_x$  will be  $u \left(\frac{1}{t} - 1\right)^{-i}$ ; we will have therefore the generating function of  $\Sigma^i y_x$  by changing  $i$  into  $-i$  in the generating function of  $\Delta^i y_x$ ; and reciprocally, we will have the variable corresponding to the function  $u \left(\frac{1}{t} - 1\right)^i$ , in which we suppose  $i$  negative, by changing  $i$  into  $-i$  in  $\Delta^i y_x$  and by supposing that the negative differences represent some integrals; but, if we have regard to the arbitrary constants, it is necessary, in passing from the positive powers to the negative powers of  $\frac{1}{t} - 1$ , to increase  $u$  by a number of terms  $\frac{A}{t} + \frac{B}{t^2} + \frac{C}{t^3} + \dots$  equal to the exponent of the negative power of  $\frac{1}{t} - 1$ . We see thence how the generating functions are formed from the law of the corresponding variables, and reciprocally, in what manner these variables are deduced from their generating functions. We apply now these results to the theory of series.

### III.

*On the interpolation of the series in one variable, and on the integration of linear differential equations.*

All the theory of the interpolation of series consists in determining, whatever be  $i$ , the value of  $y_{x+i}$  by a function of  $y_x$  and from the terms which precede or which follow  $y_x$ . For this, we must observe that  $y_{x+i}$  is equal to the coefficient of  $t^{x+i}$  in the expansion of  $u$ , and, consequently, equal to the coefficient of  $t^x$  in the expansion of  $\frac{u}{t^i}$ ; now we have

$$\begin{aligned} \frac{u}{t^i} &= u \left(1 + \frac{1}{t} - 1\right)^i \\ &= u \left[1 + i \left(\frac{1}{t} - 1\right) + \frac{i(i-1)}{1.2} \left(\frac{1}{t} - 1\right)^2 + \frac{i(i-1)(i-2)}{1.2.3} \left(\frac{1}{t} - 1\right)^3 + \dots\right]. \end{aligned}$$

Moreover, the coefficient of  $t^x$  in the expansion of  $u$  is  $y_x$ ; this coefficient in the expansion of  $u \left(\frac{1}{t} - 1\right)$  is  $\Delta y_x$ ; in the expansion of  $u \left(\frac{1}{t} - 1\right)^2$ , it is equal to  $\Delta^2 y_x$ , and thus in sequence; we will have therefore, by passing again from the generating functions to the corresponding variables,

$$y_{x+i} = y_x + i \Delta y_x + \frac{i(i-1)}{1.2} \Delta^2 y_x + \frac{i(i-1)(i-2)}{1.2.3} \Delta^3 y_x + \dots$$

This equation, holding whatever be  $i$ , will serve to interpolate the series of which the differences of the terms go by decreasing.

All the ways of expanding the power  $\frac{1}{t^i}$  will give as many different methods to interpolate the series; let, for example,

$$\frac{1}{t} = 1 + \frac{\alpha}{t^r};$$

by expanding  $\frac{1}{t^i}$ , according to the powers of  $\alpha$ , in a manner of the beautiful theorem of Mr. de la Grange (*see the Mémoires de l'Académie*, year 1777, page 115), we will find

easily

$$\frac{u}{t^i} = u \left[ 1 + ia + \frac{i(i+2r-1)}{1.2} \alpha^2 + \frac{i(i+3r-1)(i+3r-2)}{1.2.3} \alpha^3 + \frac{i(i+4r-1)(i+4r-2)(i+4r-3)}{1.2.3.4} \alpha^4 + \dots \right].$$

Now,  $\alpha$  being equal to  $t^r \left( \frac{1}{t} - 1 \right)$ , the coefficient of  $t^x$  in the expansion of  $u\alpha$  is, by the preceding article,  $\Delta y_{x-r}$ ; this same coefficient in the expansion of  $u\alpha^2$  is  $\Delta^2 y_{x-2r}$ , and thus in sequence. We will have therefore

$$y_{x+i} = y_x + i\Delta y_{x-r} + \frac{i(i+2r-1)}{1.2} \Delta^2 y_{x-2r} + \frac{i(i+3r-1)(i+3r-2)}{1.2.3} \Delta^3 y_{x-3r} + \frac{i(i+4r-1)(i+4r-2)(i+4r-3)}{1.2.3.4} \Delta^4 y_{x-4r} + \dots$$

IV.

Here is presently a general method of interpolation which has the advantage of being applicable, not only to the series of which the differences of the terms conclude by being null, but further to the series of which the last ratio of the terms is that of any recurrent series.

We suppose first that we have

$$t \left( \frac{1}{t} - 1 \right)^2 = z,$$

and we seek the value of  $\frac{1}{t^i}$  in  $z$ .

It is clear that  $\frac{1}{t^i}$  is equal to the coefficient of  $\theta^i$  in the expansion of the fraction  $\frac{1}{1-\frac{\theta}{t}}$ ; if we multiply the numerator and the denominator of this fraction by  $1-\theta t$ , we will have this here  $\frac{1-\theta t}{1-\theta(\frac{1}{t}+t)+\theta^2}$ . The equation

$$t \left( \frac{1}{t} - 1 \right)^2 = z \quad \text{gives} \quad \frac{1}{t} + t = 2 + z,$$

that which changes the preceding fraction into the following  $\frac{1-\theta t}{(1-\theta)^2 - z\theta}$ ; now we have

$$\frac{1}{(1-\theta)^2 - z\theta} = \frac{1}{(1-\theta)^2} + \frac{z\theta}{(1-\theta)^4} + \frac{z^2\theta^2}{(1-\theta)^6} + \frac{z^3\theta^3}{(1-\theta)^8} + \dots$$

Moreover, the coefficient of  $\theta^r$  in the expansion of  $\frac{1}{(1-\theta)^s}$  is equal to  $\frac{1}{1.2.3\dots r} \frac{d^r(1-\theta)^{-s}}{d\theta^r}$ , provided that we suppose  $\theta = 0$  after the differentiations, that which gives for this coefficient  $\frac{s(s+1)(s+2)\dots(s+r-1)}{1.2.3\dots r}$ ; whence it follows that the coefficient of  $\theta^i$  is:  $1^\circ$   $i+1$  in the expansion of  $\frac{1}{(1-\theta)^2}$ ;  $2^\circ$   $\frac{i(i+1)(i+2)}{1.2.3}$  in the expansion of  $\frac{\theta}{(1-\theta)^4}$ ;  $3^\circ$

$\frac{(i-1)i(i+1)(i+2)(i+3)}{1.2.3.4.5}$  in the expansion of  $\frac{\theta^2}{(1-\theta)^6}$ , and thus the rest. Therefore, if we name  $Z$  the coefficient of  $\theta^i$  in the expansion of the fraction  $\frac{1}{(1-\theta)^2-z\theta}$ , we will have

$$Z = i + 1 + \frac{i(i+1)(i+2)}{1.2.3}z + \frac{(i-1)i(i+1)(i+2)(i+3)}{1.2.3.4.5}z^2 + \frac{(i-2)(i-1)i(i+1)(i+2)(i+3)(i+4)}{1.2.3.4.5.6.7}z^3 + \dots$$

or

$$Z = i + 1 + \frac{(i+1)[(i+1)^2-1]}{1.2.3}z + \frac{(i+1)[(i+1)^2-1][(i+1)^2-4]}{1.2.3.4.5}z^2 + \frac{(i+1)[(i+1)^2-1][(i+1)^2-4][(i+1)^2-9]}{1.2.3.4.5.6.7}z^3 + \dots$$

If we name next  $Z'$  the coefficient of  $\theta^i$  in the expansion of  $\frac{\theta}{(1-\theta)^2-z\theta}$ , we will have  $Z'$ , by changing, in  $Z$ ,  $i$  into  $i-1$ , this which gives

$$Z' = i + \frac{i(i^2-1)}{1.2.3}z + \frac{i(i^2-1)(i^2-4)}{1.2.3.4.5}z^2 + \frac{i(i^2-1)(i^2-4)(i^2-9)}{1.2.3.4.5.6.7}z^3 + \dots$$

We will have thus  $Z - tZ'$  for the coefficient of  $\theta^i$  in the expansion of the fraction  $\frac{1-\theta t}{(1-\theta)^2-z\theta}$ ; this will be, consequently, the expression of  $\frac{1}{t^i}$ ; therefore

$$\frac{u}{t^i} = u(Z - tZ').$$

This put, the coefficient of  $t^x$  in  $\frac{u}{t^i}$  is  $y_{x+i}$ ; this same coefficient, in any term of  $uZ$ , such as  $Kuz^r$  or, that which comes to the same,  $Kut^r \left(\frac{1}{t} - 1\right)^{2r}$  is, by article II, equal to  $K\Delta^{2r}y_{x-r}$ ; in any term of  $utZ'$ , such as  $Kutz^r$ , this coefficient is  $K\Delta^{2r}y_{x-r-1}$ . We will have therefore, by passing again from the generating functions in the corresponding variables,

$$\begin{aligned} y_{x+i} = & (i+1)y_x + \frac{(i+1)[(i+1)^2-1]}{1.2.3}\Delta^2y_{x-1} \\ & + \frac{(i+1)[(i+1)^2-1][(i+1)^2-4]}{1.2.3.4.5}\Delta^4y_{x-2} + \dots \\ & - iy_{x-1} - \frac{i(i^2-1)}{1.2.3}\Delta^2y_{x-2} \\ & - \frac{i(i^2-1)(i^2-4)}{1.2.3.4.5}\Delta^4y_{x-3} - \dots \end{aligned}$$

We can vary again the preceding form of  $y_{x+i}$ ; for that, let  $Z''$  be that which  $Z'$  becomes when we change  $i$  into  $i-1$  and, consequently, that which  $Z$  becomes when we change  $i$  into  $i-2$ ; the equation  $\frac{1}{t^i} = Z - tZ'$  will give  $\frac{1}{t^{i-1}} = Z' - tZ''$ , hence  $\frac{1}{t^i} = \frac{Z'}{t} - Z''$ . By adding these two values of  $\frac{1}{t^i}$  and taking the half of their sum we will have

$$\frac{1}{t^i} = \frac{1}{2}Z - \frac{1}{2}Z'' + \frac{1}{2}(1+t)\left(\frac{1}{t} - 1\right)Z';$$



now we have

$$\begin{aligned} \frac{1}{2}Z - \frac{1}{2}Z'' &= \frac{1}{2} \left[ i + 1 + \frac{i(i+1)(i+2)}{1.2.3} z + \dots \right] \\ &\quad - \frac{1}{2} \left[ i - 1 + \frac{(i-2)(i-1)i}{1.2.3} z + \dots \right] \\ &= 1 + \frac{i^2}{1.2} z + \frac{i^2(i^2-1)}{1.2.3.4} z^2 + \frac{i^2(i^2-1)(i^2-4)}{1.2.3.4.5.6} z^3 + \dots \end{aligned}$$

hence

$$\begin{aligned} \frac{u}{t^i} &= u \left[ 1 + \frac{i^2}{1.2} t \left( \frac{1}{t} - 1 \right)^2 + \frac{i^2(i^2-1)}{1.2.3.4} t^2 \left( \frac{1}{t} - 1 \right)^2 \right. \\ &\quad \left. + \frac{i^2(i^2-1)(i^2-4)}{1.2.3.4.5.6} t^3 \left( \frac{1}{t} - 1 \right)^6 + \dots \right] \\ &\quad + \frac{i}{2} u(1+t) \left[ \frac{1}{t} - 1 + \frac{i^2-1}{1.2.3} t \left( \frac{1}{t} - 1 \right)^3 \right. \\ &\quad \left. + \frac{(i^2-1)(i^2-4)}{1.2.3.4.5} t^2 \left( \frac{1}{t} - 1 \right)^5 + \dots \right], \end{aligned}$$

whence we conclude, by article II, by passing again from the generating functions to the corresponding variables,

$$\begin{aligned} y_{x+1} &= y_x + \frac{i^2}{1.2} \Delta^2 y_{x-1} + \frac{i^2(i^2-1)}{1.2.3.4} \Delta^4 y_{x-2} \\ &\quad + \frac{i^2(i^2-1)(i^2-4)}{1.2.3.4.5.6} \Delta^6 y_{x-3} + \dots \\ &\quad + \frac{i}{2} \Delta(y_x + y_{x-1}) + \frac{i}{2} \frac{i^2-1}{1.2.3} \Delta^3(y_{x-1} + y_{x-2}) \\ &\quad + \frac{i}{2} \frac{(i^2-1)(i^2-4)}{1.2.3.4.5} \Delta^5(y_{x-2} + y_{x-3}) + \dots \end{aligned}$$

This formula returns to that which Newton has given in the small work entitled *Methodus differentialis*, in order to interpolate between an odd number of equidistant quantities; in this case,  $y_x$  designates the quantity of the mean and  $i$  is the distance of this quantity to that which we seek, which, consequently, is  $y_{x+i}$ , unity being supposed the common interval of the given quantities.

By differentiating in finite differences the preceding formula with respect to  $i$ , we

will have

$$\begin{aligned}
y_{x+i+1} - y_{x+i} &= \frac{1}{2} \Delta(y_x + y_{x-1}) + \frac{i(i+1)}{1.2} \Delta^3(y_{x-1} + y_{x-2}) \\
&+ \frac{(i-1)i(i+1)(i+2)}{1.2.3.4} \frac{1}{2} \Delta^5(y_{x-2} + y_{x-3}) + \dots \\
&+ (2i+1) \frac{1}{2} \Delta^2 y_{x-1} + \frac{(2i+1)(i+1)i}{1.2.3} \frac{1}{2} \Delta^4 y_{x-2} \\
&+ \frac{(2i+1)(i+2)(i+1)i(i-1)}{1.2.3.4.5} \frac{1}{2} \Delta^6 y_{x-3} + \dots
\end{aligned}$$

Let  $y_{x+1} - y_x = y'_x$  and  $i = \frac{s-1}{2}$ , we will have

$$\begin{aligned}
y_{x+\frac{s-1}{2}} &= \frac{1}{2}(y'_x + y'_{x-1}) + \frac{s^2-1}{2.4} \frac{1}{2} \Delta^2(y'_{x-1} + y'_{x-2}) \\
&+ \frac{(s^2-1)(s^2-9)}{2.4.6.8} \frac{1}{2} \Delta^4(y'_{x-2} + y'_{x-3}) + \dots \\
&+ \frac{s}{2} \Delta y'_{x-1} + \frac{s(s^2-1)}{2.4.6} \Delta^3 y'_{x-2} \\
&+ \frac{s(s^2-1)(s^2-9)}{2.4.6.8.10} \Delta^5 y'_{x-3} + \dots
\end{aligned}$$

This formula returns to that which Newton has given in the small work cited, in order to interpolate between an even number of equidistant quantities;  $y'_x$  expresses the second of the two mean quantities, and  $\frac{s-1}{2}$  expresses its distance to that which we seek and which, consequently, is  $y'_{x+\frac{s-1}{2}}$ , unity representing the common interval of the given quantities.

## V.

We suppose generally

$$(a) \quad z = a + \frac{b}{t} + \frac{c}{t^2} + \frac{e}{t^3} + \dots + \frac{p}{t^{n-1}} + \frac{q}{t^n},$$

we will have

$$\frac{1}{t^n} = \frac{z-a}{q} - \frac{b}{qt} - \frac{c}{qt^2} - \dots - \frac{p}{qt^{n-1}},$$

that which gives

$$\frac{1}{t^{n+1}} = \frac{z-a}{qt} - \frac{b}{qt^2} - \frac{c}{qt^3} - \dots - \frac{p}{qt^n};$$

by eliminating  $\frac{1}{t^n}$  from the second member of this equation, by means of the proposed (a), we will have

$$\frac{1}{t^{n+1}} = -\frac{p(z-a)}{q^2} + \frac{pb+q(z-a)}{q^2t} + \dots$$

This expression of  $\frac{1}{t^{n+1}}$  contains only some powers of  $\frac{1}{t}$  of an order inferior to  $n$ , and, by continuing to eliminate thus the power  $\frac{1}{t^n}$ , in measure as it is presented, it is clear

that we will arrive to an expression of  $\frac{1}{t^i}$ , which will contain only some powers of  $\frac{1}{t}$  less than  $n$ , and which, consequently, will have this form

$$\frac{1}{t^i} = Z + \frac{1}{t}Z^{(1)} + \frac{1}{t^2}Z^{(2)} + \frac{1}{t^3}Z^{(3)} + \dots + \frac{1}{t^{n-1}}Z^{(n-1)},$$

$Z, Z^{(1)}, Z^{(2)}, \dots, Z^{(n-1)}$  being some rational and entire functions of  $z$ , of which the first does not surpass the degree  $\frac{i}{n}$ , the second does not surpass the degree  $\frac{i}{n} - 1$ , the third the degree  $\frac{i}{n} - 2$ , and so the rest.

This manner of determining  $\frac{1}{t^i}$  is very laborious when  $i$  is a little large; it would lead besides with difficulty to the general expression of this quantity; we could attain it directly by the following method.

$\frac{1}{t^i}$  being equal to the coefficient of  $\theta^i$  in the expansion of the fraction  $\frac{1}{1-\frac{\theta}{t}}$ , we will multiply the numerator and the denominator of this fraction by

$$(a - z)\theta^n + b\theta^{n-1} + c\theta^{n-2} + \dots + p\theta + q$$

and, by substituting into the numerator in place of  $z$  its value  $a + \frac{b}{t} + \frac{c}{t^2} + \dots$ , we will have

$$\frac{b\theta^{n-1} \left(1 - \frac{\theta}{t}\right) + c\theta^{n-2} \left(1 - \frac{\theta^2}{t^2}\right) + e\theta^{n-3} \left(1 - \frac{\theta^3}{t^3}\right) + \dots + q \left(1 - \frac{\theta^n}{t^n}\right)}{\left(1 - \frac{\theta}{t}\right) (a\theta^n + b\theta^{n-1} + c\theta^{n-2} + e\theta^{n-3} + \dots + p\theta + q - z\theta^n)}.$$

The numerator of this fraction is divisible by  $a - \frac{\theta}{t}$ ; we can therefore, by making the division, put it under this form

$$(A) \quad \frac{\left\{ \begin{array}{l} b\theta^{n-1} + c\theta^{n-2} + e\theta^{n-3} + \dots + p\theta + q \\ + \frac{\theta}{t}(c\theta^{n-2} + e\theta^{n-3} + \dots + p\theta + q) \\ + \frac{\theta^2}{t^2}(e\theta^{n-3} + \dots + p\theta + q) \\ + \dots \\ + \frac{q\theta^{n-1}}{t^{n-1}} \end{array} \right\}}{a\theta^n + b\theta^{n-1} + c\theta^{n-2} + e\theta^{n-3} + \dots + p\theta + q - z\theta^n}$$

The research on the coefficient of  $\theta^i$  in the expansion of this fraction is reduced thus to determine, whatever be  $r$ , the coefficient of  $\theta^r$  in the expansion of the fraction

$$\frac{1}{a\theta^n + b\theta^{n-1} + c\theta^{n-2} + e\theta^{n-3} + \dots + p\theta + q - z\theta^n}.$$

For this, we will consider generally the fraction  $\frac{P}{Q}$ ,  $P$  and  $Q$  being some rational and entire functions of  $\theta$ , the first being of an inferior order to that of the second. We suppose that  $Q$  has a factor  $\theta - \alpha$  raised to a power  $s$  and we make  $Q = (\theta - \alpha)^s R$ ; we can always, as we know, decompose the fraction  $\frac{P}{Q}$  into two others  $\frac{A}{(\theta - \alpha)^s} + \frac{B}{R}$ ,  $A$  and

$B$  being some rational and entire functions of  $\theta$ , the first of order  $s - 1$  and the second of an order inferior to the one of  $R$ ; we will have therefore

$$\frac{A}{(\theta - \alpha)^s} + \frac{B}{R} = \frac{P}{(\theta - \alpha)^s R},$$

that which gives

$$A = \frac{P}{R} - \frac{B(\theta - \alpha)^s}{R}.$$

If we consider  $A$ ,  $B$ ,  $P$  and  $R$  as some rational and entire functions of  $\theta - \alpha$ ,  $A$  will be a function of order  $s - 1$ , and, consequently, it will be equal to the expansion of  $\frac{P}{R}$  in a series ordered with respect to the powers of  $\theta - \alpha$ , provided that we stop at the power  $s - 1$ .

Let therefore

$$\frac{P}{R} = y + y_1(\theta - \alpha) + y_2(\theta - \alpha)^2 + \dots,$$

we will have

$$\frac{A}{(\theta - \alpha)^s} = \frac{y}{(\theta - \alpha)^s} + \frac{y_1}{(\theta - \alpha)^{s-1}} + \frac{y_2}{(\theta - \alpha)^{s-2}} + \dots,$$

by rejecting the positive or null powers of  $\theta - \alpha$ ;  $\frac{A}{(\theta - \alpha)^s}$  will be consequently equal to the coefficient of  $t^{s-1}$  in the expansion of

$$\frac{y + y_1 t + y_2 t^2 + \dots}{\theta - \alpha - t}.$$

Now, if we name  $P'$  and  $R'$  that which  $P$  and  $R$  become when we change  $\theta - \alpha$  into  $t$ , or, that which returns to the same,  $\theta$  into  $t + \alpha$ , we will have

$$\frac{P'}{R'} = y + y_1 t + y_2 t^2 + \dots;$$

hence,  $\frac{A}{(\theta - \alpha)^s}$  will be equal to the coefficient of  $t^{s-1}$  in the expansion of  $\frac{P'}{R'(\theta - \alpha - t)}$ , and, consequently, it will be equal to

$$\frac{1}{1.2.3 \dots (s-1)} \frac{\partial^{s-1}}{\partial t^{s-1}} \frac{P'}{R'(\theta - \alpha - t)},$$

provided that we suppose  $t = 0$  after the differentiations. Now, the coefficient of  $\theta^r$  in  $\frac{P'}{R'(\theta - \alpha - t)}$  being equal to  $-\frac{P'}{R'(\alpha + t)^{r+1}}$ , this same coefficient in  $\frac{1}{1.2.3 \dots (s-1)} \frac{\partial^{s-1}}{\partial t^{s-1}} \frac{P'}{R'(\theta - \alpha - t)}$  will be

$$-\frac{1}{1.2.3 \dots (s-1)} \frac{\partial^{s-1}}{\partial t^{s-1}} \frac{P'}{R'(\alpha + t)^{r+1}},$$

$t$  being supposed null after the differentiations. This last quantity will be therefore the coefficient of  $\theta^r$  in the expansion of  $\frac{A}{(\theta - \alpha)^s}$ ; now, if we restore, in  $P'$  and  $R'$ ,  $\theta - \alpha$  in place of  $t$ , that which changes them into  $P$  and  $R$ , we will have

$$\frac{\partial^{s-1}}{\partial t^{s-1}} \frac{P'}{R'(t + \alpha)^{r+1}} = \frac{\partial^{s-1}}{\partial \theta^{s-1}} \frac{P}{R\theta^{r+1}},$$

provided that we suppose  $\theta = \alpha$ , after the differentiations in the second member of this equation;  $-\frac{1}{1.2.3\dots(s-1)}\frac{\partial^{s-1}}{\partial\theta^{s-1}}\frac{P}{R\theta^{r+1}}$  will be therefore, with this condition, the coefficient of  $\theta^r$  in the expansion of the fraction  $\frac{A}{(\theta-\alpha)^s}$ .

It follows thence that, if we suppose

$$Q = a(\theta - \alpha)^s(\theta - \alpha')^{s'}(\theta - \alpha'')^{s''} \dots,$$

the coefficient of  $\theta^r$  in the expansion of the fraction  $\frac{P}{Q}$  will be

$$\begin{aligned} & - \frac{1}{1.2.3\dots(s-1)}\frac{\partial^{s-1}}{\partial\theta^{s-1}}\frac{P}{a\theta^{r+1}(\theta - \alpha')^{s'}(\theta - \alpha'')^{s''}\dots}, \\ & - \frac{1}{1.2.3\dots(s'-1)}\frac{\partial^{s'-1}}{\partial\theta^{s'-1}}\frac{P}{a\theta^{r+1}(\theta - \alpha)^s(\theta - \alpha'')^{s''}\dots}, \\ & - \frac{1}{1.2.3\dots(s''-1)}\frac{\partial^{s''-1}}{\partial\theta^{s''-1}}\frac{P}{a\theta^{r+1}(\theta - \alpha)^s(\theta - \alpha')^{s'}\dots}, \end{aligned}$$

by making, after the differentiation,  $\theta = \alpha$  in the first term,  $\theta = \alpha'$  in the second term,  $\theta = \alpha''$  in the third term, and thus in sequence. This put, let

$$V = a\theta^n + b\theta^{n-1} + c\theta^{n-2} + \dots + p\theta + q,$$

and we suppose that, by putting this quantity under the form of a product, we have

$$V = a(\theta - \alpha)(\theta - \alpha')(\theta - \alpha'')\dots;$$

by expanding the fraction  $\frac{1}{V-z\theta^n}$  in a series ordered with respect to the powers of  $z$ , we will have

$$\frac{1}{V} + \frac{z\theta^n}{V^2} + \frac{z^2\theta^{2n}}{V^3} + \frac{z^3\theta^{3n}}{V^4} + \dots,$$

and the coefficient of  $\theta^r$  in the expansion of the fraction  $\frac{1}{V^s}$  will be, by that which precedes, equal to

$$-\frac{1}{1.2.3\dots(s-1)a^s}\frac{\partial^{s-1}}{\partial\theta^{s-1}}\left\{\begin{aligned} & \frac{1}{\theta^{r+1}(\theta - \alpha')^s(\theta - \alpha'')^s\dots} \\ & + \frac{1}{\theta^{r+1}(\theta - \alpha)^s(\theta - \alpha'')^s\dots} \\ & + \frac{1}{\theta^{r+1}(\theta - \alpha)^s(\theta - \alpha')^s\dots} \\ & + \dots \end{aligned}\right\}$$

provided that, after the differentiations, we suppose  $\theta = \alpha$  in the first term,  $\theta = \alpha'$  in the second term,  $\theta = \alpha''$  in the third term, etc. Let  $Z_r^{(s-1)}$  be that which this quantity becomes then, the coefficient of  $\theta^i$  in the expansion of the fraction  $\frac{1}{V-z\theta^n}$  will be

$$Z_i^{(0)} + Z_{i-n}^{(1)}z + Z_{i-2n}^{(2)}z^2 + Z_{i-3n}^{(3)}z^3 + \dots;$$

we will have therefore, for the coefficient of  $\theta^i$  in the expansion of the fraction (A) and, consequently, for the expression of  $\frac{1}{t^i}$ ,

$$(\mu) \left\{ \begin{array}{l} \frac{1}{t^i} = bZ_{i-n+1}^{(0)} + bzZ_{i-2n+1}^{(1)} + bz^2Z_{i-3n+1}^{(2)} + bz^3Z_{i-4n+1}^{(3)} + \dots \\ \quad + cZ_{i-n+2}^{(0)} + czZ_{i-2n+2}^{(1)} + cz^2Z_{i-3n+2}^{(2)} + cz^3Z_{i-4n+2}^{(3)} + \dots \\ \quad + eZ_{i-n+3}^{(0)} + ezZ_{i-2n+3}^{(1)} + ez^2Z_{i-3n+3}^{(2)} + ez^3Z_{i-4n+3}^{(3)} + \dots \\ \quad + \dots \\ \quad + \frac{1}{t} \left\{ \begin{array}{l} cZ_{i-n+1}^{(0)} + czZ_{i-2n+1}^{(1)} + cz^2Z_{i-3n+1}^{(2)} + cz^3Z_{i-4n+1}^{(3)} + \dots \\ \quad + eZ_{i-n+2}^{(0)} + ezZ_{i-2n+2}^{(1)} + ez^2Z_{i-3n+2}^{(2)} + ez^3Z_{i-4n+2}^{(3)} + \dots \\ \quad + \dots \end{array} \right\} \\ \quad + \frac{1}{t^2} \left\{ \begin{array}{l} eZ_{i-n+1}^{(0)} + ezZ_{i-2n+1}^{(1)} + ez^2Z_{i-3n+1}^{(2)} + ez^3Z_{i-4n+1}^{(3)} + \dots \\ \quad + \dots \end{array} \right\} \\ \quad + \dots \\ \quad + \frac{1}{t^{n-1}} (qZ_{i-n+1}^{(0)} + qzZ_{i-2n+1}^{(1)} + qz^2Z_{i-3n+1}^{(2)} + \dots). \end{array} \right.$$

Presently, if we designate by  $\nabla y_x$  the quantity

$$ay_x + by_{x+1} + cy_{x+2} + \dots + qy_{x+n};$$

by  $\nabla^2 y_x$  the quantity

$$a\nabla y_x + b\nabla y_{x+1} + c\nabla y_{x+2} + \dots + q\nabla y_{x+n};$$

by  $\nabla^3 y_x$  the quantity

$$a\nabla^2 y_x + b\nabla^2 y_{x+1} + c\nabla^2 y_{x+2} + \dots + q\nabla^2 y_{x+n};$$

and thus in sequence, it is clear, by article II, that the coefficient of  $t^x$  in the expansion of  $\frac{uz^s}{t^r}$  will be  $\nabla^s y_{x+r}$ ; by multiplying therefore the preceding equation by  $u$ , and by considering within each term of it only the coefficient of  $t^x$ , that is by passing again

from the generating functions to the corresponding variables, we will have

$$(B) \left\{ \begin{array}{l} y_{x+1} = y_x(bZ_{i-n+1}^{(0)} + cZ_{i-n+2}^{(0)} + eZ_{i-n+3}^{(0)} + \cdots + qZ_i^{(0)}) \\ \quad + \nabla y_x(bZ_{i-2n+1}^{(1)} + cZ_{i-2n+2}^{(1)} + eZ_{i-2n+3}^{(1)} + \cdots + qZ_{i-n}^{(1)}) \\ \quad + \nabla^2 y_x(bZ_{i-3n+1}^{(2)} + cZ_{i-3n+2}^{(2)} + eZ_{i-3n+3}^{(2)} + \cdots + qZ_{i-2n}^{(2)}) \\ \quad + \cdots \\ \quad + y_{x+1}(cZ_{i-n+1}^{(0)} + eZ_{i-n+2}^{(0)} + \cdots + qZ_{i-1}^{(0)}) \\ \quad + \nabla y_{x+1}(cZ_{i-2n+1}^{(1)} + eZ_{i-2n+2}^{(1)} + \cdots + qZ_{i-n-1}^{(1)}) \\ \quad + \cdots \\ \quad + y_{x+2}(eZ_{i-n+1}^{(0)} + \cdots + qZ_{i-2}^{(0)}) \\ \quad + \nabla y_{x+2}(eZ_{i-2n+1}^{(1)} + \cdots + qZ_{i-n-2}^{(1)}) \\ \quad + \cdots \\ \quad + qy_{x+n-1}Z_{i-n+1}^{(0)} + q\nabla y_{x+n-1}Z_{i-2n+1}^{(1)} + q\nabla^2 y_{x+n-1}Z_{i-3n+1} + \cdots \end{array} \right.$$

This formula will serve to interpolate the series of which the final ratio of the terms is that of a recurrent series; because it is clear that, in this case,  $\nabla y_x$ ,  $\nabla^2 y_x$ ,  $\nabla^3 y_x$ ,  $\dots$  will always go by diminishing and will end by being null in the infinite.

If one of these quantities is null, for example, if we have  $\nabla^r y_x = 0$ , the preceding formula will give the general expression of  $y_x$  which satisfies this equation. In order to show this, we suppose first  $\nabla y_i = 0$ , or, that which comes to the same,

$$0 = ay_i + by_{i+1} + cy_{i+2} + \cdots + qy_{i+n};$$

if we make in this case  $x = 0$  in the preceding formula, it will become

$$\begin{aligned} y_i = & y_0(bZ_{i-n+1}^{(0)} + cZ_{i-n+2}^{(0)} + eZ_{i-n+3}^{(0)} + \cdots + qZ_i^{(0)}) \\ & + y_1(cZ_{i-n+1}^{(0)} + eZ_{i-n+2}^{(0)} + \cdots + qZ_{i-1}^{(0)}) \\ & + y_2(eZ_{i-n+1}^{(0)} + \cdots + qZ_{i-2}^{(0)}) \\ & + \cdots \\ & + qy_{x+n-1}Z_{i-n+1}^{(0)}. \end{aligned}$$

$y_0, y_1, y_2, \dots, y_{n-1}$  are the first  $n$  values of  $y_i$ ; these are the  $n$  arbitrary constants that the integration of equation  $\nabla y_i = 0$  introduces.

If we have  $\nabla^2 y_i = 0$ , the general formula (B) will give, by supposing again  $x = 0$ ,

$$\begin{aligned} y_i = & y_0(bZ_{i-n+1}^{(0)} + cZ_{i-n+2}^{(0)} + \cdots + qZ_i^{(0)}) \\ & + \nabla y_0(bZ_{i-2n+1}^{(1)} + cZ_{i-2n+2}^{(1)} + \cdots + qZ_{i-n}^{(1)}) \\ & + y_1(cZ_{i-n+1}^{(0)} + \cdots + qZ_{i-1}^{(0)}) \\ & + \nabla y_1(cZ_{i-2n+1}^{(1)} + \cdots + qZ_{i-n+1}^{(1)}) \\ & + \cdots \\ & + qZ_{i-n+1}^{(0)}y_{n-1} + qZ_{i-2n+1}^{(1)}\nabla y_{n-1}, \end{aligned}$$

$y_0, \nabla y_0, y_1, \nabla y_1, \dots, y_{n-1}, \nabla y_{n-1}$  being the  $2n$  arbitrary constants which the integration of the equation  $\nabla^2 y_i = 0$  introduces. We will have in the same manner the value of  $y_i$  in the case of  $\nabla^3 y_i = 0, \nabla^4 y_i = 0, \dots$ , and we see thus the analogy which exists between interpolation of series and the integration of equations linear in the finite differences.

## VI.

Let  $y_x = y'_x + y''_x$ , and we suppose that  $u'$  is the generating function of  $y'_x$ , and  $u''$  of  $y''_x$ ; we will have

$$u = u' + u''.$$

Let further  $u'' z^s = \lambda$  or  $u'' = \frac{\lambda}{z^s}$ ; if we designate by  $X_{x+i}$  the coefficient of  $t^{x+i}$  in the expansion of  $\lambda$ , we will have, by article II,

$$X_{x+i} = \nabla^s y''_{x+i};$$

presently, we have

$$\frac{1}{z^s} = \frac{t^{ns}}{(at^n + bt^{n-1} + ct^{n-2} + \dots + q)^s}.$$

Now the coefficient of  $t^{x+i}$ , in the expansion of the second member of this equation, is equal to the one of  $\theta^{x+i-ns}$  in the expansion of  $\frac{1}{(\theta^n + b\theta^{n-1} + c\theta^{n-2} + \dots + q)^s}$ , and, by the preceding article, this last coefficient is equal to  $Z_{x+i-ns}^{(s-1)}$ ; therefore the coefficient of  $t^{x+i}$ , in the expansion of  $\frac{\lambda}{z^s}$ , will be

$$X_{x+i-ns} Z_0^{(s-1)} + X_{x+i-ns-1} Z_1^{(s-1)} + \dots + X_0 Z_{x+i-ns}^{(s-1)} \quad \text{or} \quad \Sigma X_r Z_{x+i-ns-r}^{(s-1)},$$

the integral being taken relatively to  $r$  and from  $r = 0$  to  $r = x + i - ns$ ; this integral will be the expression of  $y''_{x+i}$ .

In the present case, it is easy to reduce it to some integrals relative to  $i$ , because it results from the expression which we have given of  $Z_i^{(s-1)}$  in the preceding article, as that of  $Z_{x+i-ns-r}^{(s-1)}$  is reducible to some terms of this form  $K\beta_r r^\mu$ , so that the term corresponding to  $\Sigma X_r Z_{x+i-ns-r}^{(s-1)}$  will be  $K\Sigma\beta^r r^\mu X_r$ ,  $K$  being a function of  $x+i-ns$ ; now, if we designate by the characteristic  $\Sigma'$  the integral relative to  $i$ , we will have

$$K\Sigma\beta^r r^\mu X_r = K\Sigma' \beta^{x+i-ns} (x+i-ns)^\mu X_{x+i-ns},$$

provided that we terminate the integral relative to  $r$ , when  $r$  equals  $x + i - ns$ ; we will reduce thus the integral  $\Sigma X_r Z_{x+i-ns-r}^{(s-1)}$  to some integrals uniquely relative to the



variable  $i$ . This put, if in formula (B) we make  $x = 0$  and  $\nabla^s y_i = 0$ , it will become

$$y'_{i+\Sigma} X_r Z_{i-n_s-r}^{(s-1)} = \left\{ \begin{array}{l} y_0(bZ_{i-n+1}^{(0)} + cZ_{i-n+2}^{(0)} + \dots + qZ_i^{(0)}) \\ + \nabla y_0(bZ_{i-2n+1}^{(1)} + cZ_{i-2n+2}^{(1)} + \dots + qZ_{i-n}^{(1)}) \\ + \dots \\ + \nabla^{s-1} y_0(bZ_{i-sn+1}^{(s-1)} + cZ_{i-sn+2}^{(s-1)} + \dots + qZ_{i-sn+n}^{(s-1)}) \\ y_1(cZ_{i-n+1}^{(0)} + \dots + qZ_{i-1}^{(0)}) \\ + \dots \\ + \nabla^{s-1} y_1(cZ_{i-sn+1}^{(s-1)} + \dots + qZ_{i-sn+n-1}^{(s-1)}) \\ + \dots \\ + qZ_{i-n+1}^{(0)} y_{n-1} + qZ_{i-2n+1}^{(1)} \nabla y_{n-1} + \dots + qZ_{i-sn+1}^{(s-1)} \nabla^{s-1} y_{n-1}, \end{array} \right.$$

$y_0, \nabla y_0, \dots, \nabla^{s-1} y_0, y_1, \dots, \nabla y_1, \dots, \nabla^{s-1} y_1, y_{n-1}, \nabla y_{n-1}, \dots, \nabla^{s-1} y_{n-1}$  being the  $sn$  arbitraries of the integral of the equation

$$\nabla^s y_i = 0 \quad \text{or} \quad \nabla^s y'_i + \nabla^s y''_i = 0;$$

now,  $\nabla^s y''_i$  being equal to  $X_i$ , this equation becomes

$$0 = \nabla^s y'_i + X_i.$$

We will have therefore, by the preceding formula, the integral of all the equations linear in the finite differences of which the coefficients are constants, in the case where they have a last term which is a function of  $i$ .

## VII.

We can give to the expression of  $\frac{1}{t^i}$  an infinity of other forms among which there is found what can be utile in many cases. Here is how we can attain it.

For this, we suppose that, instead of giving, as above, to  $\frac{1}{t^i}$  this form

$$\frac{1}{t^i} = Z + \frac{1}{t} Z^{(1)} + \frac{1}{t^2} Z^{(2)} + \dots + \frac{1}{t^{n-1}} Z^{(n-1)},$$

we give it this one

$$\frac{1}{t^i} = Z + \left(\frac{1}{t} - 1\right) Z^{(1)} + \left(\frac{1}{t} - 1\right)^2 Z^{(2)} + \dots + \left(\frac{1}{t} - 1\right)^n Z^{(n-1)},$$

the question is reduced to determining  $Z, Z^{(1)}, Z^{(2)}, \dots$

We put first the equation

$$z = a + \frac{b}{t} + \frac{c}{t^2} + \dots + \frac{p}{t^{n-1}} + \frac{q}{t^n}$$

under this form

$$z = a' + b' \left( \frac{1}{t} - 1 \right) + c' \left( \frac{1}{t} - 1 \right)^2 + \cdots + p' \left( \frac{1}{t} - 1 \right)^{n-1} + q' \left( \frac{1}{t} - 1 \right)^n,$$

and one will have

$$a = a' - b' + c' - \cdots \mp p' \pm q',$$

the upper signs having place if  $n$  is even and the lower signs if  $n$  is odd. We will multiply next, as previously, the numerator and the denominator of the fraction  $\frac{1}{1-\frac{\theta}{t}}$  by

$$(a - z)\theta^n + b\theta^{n-1} + c\theta^{n-2} + \cdots + p\theta + q,$$

by observing to substitute into the numerator:  $1^\circ$  in place of  $z$ ,

$$a' + b' \left( \frac{1}{t} - 1 \right) + c' \left( \frac{1}{t} - 1 \right)^2 + \cdots;$$

$2^\circ$  in place of  $a\theta^n + b\theta^{n-1} + c\theta^{n-2} + \cdots$  the quantity

$$\theta^n \left[ a' + b' \left( \frac{1}{t} - 1 \right) + c' \left( \frac{1}{t} - 1 \right)^2 + \cdots \right].$$

If moreover we make, for brevity,  $\frac{1}{t} - 1 = \frac{1}{t'}$ , we will have

$$\frac{b'\theta^{n-1} \left( 1 - \theta - \frac{\theta}{t'} \right) + c'\theta^{n-2} \left[ (1 - \theta)^2 - \frac{\theta^2}{t'^2} \right] + \cdots + q \left[ (1 - \theta)^n - \frac{\theta^n}{t'^n} \right]}{\left( 1 - \frac{\theta}{t} \right) (a\theta^n + b\theta^{n-1} + c\theta^{n-2} + \cdots + p\theta + q - z\theta^n)};$$

now we have

$$1 - \frac{\theta}{t} = 1 - \theta - \frac{\theta}{t'}.$$

By dividing therefore the numerator of the preceding fraction by this quantity, it will be reduced to this one

$$\frac{\left\{ \begin{array}{l} b'\theta^{n-1} + c'\theta^{n-2} \left( 1 - \theta + \frac{\theta}{t'} \right) + e'\theta^{n-3} \left[ (1 - \theta)^2 + (1 - \theta)\frac{\theta}{t'} + \frac{\theta^2}{t'^2} \right] + \cdots \\ + q \left[ (1 - \theta)^{n-1} + (1 - \theta)^{n-2}\frac{\theta}{t'} + (1 - \theta)^{n-3}\frac{\theta^2}{t'^2} + \cdots \right] \end{array} \right\}}{a\theta^n + b\theta^{n-1} + c\theta^{n-2} + \cdots + p\theta + q - z\theta^n}$$

whence, that which returns to the same, to

$$\left\{ \begin{array}{l} b'\theta^{n-1} + c'\theta^{n-1} \left( \frac{1}{\theta} - 1 \right) + e'\theta^{n-1} \left( \frac{1}{\theta} - 1 \right)^2 + \dots + q\theta^{n-1} \left( \frac{1}{\theta} - 1 \right)^{n-1} \\ + \frac{\theta^{n-1}}{t'} \left[ c' + e' \left( \frac{1}{\theta} - 1 \right) + \dots + q \left( \frac{1}{\theta} - 1 \right)^{n-2} \right] \\ + \frac{\theta^{n-1}}{t'^2} \left[ e' + \dots + q \left( \frac{1}{\theta} - 1 \right)^{n-3} \right] \\ + \dots \\ + \frac{q\theta^{n-1}}{t'^{n-1}} \end{array} \right\} \\ \hline a\theta^n + b\theta^{n-1} + c\theta^{n-2} + \dots + p\theta + q - z\theta^n$$

Thence is easy to conclude that, if we conserve to  $Z_r^{(s-1)}$  the same signification that we have given to it in article V and if we consider that, by designating by  $q_i$  the coefficient of  $\theta^i$  in the expansion of any function of  $\theta$ , this same coefficient in the expansion of this function multiplied by  $\left( \frac{1}{\theta} - 1 \right)^\mu$  will be, by article II,  $\Delta^\mu q_i$ ; we will have

$$(\mu') \left\{ \begin{array}{l} \frac{1}{t^i} = b'Z_{i-n+1}^{(0)} + b'zZ_{i-2n+1}^{(1)} + b'z^2Z_{i-3n+1}^{(2)} + \dots \\ + c'\Delta Z_{i-n+1}^{(0)} + c'z\Delta Z_{i-2n+1}^{(1)} + c'z^2\Delta Z_{i-3n+1}^{(2)} + \dots \\ + e'\Delta^2 Z_{i-n+1}^{(0)} + e'z\Delta^2 Z_{i-2n+1}^{(1)} + e'z^2\Delta^2 Z_{i-3n+1}^{(2)} + \dots \\ + \dots \\ + q\Delta^{n-1} Z_{i-n+1}^{(0)} + qz\Delta^{n-1} Z_{i-2n+1}^{(1)} + qz^2\Delta^{n-1} Z_{i-3n+1}^{(2)} + \dots \\ + \frac{1}{t'} \left\{ \begin{array}{l} c'Z_{i-n+1}^{(0)} + c'zZ_{i-2n+1}^{(1)} + \dots \\ + e'\Delta Z_{i-n+1}^{(0)} + e'z\Delta Z_{i-2n+1}^{(1)} + \dots \\ + \dots \end{array} \right\} \\ + \frac{1}{t'^2} \left\{ \begin{array}{l} e'Z_{i-n+1}^{(0)} + e'zZ_{i-2n+1}^{(1)} + \dots \\ + \dots \end{array} \right\} \\ + \dots \\ + \frac{q}{t'^{n-1}} (Z_{i-n+1}^{(0)} + zZ_{i-2n+1}^{(1)} + \dots). \end{array} \right.$$

Presently, it is clear, by article II, that the coefficient of  $t^x$  in the expansion of the function  $\frac{uz^s}{t'^\mu}$  is  $\Delta^\mu \nabla^s y_x$ ; the preceding equation will give therefore, by multiplying it

by  $u$  and by passing again from the generating functions to the corresponding variables,

$$\begin{aligned}
y_{x+i} = & y_x (b' Z_{i-n+1}^{(0)} + c' \Delta Z_{i-n+1}^{(0)} + e' \Delta^2 Z_{i-n+1}^{(0)} + \cdots + q \Delta^{n-1} Z_{i-n+1}^{(0)}) \\
& + \nabla y_x (b' Z_{i-2n+1}^{(1)} + c' \Delta Z_{i-2n+1}^{(1)} + e' \Delta^2 Z_{i-2n+1}^{(1)} + \cdots + q \Delta^{n-1} Z_{i-2n+1}^{(1)}) \\
& + \nabla^2 y_x (b' Z_{i-3n+1}^{(2)} + c' \Delta Z_{i-3n+1}^{(2)} + e' \Delta^2 Z_{i-3n+1}^{(2)} + \cdots + q \Delta^{n-1} Z_{i-3n+1}^{(2)}) \\
& + \cdots \\
& + \Delta y_x (c' Z_{i-n+1}^{(0)} + e' \Delta Z_{i-n+1}^{(0)} + \cdots + q \Delta^{n-2} Z_{i-n+1}^{(0)}) \\
& + \Delta \nabla y_x (c' Z_{i-2n+1}^{(1)} + e' \Delta Z_{i-2n+1}^{(1)} + \cdots + q \Delta^{n-2} Z_{i-2n+1}^{(1)}) \\
& + \Delta \nabla^2 y_x (c' Z_{i-3n+1}^{(2)} + e' \Delta Z_{i-3n+1}^{(2)} + \cdots + q \Delta^{n-2} Z_{i-3n+1}^{(2)}) \\
& + \cdots \\
& + \Delta^2 y_x (e' Z_{i-n+1}^{(0)} + \cdots + q \Delta^{n-3} Z_{i-n+1}^{(0)}) \\
& + \Delta^2 \nabla y_x (e' Z_{i-2n+1}^{(1)} + \cdots + q \Delta^{n-3} Z_{i-2n+1}^{(1)}) \\
& + \cdots \\
& + q Z_{i-n+1}^{(0)} \Delta^{n-1} y_x + q Z_{i-2n+1}^{(1)} \Delta^{n-1} \nabla y_x + q Z_{i-3n+1}^{(2)} \Delta^{n-1} \nabla^2 y_x + \cdots
\end{aligned}$$

### VIII.

We suppose, in the preceding formula,  $x$  and  $i$  infinitely great, in a way that we have

$$i = \frac{x_1}{dx_1} \quad \text{and} \quad x = \frac{\varpi}{dx_1};$$

$y_{x+i}$  becomes a function of  $\varpi + x_1$ , which we will designate by  $\phi(\varpi + x_1)$ . We suppose moreover

$$a_1 = a_2, \quad b_1 = \frac{b_2}{dx_1}, \quad c_1 = \frac{c_2}{dx_1^2}, \quad \dots, \quad q = \frac{q_2}{dx_1^n},$$

the equation

$$0 = a_1 + b_1 \left( \frac{1}{\theta} - 1 \right) + c_1 \left( \frac{1}{\theta} - 1 \right)^2 + \cdots + q \left( \frac{1}{\theta} - 1 \right)^n$$

will give, for  $\theta$ ,  $n$  roots of this form

$$\theta = 1 + f dx_1, \quad \theta = 1 + f_1 dx_1, \quad \theta = 1 + f_2 dx_1, \quad \dots;$$

these will be the quantities which we have named  $\alpha, \alpha', \alpha'', \dots$  in the expression  $Z_r^{(s-1)}$  of article V, and the values of  $f, f_1, f_2, \dots$ , will be given by the  $n$  roots of the equation

$$0 = a_2 - b_2 f + c_2 f^2 + \cdots \pm q_2 f^n.$$

Now, if we make  $\theta = 1 + h dx_1$ , we will have

$$\frac{1}{\theta^i} = \frac{1}{(1 + h dx_1)^i};$$

the hyperbolic logarithm of this last quantity is

$$-i \log(1 + hdx_1) = -ihdx_1 = -hx_1,$$

whence we deduce  $\frac{1}{\theta^i} = e^{-hx_1}$ ,  $e$  being here the number of which the hyperbolic logarithm is unity; we have besides

$$a = a_1 - b_1 + c_1 - \dots \pm q = a_2 - \frac{b_2}{dx_1} + \frac{c_2}{dx_1^2} + \dots \pm \frac{q_2}{dx_1^n},$$

and this value of  $a$  is reduced to the term  $\pm \frac{q_2}{dx_1^n}$ , because it is infinitely greater than the others; the expression of  $Z_r^{(s-1)}$  of article V will give therefore, by changing  $r$  into  $i - 1$ ,

$$Z_{i-1}^{(s-1)} = -\frac{dx_1}{1.2.3 \dots (s-1)(\pm q_2)^s} \frac{\partial^{s-1}}{\partial h^{s-1}} \left\{ \begin{array}{l} \frac{e^{-hx_1}}{(h-f_1)^s (h-f_2)^s \dots} \\ + \frac{e^{-hx_1}}{(h-f)^s (h-f_2)^s \dots} \\ + \frac{e^{-hx_1}}{(h-f)^s (h-f_1)^s \dots} \\ + \dots \end{array} \right\},$$

the difference  $\partial^{s-1}$  being taken by making  $h$  vary only and by substituting, after the differentiations,  $f$  in place of  $h$  in the first term,  $f_1$  in place of  $h$  in the second term, and thus in sequence. We name  $X^{(s-1)} dx_1$  the preceding quantity, we will have, to the infinitely small nearly,

$$Z_{i \pm \mu}^{(s-1)} = Z_{i-1}^{(s-1)} = X^{(s-1)} dx_1;$$

moreover we have  $y_x = \phi(\varpi)$ , and the characteristic  $\Delta$  of the finite differences must be changed here into the characteristic  $\partial$  of the infinitely small differences, so that the equation

$$\nabla y_x = ay_x + by_{x+1} + cy_{x+2} + \dots$$

or, that which returns to the same, this here

$$\nabla y_x = a_2 + \frac{b_2}{dx_1} \Delta y_x + \frac{c_2}{dx_1^2} \Delta^2 y_x + \dots$$

becomes, by changing  $dx_1$  into  $d\varpi$ ,

$$\nabla y_x = a_2 + b_2 \frac{d\phi(\varpi)}{d\varpi} + c_2 \frac{d^2\phi(\varpi)}{d\varpi^2} + e_2 \frac{d^3\phi(\varpi)}{d\varpi^3} + \dots + q_2 \frac{d^n\phi(\varpi)}{d\varpi^n};$$

the expression of  $y_{x+i}$ , found in the preceding article, will become therefore

$$(C) \left\{ \begin{aligned} & \phi(\varpi + x_1) = \phi(\varpi) \left( b_2 X^{(0)} + c_2 \frac{dX^{(0)}}{dx_1} + e_2 \frac{d^2 X^{(0)}}{dx_1^2} + \dots \right) \\ & \quad + \nabla \phi(\varpi) \left( b_2 X^{(1)} + c_2 \frac{dX^{(1)}}{dx_1} + e_2 \frac{d^2 X^{(1)}}{dx_1^2} + \dots \right) \\ & \quad + \nabla^2 \phi(\varpi) \left( b_2 X^{(2)} + c_2 \frac{dX^{(2)}}{dx_1} + e_2 \frac{d^2 X^{(2)}}{dx_1^2} + \dots \right) \\ & \quad + \dots \\ & \quad + \frac{d\phi(\varpi)}{d\varpi} \left( c_2 X^{(0)} + e_2 \frac{dX^{(0)}}{dx_1} + \dots \right) \\ & \quad + \frac{d\nabla\phi(\varpi)}{d\varpi} \left( c_2 X^{(1)} + e_2 \frac{dX^{(1)}}{dx_1} + \dots \right) \\ & \quad + \frac{d\nabla^2\phi(\varpi)}{d\varpi} \left( c_2 X^{(2)} + e_2 \frac{dX^{(2)}}{dx_1} + \dots \right) \\ & \quad + \dots \\ & \quad + \frac{d^2\phi(\varpi)}{d\varpi^2} (e_2 X^{(0)} + \dots) \\ & \quad + \frac{d^2\nabla\phi(\varpi)}{d\varpi^2} (e_2 X^{(1)} + \dots) \\ & \quad + \dots \\ & \quad + q_2 \frac{d^{n-1}\phi(\varpi)}{d\varpi^{n-1}} X^{(0)} + q_2 \frac{d^{n-1}\nabla\phi(\varpi)}{d\varpi^{n-1}} X^{(1)} \\ & \quad + q_2 \frac{d^{n-1}\nabla^2\phi(\varpi)}{d\varpi^{n-1}} X^{(2)} + \dots \end{aligned} \right.$$

This formula will serve to interpolate the series, of which the last ratio of the terms is that of a linear equation in the infinitely small differences of which the coefficients are constants.

If we have

$$\nabla\phi(\varpi) = a_2\phi(\varpi) + b_2 \frac{d\phi(\varpi)}{d\varpi},$$

we will have

$$f = \frac{a_2}{b_2}$$

and

$$\nabla\phi(\varpi) = b_2 e^{-f\varpi} \frac{d[e^{f\varpi}\phi(\varpi)]}{d\varpi};$$

the expression of  $X^{(s-1)}$  becomes, in this particular case,

$$\frac{1}{1.2.3 \dots (s-1)b_2^s} x^{s-1} e^{-fx_1};$$

we will have therefore

$$\phi(\varpi + x_1) = e^{-f\varpi - fx_1} \left\{ e^{f\varpi} \phi(\varpi) + \frac{x_1}{b_2} \frac{d[e^{f\varpi} \phi(\varpi)]}{d\varpi} + \frac{x_1^2}{b_2^2} \frac{d^2[e^{f\varpi} \phi(\varpi)]}{d\varpi^2} + \dots \right\}.$$

By supposing  $b_2 = 1$  and  $f = 0$ , consequently  $a_2 = 0$ , we will have the known formula of Taylor.

Formula (C) will be terminated anytime we have  $\nabla^s \phi(\varpi) = 0$ ; if, for example,  $\nabla \phi(\varpi) = 0$ , we will have

$$\begin{aligned} \phi(\varpi + x_1) = & \phi(\varpi) \left( b_2 X^{(0)} + c_2 \frac{dX^{(0)}}{dx_1} + \dots + q_2 \frac{d^{n-1} X^{(0)}}{dx_1^{n-1}} \right) \\ & + \frac{d\phi(\varpi)}{d\varpi} \left( c_2 X^{(0)} + e_2 \frac{dX^{(0)}}{dx_1} + \dots + q_2 \frac{d^{n-2} X^{(0)}}{dx_1^{n-2}} \right) \\ & + \dots \\ & + q_2 X^{(0)} \frac{d^{n-1} \phi(\varpi)}{d\varpi^{n-1}}; \end{aligned}$$

this will be the integral of the equation  $0 = \nabla \phi(\varpi + x_1)$  or, that which returns to the same, of this

$$0 = a_2 \phi(\varpi + x_1) + b_2 \frac{d\phi(\varpi + x_1)}{dx_1} + c_2 \frac{d^2 \phi(\varpi + x_1)}{dx_1^2} + \dots + q_2 \frac{d^n \phi(\varpi + x_1)}{dx_1^n},$$

$\phi(\varpi)$ ,  $\frac{d\phi(\varpi)}{d\varpi}$ ,  $\frac{d^2 \phi(\varpi)}{d\varpi^2}$ ,  $\dots$ ,  $\frac{d^{n-1} \phi(\varpi)}{d\varpi^{n-1}}$  being the  $n$  arbitrary constants which the integration introduces. We will have, by the same formula, the integrals of the equations

$$\nabla^2 \phi(\varpi + x_1) = 0, \quad \nabla^3 \phi(\varpi + x_1) = 0, \quad \dots$$

If we make

$$\phi(\varpi + x_1) = y_1 x_1 + y_2 x_1$$

and if we suppose  $\nabla^s y_2 x_1 = V$ ,  $V$  being a given function of  $x_1$ , we will find easily, by article VI, that if we change, in the expression of  $X^{(s-1)}$ ,  $x_1$  into  $\varpi + x_1 - r$  and, in  $V$ ,  $x_1$  into  $r - \varpi$ , and if we name  $R$  that which the first of these two quantities becomes and  $S$  that which the second becomes, we will have  $y_2 x_1 = \int RS dr$ , the integral being taken from  $r = 0$  to  $r = \varpi + x_1$ ; if we suppose, moreover, in formula

(C),  $\nabla^s \phi(\varpi + x_1) = 0$ , it will become

$$\begin{aligned}
y_1 x_1 + \int RS dr = & y_0 \left( b_2 X^{(0)} + c_2 \frac{dX^{(0)}}{dx_1} + \dots \right) \\
& + \nabla y_0 \left( b_2 X^{(1)} + c_2 \frac{dX^{(1)}}{dx_1} + \dots \right) \\
& + \dots \\
& + \nabla^{(s-1)} y_0 \left( b_2 X^{(s-1)} + c_2 \frac{dX^{(s-1)}}{dx_1} + \dots \right) \\
& + \frac{dy_0}{d\varpi} (c_2 X^{(0)} + \dots) \\
& + \nabla \left( \frac{dy_0}{d\varpi} \right) (c_2 X^{(1)} + \dots) \\
& + \dots \\
& + \nabla^{(s-1)} \left( \frac{dy_0}{d\varpi} \right) (c_2 X^{(s-1)} + \dots) \\
& + \dots \\
& + q_2 \frac{d^{n-1} y_0}{d\varpi^{n-1}} X^{(0)} + q_2 \nabla \left( \frac{d^{n-1} y_0}{d\varpi^{n-1}} \right) X^{(1)} + \dots \\
& + q_2 \nabla^{s-1} \left( \frac{d^{n-1} y_0}{d\varpi^{n-1}} \right) X^{(s-1)},
\end{aligned}$$

$y_0, \frac{dy_0}{d\varpi}, \dots, \nabla y_0, \nabla \left( \frac{dy_0}{d\varpi} \right), \dots$  being the  $sn$  arbitrary constants of the integral of the equation

$$0 = \nabla^s \phi(\varpi + x_1) \quad \text{or} \quad \nabla^s y_1 x_1 + V = 0;$$

the preceding formula will serve therefore to integrate all the equations linear in the infinitely small finite differences, of which the coefficients are constants, when they have a last term which is a function of  $x_1$  alone.

## IX.

### *On the transformation of the series.*

We see, by that which precedes, with what facility all theory of the recurrent series results by consideration of generating functions; this consideration can yet serve to transform, in a more general and more simple manner than by known methods, a series into another of which the terms follow a known law.

For this, we will consider the series

$$(\gamma) \quad y_0 + y_1 + y_2 + y_3 + \dots + y_x + y_{x+1} + \dots + y_\infty,$$

and we name, as above,  $u$  the sum of the series

$$y_0 + y_1 t + y_2 t^2 + y_3 t^3 + \dots + y_x t^x + y_{x+1} t^{x+1} + \dots + y_\infty t^\infty,$$



it is clear that the coefficient of  $t^x$ , in the expansion of the fraction  $\frac{u}{1-\frac{1}{t}}$ , will be equal to the sum of the proposed series ( $\gamma$ ), from the term  $y_x$  to infinity; now, if we multiply the numerator and the denominator of this fraction by

$$a + b + c + e + \dots - \left( a + \frac{b}{t} + \frac{c}{t^2} + \frac{e}{t^3} + \dots \right),$$

the numerator will be divisible by  $1 - \frac{1}{t}$  and the quotient of the division will be

$$u \left[ b + c + e + \dots + \frac{1}{t}(c + e + \dots) + \frac{1}{t^2}(e + \dots) + \dots \right];$$

therefore, if we make, for brevity,

$$\begin{aligned} a + b + c + e + \dots &= K, \\ a + \frac{b}{t} + \frac{c}{t^2} + \frac{e}{t^3} + \dots &= z, \end{aligned}$$

we will have

$$\frac{u}{1 - \frac{1}{t}} = \frac{u}{K - z} \left[ b + c + e + \dots + \frac{1}{t}(c + e + \dots) + \frac{1}{t^2}(e + \dots) + \dots \right];$$

by expanding the second member of this equation with respect to the powers of  $z$ , we will have

$$u \left[ b + c + e + \dots + \frac{1}{t}(c + e + \dots) + \frac{1}{t^2}(e + \dots) + \dots \right] \left( \frac{1}{K} + \frac{z}{K^2} + \frac{z^2}{K^3} + \frac{z^3}{K^4} + \dots \right).$$

Now, the coefficient of  $t^x$ , in any term such as  $\frac{uz^s}{t^r}$ , is, by article II, equal to  $\nabla^s y_{x+r}$ ; this coefficient will be therefore, in the preceding quantity, equal to

$$\begin{aligned} &(b+c+e+\dots) \left( \frac{y_x}{K} + \frac{\nabla y_x}{K^2} + \frac{\nabla^2 y_x}{K^3} + \frac{\nabla^3 y_x}{K^4} + \dots \right) \\ &+ (c+e+\dots) \left( \frac{y_{x+1}}{K} + \frac{\nabla y_{x+1}}{K^2} + \frac{\nabla^2 y_{x+1}}{K^3} + \frac{\nabla^3 y_{x+1}}{K^4} + \dots \right) \\ &+ (e+\dots) \left( \frac{y_{x+2}}{K} + \frac{\nabla y_{x+2}}{K^2} + \frac{\nabla^2 y_{x+2}}{K^3} + \frac{\nabla^3 y_{x+2}}{K^4} + \dots \right) \\ &+ \dots \end{aligned}$$

this will be the value of the proposed series ( $\gamma$ ) from the term  $y_x$  to infinity.

If we make  $x = 0$ , we will have a new series equal to the proposed, but in which the terms follow another law; and, if the quantities  $\nabla y_x, \nabla^2 y_x, \dots$  go by decreasing, this new series will be convergent; it will terminate itself anytime that we have  $\nabla^s y_x = 0$ , that which will take place when the proposed series will be recurrent; we will have in this manner the sum of the recurrent series.

The transformation of the series is reduced to determining the integral  $\Sigma y_x$ , taken from  $x = 0$  to  $x = \infty$ , and all the ways to express this integral will give as many different transformations; that which consists, by that which precedes, in determining the

coefficient of  $t^x$  in the expansion of  $\frac{u}{1-\frac{1}{t}}$ . For that, let generally  $z$  be any function of  $\frac{1}{t}$ , and we name  $\nabla y_x$  the coefficient of  $t^x$  in  $uz$ ; the coefficients of  $t^x$  in  $uz^2, uz^3, uz^4, \dots$  will be  $\nabla^2 y_x, \nabla^3 y_x, \nabla^4 y_x, \dots$ . This put, we will multiply the numerator and the denominator of the fraction  $\frac{u}{1-\frac{1}{t}}$  by  $K - z$ , and we will take  $K$  in a way that it will be equal to  $z$ , when we make  $t$  equal to 1 in this last quantity;  $K - z$  will thus be divisible by  $1 - \frac{1}{t}$ . Let  $q + \frac{q^{(1)}}{t} + \frac{q^{(2)}}{t^2} + \frac{q^{(3)}}{t^3} + \dots$  be the quotient of the division; we will have

$$\begin{aligned} \frac{u}{1-\frac{1}{t}} &= \frac{uq}{K} \left( 1 + \frac{z}{K} + \frac{z^2}{K^2} + \frac{z^3}{K^3} + \dots \right) \\ &\quad + \frac{uq^{(1)}}{Kt} \left( 1 + \frac{z}{K} + \frac{z^2}{K^2} + \frac{z^3}{K^3} + \dots \right) \\ &\quad + \dots \end{aligned}$$

that which gives, by passing from the generating functions to their corresponding variables,

$$\begin{aligned} \Sigma y_x &= \frac{qy_x}{K} + \frac{q\nabla y_x}{K^2} + \frac{q\nabla^2 y_x}{K^3} + \dots \\ &\quad + \frac{q^{(1)}y_{x+1}}{K} + \frac{q^{(1)}\nabla y_{x+1}}{K^2} + \frac{q^{(1)}\nabla^2 y_{x+1}}{K^3} + \dots \\ &\quad + \frac{q^{(2)}y_{x+2}}{K} + \frac{q^{(2)}\nabla y_{x+2}}{K^2} + \frac{q^{(2)}\nabla^2 y_{x+2}}{K^3} + \dots \\ &\quad + \dots \end{aligned}$$

the integral  $\Sigma y_x$  being taken from  $y_x$  to  $y_\infty$ ; and, if we make in the preceding equation  $x = 0$ , we will have a new series equal to the proposed and which will be, consequently, its transformed.

## X.

*Theorems on the expansion of functions and of their differences in series.*

By applying to some particular cases the results which we have given in article II, we have an infinity of theorems on the expansion of functions in series; we are going to present here the most remarkable.

We have generally

$$u \left( \frac{1}{t^i} - 1 \right)^n = u \left[ \left( 1 + \frac{1}{t} - 1 \right)^i - 1 \right]^n ;$$

now it is clear that the coefficient of  $t^x$ , in the first member of this equation, is the  $n^{\text{th}}$  difference of  $y_x$ ,  $x$  varying with  $i$ ; because this coefficient in  $u \left( \frac{1}{t^i} - 1 \right)$  is  $y_{x+i} - y_x$  or  ${}^1\Delta y_x$ , by designating by the characteristic  ${}^1\Delta$  the finite differences, when  $x$  varies from the quantity  $i$ ; whence it is easy to conclude that this same coefficient in the expansion of  $u \left( \frac{1}{t^i} - 1 \right)^n$  is  ${}^1\Delta^n y_x$ . Moreover, if we expand  $u \left[ \left( 1 + \frac{1}{t} - 1 \right)^i - 1 \right]^n$  according to the powers of  $\frac{1}{t} - 1$ , the coefficients of  $t^x$  in the expansions of  $u \left( \frac{1}{t} - 1 \right)$ ,

$u \left(\frac{1}{t} - 1\right)^2, u \left(\frac{1}{t} - 1\right)^3, \dots$  will be, by article II,  $\Delta y_x, \Delta^2 y_x, \Delta^3 y_x, \dots$ ; so that this coefficient in  $u \left[\left(1 + \frac{1}{t} - 1\right)^i - 1\right]^n$  will be  $[(1 + \Delta y_x)^i - 1]$ , provided that, in the expansion of this quantity, we apply to the characteristic  $\Delta$  the exponents of the powers of  $\Delta y_x$ , and that thus, in place of any power  $(\Delta y_x)^m$ , we write  $\Delta^m y_x$ ; we will have therefore

$$(1) \quad {}^1\Delta^n y_x = [(1 + \Delta y_x)^i - 1]^n.$$

If we designate by the characteristic  ${}^1\Sigma$  the finite integral when  $x$  varies from  $i$ ,  ${}^1\Sigma^n y_x$  will be clearly equal, by article II, to the coefficient of  $t^x$  in the expansion of the function  $u \left(\frac{1}{t^i} - 1\right)^{-n}$ , by setting aside here some arbitrary constants which the integration must introduce; now we have

$$u \left(\frac{1}{t^i} - 1\right)^{-n} = u \left[\left(1 + \frac{1}{t}\right)^i - 1\right]^{-n};$$

moreover, the coefficient of  $t^x$  in  $u \left(\frac{1}{t} - 1\right)^{-m}$  is, whatever be  $m$ ,  $\Sigma^m y_x$ , by setting aside some arbitrary constants, and this coefficient in  $u \left(\frac{1}{t} - 1\right)^m$  is  $\Delta^m y_x$ ; we will have therefore, by always setting aside some arbitrary constants,

$$(2) \quad {}^1\Sigma^n y_x = [(1 + \Delta y_x)^i - 1]^{-n},$$

provided that, in the expansion of the second member of this equation, we apply to the characteristic  $\Delta$  the exponents of the powers of  $\Delta y_x$  and that we change the negative differences to integrals; and, as, in this expansion, the integral  $\Sigma^n y_x$  is encountered, and as this integral can be counted to contain  $n$  arbitrary constants, equation (2) is again true by having regard to the arbitrary constants.

We can observe here that this equation is deduced from equation (1), by making  $n$  negative and by changing the negative differences to integrals, that is by writing  ${}^1\Sigma^n y_x$  in place of  ${}^1\Delta^{-n} y_x$  and  $\Sigma^m y_x$  in place of  $\Delta^{-m} y_x$ .

Equations (1) and (2) would equally hold if  $x$ , instead of varying from unity in  $\Delta y_x$ , varied from any quantity  $\varpi$ ; but then the variation of  $x$  in  ${}^1\Delta y_x$ , instead of being  $i$ , would be  $i\varpi$ . Indeed, it is clear that, if in  $y_x$  we make  $x = \frac{x_1}{\varpi}$ ,  $x_1$  will vary from  $\varpi$  when  $x$  will vary from unity;  $\Delta y_x$  will be changed thus into  $\Delta y_{x_1}$ , the variation of  $x_1$  being  $\varpi$ , and  ${}^1\Delta y_x$  will be changed into  ${}^1\Delta y_{x_1}$ , the variation of  $x_1$  being  $i\varpi$ . This put, if we suppose in these equations that the variation of  $x$  is infinitely small and equal to  $dx$  in  $\Delta y_x$ , this difference will be changed into the infinitely small differential  $dy_x$ ; if, moreover, we make  $i$  infinite and  $idx = \alpha$ ,  $\alpha$  being a finite quantity, the variation of  $x$  in  ${}^1\Delta y_x$  will be  $\alpha$ . We will have therefore

$$\begin{aligned} {}^1\Delta^n y_x &= [(1 + dy_x)^i - 1]^n, \\ {}^1\Sigma^n y_x &= \frac{1}{[(1 + dy_x)^i - 1]^n}; \end{aligned}$$

now we have

$$\log(1 + dy_x)^i = i \log(1 + dy_x) = i dy_x = i dx \frac{dy_x}{dx} = \alpha \frac{dy_x}{dx},$$

that which gives

$$(1 + dy_x)^i = e^{\alpha \frac{dy_x}{dx}},$$

$e$  being the number of which the hyperbolic logarithm is unity; therefore

$$(3) \quad {}^1\Delta^n y_x = \left( e^{\alpha \frac{dy_x}{dx}} - 1 \right)^n,$$

$$(4) \quad {}^1\Sigma^n y_x = \frac{1}{\left( e^{\alpha \frac{dy_x}{dx}} - 1 \right)^n},$$

by taking care to apply to the characteristic  $d$  the exponents of the powers of  $dy_x$  and to change the negative differences to integrals.

If, in equations (1) and (2), we suppose further  $i$  infinitely small and equal to  $dx$ , we will have

$${}^1\Delta^n y_x = d^n y_x \quad \text{and} \quad {}^1\Sigma^n y_x = \frac{1}{dx^n} \int^n y dx^n.$$

We have besides

$$(1 + \Delta y_x)^i = e^{dx \log(1 + \Delta y_x)} = 1 + dx \log(1 + \Delta y_x);$$

these equations will become thus

$$(5) \quad \frac{d^n y_x}{dx^n} = [\log(1 + \Delta y_x)]^n,$$

$$(6) \quad \int^n y_x dx^n = \frac{1}{[\log(1 + \Delta y_x)]^n}.$$

We can remark here a singular analogy between the positive powers and the differences; the equation

$${}^1\Delta y_x = (1 + \Delta y_x)^i - 1$$

holds yet in raising its two members to the power  $n$ , provided that we apply to the characteristics  $\Delta$  and  ${}^1\Delta$  the powers of  $\Delta y_x$  and of  ${}^1\Delta y_x$ , because it is clear that in this case we will have equation (1).

The same analogy subsists between the negative powers and the integrals, and the preceding equation holds still in raising its two members to the power  $-n$ , provided that we change to integrals of the same order the negative powers of  $\Delta y_x$  and of  ${}^1\Delta y_x$ ; we will form thus equation (2).

It is likewise in the equation

$${}^1\Delta y_x = e^{\alpha \frac{dy_x}{dx}} - 1;$$

in raising its two members to the powers  $n$  and  $-n$ , it will still be true and it will be changed into equations (3) and (4), provided that we change the positive powers

of  ${}^1\Delta y_x$  and of  $dy_x$  into differences of the same order, and the negative powers into integrals of the same order. We see, besides, that these analogies hold to that which the products of the function  $u$ , generator of  $y_x$ , with the successive powers of  $\frac{1}{t} - 1$ , are the generating functions of the successive finite differences of  $y_x$ , while the quotients of  $u$  with these same powers are the generating functions of the finite integrals of  $y_x$ .

## XI.

The preceding formulas are able to be of use only in the case where the finite and infinitely small differences of  $y_x$  proceed by decreasing; but there is an infinity of cases in which this does not take place and where it is however useful to have the expression of the differences and of the integrals in convergent series; the simplest of all is that in which the terms of one series, of which the differences are convergent, are multiplied by the terms of a geometric progression: we are going to occupy ourselves with it first.

The general term of the series thus formed can be represented by  $h^x y_x$ ,  $y_x$  being the general term of a series of which the differences are convergent. This put, we name  $u$  the sum of the infinite series

$$y_0 + y_1 h t + y_2 h^2 t^2 + y_3 h^3 t^3 + \cdots + y_{\infty} h^{\infty} t^{\infty};$$

we have

$$u \left( \frac{1}{t^i} - 1 \right)^n = u \left[ h^i \left( 1 + \frac{1}{h t} - 1 \right)^i - 1 \right]^n.$$

The coefficient of  $t^x$ , in the first member of this equation, is the  $n^{\text{th}}$  finite difference of  $h^x y_x$ ,  $x$  varying with the quantity  $i$ ; besides, if we expand the second member with respect to the powers of  $\frac{1}{h t} - 1$ , the coefficient of  $t^x$ , in  $u \left( \frac{1}{h t} - 1 \right)^r$ , will be, whatever be  $r$ ,  $h^x \Delta^r y_x$ . The preceding equation will give therefore, by passing again by article II, from the generating functions to their corresponding variables

$$(7) \quad {}^1\Delta^n h^x y_x = h^x [h^i (1 + \Delta y_x)^i - 1]^n,$$

provided that, in the expansion of the second member of this equation, we apply to the characteristic  $\Delta$  the exponents of the powers of  $\Delta y_x$  and that thus, in place of  $(\Delta y_x)^0$ , we write  $\Delta^0 y_x$ , that is  $y_x$ .

By changing  $n$  into  $-n$ , we will have, as in the preceding article

$$(8) \quad {}^1\Sigma^n (h^x y_x) = \frac{h^x}{[h^i (1 + \Delta y_x)^i - 1]^n} + a x^{n-1} + b x^{n-2} + \cdots + f,$$

$a, b, \dots, f$  being the  $n$  arbitrary constants of the integral of the first member, of which the addition becomes useless in the case where  $h = 1$ , because then the second member contains the integral  $\Sigma^n y_x$ , which it no longer contains when  $h$  differs from unity.

If we suppose  $y_x$  equal to a function  $y_1$  of  $x_1$ ,  $x_1$  being equal to  $\frac{x}{r}$  and  $r$  being supposed infinite, we will have  $\Delta y_x = dy_1$ , the difference  $dx_1$  being equal to  $\frac{1}{r}$ ; moreover, if we make  $h^r = p$ , we will have  $h^x = p^x$ , and the function  $h^x y_x$  will be changed into  $p^{x_1} y_1$ ; now, if we suppose  $i$  infinitely great and  $\frac{i}{r} = \alpha$ , it is clear that,  $x$  varying with  $i$ ,  $x_1$  will vary with  $\alpha$ , in a way that  ${}^1\Delta^n (p^{x_1} y_1)$  and  ${}^1\Sigma^n (p^{x_1} y_1)$  will

be the difference and the  $n^{\text{th}}$  finite integral of  $p^{x_1}y_1$ ,  $x_1$  varying with the quantity  $\alpha$ . We have besides  $h^i = p^\alpha$ ; equations (7) and (8) will become consequently

$$\begin{aligned} {}^1\Delta^n(p^{x_1}y_1) &= p^{x_1}[p^\alpha(1+dy_1)^i - 1]^n, \\ {}^1\Sigma^n(p^{x_1}y_1) &= \frac{p^{x_1}}{[p^\alpha(1+dy_1)^i - 1]^n} + ax_1^{n-1} + bx_1^{n-2} + \dots; \end{aligned}$$

now we have

$$(1+dy_1)^i = e^{\alpha \frac{dy_1}{dx_1}},$$

therefore

$$(9) \quad {}^1\Delta^n(p^{x_1}y_1) = p^{x_1} \left( p^\alpha e^{\alpha \frac{dy_1}{dx_1}} - 1 \right)^n,$$

$$(10) \quad {}^1\Sigma^n(p^{x_1}y_1) = \frac{p^{x_1}}{\left( p^\alpha e^{\alpha \frac{dy_1}{dx_1}} - 1 \right)^n} + ax_1^{n-1} + bx_1^{n-2} + \dots + f,$$

by taking care, in the expansion of these equations, to write  $y_1$  instead of  $\left(\frac{dy_1}{dx_1}\right)^0$  and  $\frac{d^\mu y_1}{dx_1^\mu}$  instead of  $\left(\frac{dy_1}{dx_1}\right)^\mu$ ,  $\mu$  being any whatsoever.

If, in formulas (7) and (8), we suppose  $i$  infinitely small and equal to  $dx$ ,  ${}^1\Delta^n(h^x y_x)$  will be changed into  $d^n(h^x y_x)$  and  ${}^1\Sigma^n(h^x y_x)$  into  $\int^n(h^x y_x)$ ; we have besides

$$h^i(1+\Delta y_x)^i = 1 + dx \log[h(1+\Delta y_x)];$$

hence, we will have

$$(11) \quad \frac{d^n(h^x y_x)}{dx^n} = h^x [\log(1+\Delta y_x)]^n,$$

$$(12) \quad \int^n h^x y_x dx^n = \frac{h^x}{[\log h(1+\Delta y_x)]^n} + ax^{n-1} + bx^{n-2} + \dots + f.$$

I must observe here that equations (1), (2), (3), (4), (5) and (6) of the preceding article have been found by Mr. de la Grange, in the *Mémoires de Berlin* for the year 1771, by means of the analogy which exists between the positive powers and the differences, and between the negative powers and the integrals; but this illustrious author is content to suppose it without giving the demonstration of it, which he regards as very difficult. As for equations (7), (8), (9), (10), (11) and (12), they are new, with the exception of equation (10), of which Mr. Euler has given the particular case where  $n = 1$  in his *Institutions de Calcul différentiel*.

## XII.

We will have an infinity of analogous theorems to those of the preceding articles if, instead of considering the differences and the integrals of  $y_x$ , we considered any other function of this variable; it will be easy to deduce them from the general solution of the following problem:

$\Gamma(y_x)$  representing any linear function of  $y_x, y_{x+1}, y_{x+2}, \dots$ , and  $\nabla y_x$  another linear function of these same variables, we propose to find the expression of  $\Gamma(y_x)$  in a series ordered according to the quantities  $\nabla y_x, \nabla^2 y_x, \nabla^3 y_x, \dots$

For this, let  $u$  be the generating function of  $y_x$ ,  $us$  that of  $\Gamma(y_x)$  and  $uz$  that of  $\nabla y_x$ ,  $s$  and  $z$  being functions of  $\frac{1}{t}$ ; we will begin by drawing from the equation which expresses the relation of  $\frac{1}{t}$  and of  $z$  the value of  $\frac{1}{t}$  in  $z$ , and, by substituting it into  $s$ , we will have the value of  $s$  in  $z$ , but, as it can happen that we have many values of  $\frac{1}{t}$  in  $z$ , we will have as many different expressions of  $s$ . In order to have one which can belong indifferently to all these values of  $s$ , we will suppose that the number of values of  $\frac{1}{t}$  in  $z$  be  $n$ , and we will give to the expression of  $s$  the following form

$$s = Z + \frac{1}{t}Z^{(1)} + \frac{1}{t^2}Z^{(2)} + \dots + \frac{1}{t^{n-1}}Z^{(n-1)},$$

$Z, Z^{(1)}, Z^{(2)}, \dots$  being some functions of  $z$  which the question is to determine; now, if we substitute successively into this equation, in place of  $\frac{1}{t}$ , its  $n$  values in  $z$ , we will form  $n$  equations by means of which we will determine the  $n$  quantities  $Z, Z^{(1)}, Z^{(2)}, \dots$ ; there will no longer be a question next but to reduce these quantities to a series ordered with respect to the powers of  $z$  and to substitute them into the preceding equation. This put, if we multiply this equation by  $u$ , the coefficient of  $t^x$ , in  $us$ , will be  $\Gamma(y_x)$ ; this same coefficient, in any term such as  $\frac{uz^s}{t^r}$ , will be, by article II, equal to  $\nabla^s y_{x+r}$ . The preceding equation will give therefore, by passing again from the generating functions to the corresponding variables, an expression for  $\Gamma(y_x)$  by a series ordered according to the quantities  $\nabla y_x, \nabla^2 y_x, \nabla^3 y_x, \dots, \nabla y_{x+1}, \nabla^2 y_{x+1}, \dots, \nabla y_{x+n-1}, \dots$

We can suppose next, for more generality, that the quantities  $Z^{(1)}, Z^{(2)}, Z^{(3)}, \dots$ , instead of being multiplied by  $\frac{1}{t}, \frac{1}{t^2}, \frac{1}{t^3}, \dots$ , are multiplied by some functions whatever of  $\frac{1}{t}$ , and we will have by this means an infinity of different expressions of  $\Gamma(y_x)$ .

If we suppose

$$s = \frac{1}{t^i}, \quad z = a + \frac{b}{t} + \frac{c}{t^2} + \dots + \frac{q}{t^n},$$

$\Gamma(y_x)$  will be changed into  $y_{x+i}$ ; we will have therefore, by this process, the value of  $y_{x+i}$  in a function of  $\nabla y_x, \nabla^2 y_x, \dots$ ; but the method that we have given for this object in article V is of a much more easy use.

XIII.  
Of series of two variables.

We consider a function  $y_{x,x_1}$  of two variables  $x$  and  $x_1$ , and we name  $u$  the infinite series

$$\begin{aligned} & y_{0,0} + y_{1,0}t + y_{2,0}t^2 + y_{3,0}t^3 + \cdots + y_{x,0}t^x + y_{x+1,0}t^{x+1} + \cdots + y_{\infty,0}t^\infty \\ & + y_{0,1}t_1 + y_{1,1}t_1t + y_{2,1}t_1t^2 + \cdots + y_{x-1,1}t_1t^{x-1} + y_{x,1}t_1t^x + \cdots + y_{\infty,1}t_1t^\infty \\ & + y_{0,2}t_1^2 + y_{1,2}t_1^2t + \cdots + y_{x-2,2}t_1^2t^{x-2} + \cdots + y_{\infty,2}t_1^2t^\infty \\ & + \cdots, \end{aligned}$$

the coefficient of  $t^x t_1^{x_1}$  will be  $y_{x,x_1}$ ; thus  $u$  will be the generating function of  $y_{x,x_1}$ , and, if we designate by the characteristic  $\Delta$  the finite differences when  $x$  alone varies and by the characteristic  $\Delta_1$  those differences when  $x_1$  alone varies, the generating function of  $\Delta y_{x,x_1}$  will be, by article II,  $u\left(\frac{1}{t} - 1\right)$  and that of  $\Delta_1 y_{x,x_1}$  will be  $u\left(\frac{1}{t_1} - 1\right)$ : hence the generating function of  $\Delta\Delta_1 y_{x,x_1}$  will be  $u\left(\frac{1}{t} - 1\right)\left(\frac{1}{t_1} - 1\right)$ , whence it is easy to conclude that that of  $\Delta^i \Delta_1^{i_1} y_{x,x_1}$  will be  $u\left(\frac{1}{t} - 1\right)^i \left(\frac{1}{t_1} - 1\right)^{i_1}$ .

In general, if we designate by  $\nabla y_{x,x_1}$  the quantity

$$\begin{aligned} & Ay_{x,x_1} + By_{x+1,x_1} + Cy_{x+2,x_1} + \cdots \\ & + B_1y_{x,x_1+1} + C_1y_{x+1,x_1+1} + \cdots \\ & + C_2y_{x,x_1+2} + \cdots \\ & + \cdots; \end{aligned}$$

if we designate similarly by  $\nabla^2 y_{x,x_1}$  a function in which  $\nabla y_{x,x_1}$  enters in the same manner as  $y_{x,x_1}$  enters in  $\nabla y_{x,x_1}$ ; if we designate further by  $\nabla^3 y_{x,x_1}$  a function in which  $\nabla^2 y_{x,x_1}$  enters in the same manner as  $y_{x,x_1}$  in  $\nabla y_{x,x_1}$  and thus in sequence, the generating function of  $\nabla^n y_{x,x_1}$  will be

$$u \left( A + \frac{B}{t} + \frac{C}{t^2} + \cdots + \frac{B_1}{t_1} + \frac{C_1}{tt_1} + \cdots + \frac{C_2}{t_1^2} + \cdots \right)^n;$$

hence

$$ut^r t_1^{r_1} \left(\frac{1}{t} - 1\right)^i \left(\frac{1}{t_1} - 1\right)^{i_1} \left( A + \frac{B}{t} + \cdots + \frac{B_1}{t_1} + \cdots \right)^n$$

is the generating function of  $\Delta^i \Delta_1^{i_1} \nabla^n y_{x-r,x_1-r_1}$ .

$s$  being supposed any function of  $\frac{1}{t}$  and of  $\frac{1}{t_1}$ , if we expand  $s^i$  according to the powers of these variables and if we designate by  $\frac{K}{t^m t_1^{m_1}}$  any term of this expansion, the coefficient of  $t^x t_1^{x_1}$  in  $\frac{K u}{t^m t_1^{m_1}}$  will be  $K y_{x+m,x_1+m_1}$ ; we will have therefore the coefficient of  $t^x t_1^{x_1}$  in  $us^i$  or, that which returns to the same, we will have  $\nabla^i y_{x,x_1}$ : 1° by substituting, in  $s$ ,  $y_x$  in place of  $\frac{1}{t}$  and  $y_{x_1}$  in place of  $\frac{1}{t_1}$ ; 2° by expanding that which  $us^i$  then becomes according to the powers of  $y_x$  and of  $y_{x_1}$  and by writing in it



in the place of any term, such as  $K(y_x)^m(y_{x_1})^{m_1}$ ,  $Ky_{x+m,x_1+m_1}$  and, consequently, by substituting  $Ky_{x,x_1}$  in the place of entirely constant term  $K$  or  $K(y_x)^0(y_{x_1})^0$ .

If, instead of expanding  $s^i$  according to the powers of  $\frac{1}{t}$  and  $\frac{1}{t_1}$ , we expand it according to the powers of  $\frac{1}{t}-1$  and  $\frac{1}{t_1}-1$ , and if we designate by  $K\left(\frac{1}{t}-1\right)^m\left(\frac{1}{t_1}-1\right)^{m_1}$  any term of this expansion, the coefficient of  $t^x t_1^{x_1}$  in  $Ku\left(\frac{1}{t}-1\right)^m\left(\frac{1}{t_1}-1\right)^{m_1}$  will be  $K\Delta^m\Delta_1^{m_1}y_{x,x_1}$ ; we will have therefore  $\nabla^i y_{x,x_1}$ : 1° by substituting, in  $s$ ,  $\Delta y_{x,x_1}$  in place of  $\frac{1}{t}-1$  and  $\Delta_1 y_{x,x_1}$  in place of  $\frac{1}{t_1}-1$ ; 2° by expanding that which  $s^i$  then becomes according to the powers of  $\Delta y_{x,x_1}$  and of  $\Delta_1 y_{x,x_1}$  and by applying to the characteristics  $\Delta$  and  $\Delta_1$  the exponents of these powers, that is by writing, in the place of any term such as  $K(\Delta y_{x,x_1})^m(\Delta_1 y_{x,x_1})^{m_1}$ , this one  $K\Delta^m\Delta_1^{m_1}y_{x,x_1}$ .

Let  $\Sigma$  be the characteristic of the finite integrals relative to  $x$  and  $\Sigma_1$  that of the integrals relative to  $x_1$ ; let moreover  $z$  be the generating function of  $\Sigma^i \Sigma_1^{i_1} y_{x,x_1}$ ; we will have  $z\left(\frac{1}{t}-1\right)^i\left(\frac{1}{t_1}-1\right)^{i_1}$  for the generating function of  $y_{x,x_1}$ ; this generating function must, by having regard only to the positive or null powers of  $t$  and  $t_1$ , be reduced to  $u$ ; we will have thus

$$z\left(\frac{1}{t}-1\right)^i\left(\frac{1}{t_1}-1\right)^{i_1} = u + \frac{a}{t} + \frac{b}{t^2} + \frac{c}{t^3} + \dots + \frac{q}{t^i} \\ + \frac{a_1}{t_1} + \frac{b_1}{t_1^2} + \frac{c_1}{t_1^3} + \dots + \frac{q_1}{t_1^{i_1}},$$

$a, b, c, \dots, q$  being some arbitrary functions of  $t_1$  and  $a_1, b_1, c_1, \dots, q_1$  being some arbitrary functions of  $t$ , hence

$$z = \frac{ut^i t_1^{i_1} + at^{i-1} t_1^{i_1} + bt^{i-2} t_1^{i_1} + \dots + qt^{i_1} + a_1 t^i t_1^{i_1-1} + b_1 t^i t_1^{i_1-2} + \dots + q_1 t^i}{(1-t)^i (1-t_1)^{i_1}}.$$

#### XIV.

*On the interpolation of series in two variables and on the integration of equations linear in finite and infinitely small partial differences.*

$y_{x+i,x_1+i_1}$  is evidently equal to the coefficient of  $t^x t_1^{x_1}$  in the expansion of  $\frac{u}{t^i t_1^{i_1}}$ ; now we have

$$\frac{u}{t^i t_1^{i_1}} = u \left(1 + \frac{1-t}{t}\right)^i \left(1 + \frac{1-t_1}{t_1}\right)^{i_1} \\ = u \left\{ \begin{array}{l} 1 + i \frac{1-t}{t} + \frac{i(i-1)}{1.2} \left(\frac{1-t}{t}\right)^2 + \frac{i(i-1)(i-2)}{1.2.3} \left(\frac{1-t}{t}\right)^3 + \dots \\ + i_1 \frac{1-t_1}{t_1} + i_1 i \frac{1-t_1}{t_1} \frac{1-t}{t} + i_1 \frac{i(i-1)}{1.2} \frac{1-t_1}{t_1} \left(\frac{1-t}{t}\right)^2 + \dots \\ + \frac{i_1(i_1-1)}{1.2} \left(\frac{1-t_1}{t_1}\right)^2 + \frac{i_1(i_1-1)}{1.2} \left(\frac{1-t_1}{t_1}\right)^2 \frac{1-t}{t} + \dots \\ + \dots \end{array} \right\},$$

the coefficient of  $u \left(\frac{1}{t} - 1\right)^r \left(\frac{1}{t_1} - 1\right)^{r_1}$  being equal to

$$\frac{i(i-1)(i-2)\cdots(i-r+1)}{1.2.3\dots r} \frac{i_1(i_1-1)(i_1-2)\cdots(i_1-r_1+1)}{1.2.3\dots r_1}.$$

Now, the coefficient of  $t^x t_1^{x_1}$ , in the expansion of  $u \left(\frac{1}{t} - 1\right)^r \left(\frac{1}{t_1} - 1\right)^{r_1}$ , is  $\Delta^r \Delta_1^{r_1} y_{x,x_1}$ ; we will have therefore, by passing from the generating functions to the corresponding variables,

$$\begin{aligned} y_{x+i,x_1+i_1} &= y_{x,x_1} + i\Delta y_{x,x_1} + \frac{i(i-1)}{1.2} \Delta^2 y_{x,x_1} + \cdots \\ &\quad + i_1 \Delta_1 y_{x,x_1} + i_1 i \Delta_1 \Delta y_{x,x_1} + \cdots \\ &\quad + \frac{i_1(i_1-1)}{1.2} \Delta_1^2 y_{x,x_1} + \cdots \\ &\quad + \cdots \end{aligned}$$

an equation which can be put under this very simple form

$$y_{x+i,x_1+i_1} = (1 + \Delta y_{x,x_1})^i (1 + \Delta_1 y_{x,x_1})^{i_1},$$

provided that, in the expansion of the second member of this last equation, we apply to the characteristics  $\Delta$  and  $\Delta_1$  the exponents of the powers of  $\Delta y_{x,x_1}$  and of  $\Delta_1 y_{x,x_1}$  and, consequently, that in the place of the entirely constant term or the term multiplied by  $(\Delta y_{x,x_1})^0 (\Delta_1 y_{x,x_1})^0$ , we write  $y_{x,x_1}$ .

## XV.

We suppose now that, instead of interpolating according to the differences of the function  $y_{x,x_1}$ , we wish to interpolate according to other laws; for that, let

$$\begin{aligned} z &= A + \frac{B}{t} + \frac{C}{t^2} + \frac{D}{t^3} + \cdots + \frac{p}{t^{n-1}} + \frac{q}{t^n} \\ &\quad + \frac{B_1}{t_1} + \frac{C_1}{t_1 t} + \frac{D_1}{t_1 t^2} + \cdots \\ &\quad \quad + \frac{C_2}{t_1^2} + \frac{D_2}{t_1^2 t} + \cdots \\ &\quad \quad \quad + \cdots \\ &\quad \quad \quad \quad + \frac{1}{t_1^{n_1}} \end{aligned}$$

If we make

$$\begin{aligned} A + \frac{B_1}{t_1} + \frac{C_2}{t_1^2} + \cdots + \frac{1}{t_1^{n_1}} &= a, \\ B + \frac{C_1}{t_1} + \frac{D_2}{t_1^2} + \cdots &= b, \\ C + \frac{D_1}{t_1} + \cdots &= c, \\ \cdots, & \end{aligned}$$

we will have

$$z = a + \frac{b}{t} + \frac{c}{t^2} + \cdots + \frac{q}{t^n}.$$

It is easy to conclude from it, as in article V, the successive values of  $\frac{1}{t^{n+1}}, \frac{1}{t^{n+2}}, \frac{1}{t^{n+3}}, \dots$  as functions of  $a, b, c, \dots$  and  $z$ , and it is clear that, in any term of the expression of  $\frac{1}{t^i}$ , the sum of the powers of  $\frac{1}{t}$  and  $\frac{1}{t_1}$  will not surpass  $i$  when  $i$  will be a positive whole number,  $n_1$  being supposed equal or less than  $n$ .

We consider now formula ( $\mu$ ) of article V and we suppose that by expanding it according to the powers of  $\frac{1}{t_1}$  the quantity

$$\begin{aligned} & bZ_{i-n+1}^{(0)} + bzZ_{i-2n+1}^{(1)} + \cdots \\ & + cZ_{i-n+2}^{(0)} + czZ_{i-2n+2}^{(1)} + \cdots \\ & + \cdots \end{aligned}$$

we have

$$M + Nz + \cdots + \frac{1}{t_1}(M^{(1)} + N^{(1)}z + \cdots) + \frac{1}{t_1^2}(M^{(2)} + N^{(2)}z + \cdots) + \cdots + \frac{1}{t_1^i}M^{(i)},$$

the ulterior powers of  $\frac{1}{t^i}$  are destroyed reciprocally, since the expression of  $\frac{1}{t^i}$  must not contain them at all. We suppose similarly that by expanding the quantity

$$cZ_{i-n+1}^{(0)} + czZ_{i-2n+1}^{(1)} + \cdots + eZ_{i-n+2}^{(0)} + ezZ_{i-2n+2}^{(1)} + \cdots$$

we have

$$M_1 + N_1z + \cdots + \frac{1}{t_1}(M_1^{(1)} + N_1^{(1)}z + \cdots) + \frac{1}{t_1^2}(M_1^{(2)} + N_1^{(2)}z + \cdots) + \cdots + \frac{1}{t_1^{i-1}}M_1^{(i-1)};$$

which by expanding the quantity

$$\begin{aligned} & eZ_{i-n+1}^{(1)} + \cdots \\ & + \cdots \end{aligned}$$

we have

$$M_2 + N_2z + \cdots + \frac{1}{t_1}(M_2^{(1)} + N_2^{(1)}z + \cdots) + \cdots + \frac{1}{t_1^{i-2}}M_2^{(i-2)},$$

and thus in sequence; we will have

$$\begin{aligned} \frac{1}{t^i} = & M + Nz + \cdots + \frac{1}{t_1}(M^{(1)} + N^{(1)}z + \cdots) + \frac{1}{t_1^2}(M^{(2)} + N^{(2)}z + \cdots) + \frac{1}{t_1^i}M^{(i)} \\ & + \frac{1}{t} \left[ M_1 + N_1z + \cdots + \frac{1}{t_1}(M_1^{(1)} + N_1^{(1)}z + \cdots) + \frac{1}{t_1^2}(M_1^{(2)} + N_1^{(2)}z + \cdots) + \cdots + \frac{1}{t_1^{i-1}}M_1^{(i-1)} \right] \\ & + \frac{1}{t^2} \left[ M_2 + N_2z + \cdots + \frac{1}{t_1}(M_2^{(1)} + N_2^{(1)}z + \cdots) + \cdots + \frac{1}{t_1^{i-2}}M_2^{(i-2)} \right] \\ & + \cdots \\ & + \frac{1}{t^{n-1}} \left[ M_{n-1} + N_{n-1}z + \cdots + \frac{1}{t_1}(M_{n-1}^{(1)} + N_{n-1}^{(1)}z + \cdots) + \cdots + \frac{1}{t_1^{i-n+1}}M_{n-1}^{(i-n+1)} \right]. \end{aligned}$$

This put, if we name  $\nabla y_{x,x_1}$  the quantity

$$\begin{aligned} & Ay_{x,x_1} + By_{x+1,x_1} + Cy_{x+2,x_1} + \cdots \\ & + B_1 y_{x,x_1+1} + C_1 y_{x+1,x_1+1} + \cdots \\ & + C_2 y_{x,x_1+2} + \cdots \\ & + \cdots ; \end{aligned}$$

the coefficient of  $t^x t_1^{x_1}$  in the expansion of the quantity  $\frac{uz^\mu}{t^r t_1^{r_1}}$  will be, by article XIII,  $\nabla^\mu y_{x+r,x_1+r_1}$ ; the preceding equation will give consequently, by multiplying it by  $u$  and by passing from the generating functions to the corresponding variables,

$$\begin{aligned} y_{x+i,x_1} &= My_{x,x_1} + N\nabla y_{x,x_1} + \cdots \\ & + M^{(1)} y_{x,x_1+1} + N^{(1)} \nabla y_{x,x_1+1} + \cdots \\ & + M^{(2)} y_{x,x_1+2} + N^{(2)} \nabla y_{x,x_1+2} + \cdots \\ & + \cdots \\ & + M^{(i)} y_{x,x_1+i} \\ & + M_1 y_{x+1,x_1} + N_1 \nabla y_{x+1,x_1} + \cdots \\ & + M_1^{(1)} y_{x+1,x_1+1} + N_1^{(1)} \nabla y_{x+1,x_1+1} + \cdots \\ & + \cdots \\ & + M_1^{(i-1)} y_{x+1,x_1+i-1} \\ & + \cdots \\ & + M_{n-1} y_{x+n-1,x_1} + N_{n-1} \nabla y_{x+n-1,x_1} + \cdots \\ & + M_{n-1}^{(1)} y_{x+n-1,x_1+1} + N_{n-1}^{(1)} \nabla y_{x+n-1,x_1+1} + \cdots \\ & + \cdots \\ & + M_{n-1}^{(i-n+1)} y_{x+n-1,x_1+i-n+1}. \end{aligned}$$

## XVI.

If we suppose  $\nabla y_{i,x_1} = 0$ , we will have, by making  $x = 0$  in the preceding equation,

$$\begin{aligned} y_{i,x_1} &= My_{0,x_1} + M^{(1)} y_{0,x_1+1} + M^{(2)} y_{0,x_1+2} + \cdots + M^{(i)} y_{0,x_1+i} \\ & + M_1 y_{1,x_1} + M_1^{(1)} y_{1,x_1+1} + M_1^{(2)} y_{1,x_1+2} + \cdots + M_1^{(i-1)} y_{1,x_1+i-1} \\ & + \cdots \\ & + M_{n-1} y_{n-1,x_1} + M_{n-1}^{(1)} y_{n-1,x_1+1} + \cdots + M_{n-1}^{(i-n+1)} y_{n-1,x_1+i-n+1} \end{aligned}$$

$M^{(r)}$ ,  $M_1^{(r)}$ ,  $M_2^{(r)}$ ,  $\dots$  being some functions of  $i$  and of  $r$ , the preceding expression of  $y_{i,x_1}$  can be taken under this very simple form

$$(\lambda) \quad y_{i,x_1} = \Sigma (M^{(r)} y_{0,x_1+r} + M_1^{(r-1)} y_{1,x_1+r-1} + M_2^{(r-2)} y_{2,x_1+r-2} + \cdots + M_{n-1}^{(r-n+1)} y_{n-1,x_1+r-n+1},$$

the integral being taken with respect to  $r$ , from  $r = 0$  to  $r = i + 1$ , with respect to the first term; from  $r = 1$  to  $r = i + 1$  with respect to the second term, and thus in sequence. This expression for  $y_{i,x_1}$  will be the complete integral of the equation  $\nabla y_{x,x_1} = 0$ , or, that which returns to the same, of this

$$\begin{aligned} 0 = & Ay_{i,x_1} + By_{i+1,x_1} + Cy_{i+2,x_1} + \cdots + Py_{i+n-1,x_1} + qy_{i+n,x_1} \\ & + B_1y_{i,x_1+1} + C_1y_{i+1,x_1+1} + \cdots \\ & + C_2y_{i,x_1+2} + \cdots \\ & + \cdots \\ & + y_{i,x_1+n}. \end{aligned}$$

It is clear that in this integral the quantities  $y_{0,x_1}, y_{1,x_1}, y_{2,x_1}, \dots, y_{n-1,x_1}$  are the  $n$  arbitrary functions which the integration of the equations  $\nabla y_{i,x_1} = 0$  introduces, it is necessary to know immediately, or at least to be able to conclude from conditions of the problem the first  $n$  vertical ranks of the following Table:

$$(Q) \quad \begin{cases} y_{0,0}, & y_{1,0}, & y_{2,0}, & y_{3,0}, & \cdots, & y_{x,0}, & y_{x+1,0}, & \cdots, & y_{\infty,0}, \\ y_{0,1}, & y_{1,1}, & y_{2,1}, & y_{3,1}, & \cdots, & y_{x,1}, & y_{x+1,1}, & \cdots, & y_{\infty,1}, \\ y_{0,2}, & y_{1,2}, & y_{2,2}, & y_{3,2}, & \cdots, & y_{x,2}, & y_{x+1,2}, & \cdots, & y_{\infty,2}, \\ \cdots, & \cdots, & \cdots, & \cdots, & \cdots, & \cdots, & \cdots, & \cdots, & \cdots, \\ y_{0,x_1}, & y_{1,x_1}, & y_{2,x_1}, & y_{3,x_1}, & \cdots, & y_{x,x_1}, & y_{x+1,x_1}, & \cdots, & y_{\infty,x_1}, \\ \cdots, & \cdots, & \cdots, & \cdots, & \cdots, & \cdots, & \cdots, & \cdots, & \cdots, \end{cases}$$

*Remark.*— In a great number of problems, and principally in those which concern the analysis of chances, the first  $n$  vertical ranks are recurrent series of which the law is known; in this case  $y_{0,x_1}, y_{1,x_1}, \dots$  are given by some terms of the form  $Ap^{x_1}$ . We suppose consequently that the expression of  $y_{0,x_1}$  contains the term  $Ap^{x_1}$ , the corresponding part of  $\Sigma M^{(r)}y_{0,x_1+r}$  will be

$$Ap^{x_1}(M^{(0)} + M^{(1)}p + M^{(2)}p^2 + M^{(3)}p^3 + \cdots + M^{(i)}p^i);$$

but

$$M^{(0)} + \frac{M^{(1)}}{t} + \frac{M^{(2)}}{t_1} + \frac{M^{(3)}}{t_1^2} + \cdots + \frac{M^{(i)}}{t_1^i}$$

is the expansion of

$$bZ_{i-n+1}^{(0)} + cZ_{i-n+2}^{(0)} + \cdots$$

according to the powers of  $\frac{1}{t_1}$ . By changing therefore  $\frac{1}{t_1}$  in this last quantity into  $p$  and naming  $P$  that which it then becomes, we will have  $APp^{x_1}$  for the part of  $\Sigma M^{(r)}y_{0,x_1+r}$  which corresponds to the term  $Ap^{x_1}$ . It follows thence that, if the value of  $y_{0,x_1}$  is equal to  $Ap^{x_1} + A_1p_1^{x_1} + A_2p_2^{x_1} + \cdots$  and if we name  $P_1, P_2, \dots$  that which  $P$  becomes, by changing successively  $p$  into  $p_1, p_2, \dots$ , we will have

$$\Sigma M^{(r)}y_{0,x_1+r} = APp^{x_1} + A_1P_1p_1^{x_1} + A_2P_2p_2^{x_1} + \cdots$$

We will find similarly that, if the value of  $y_{1,x_1}$  is expressed by  $Bq^{x_1} + B_1q_1^{x_1} + B_2q_2^{x_1} + \cdots$ , and if we name  $Q, Q_1, Q_2, \dots$  that which the quantities  $cZ_{i-n+1}^{(0)} + eZ_{i-n+2}^{(0)} + \cdots$

become when we change successively  $\frac{1}{t_1}$  into  $q, q_1, q_2, \dots$ , we will have

$$\Sigma M^{(r-1)} y_{1,x_1+r-1} = BQq^{x_1} + B_1Q_1q_1^{x_1} + B_2Q_2q_2^{x_1} + \dots$$

and thus in sequence; we will have thus the most simple expression of  $y_{i,x_1}$  to which we can arrive.

If we have  $\nabla^2 y_{x,x_1} = 0$ , we will have, by making  $x = 0$  in the general expression of  $y_{x+i,x_1}$  of the preceding article,

$$\begin{aligned} y_{i,x_1} = & My_{0,x_1} + M^{(1)}y_{0,x_1+1} + \dots + M^{(i)}y_{0,x_1+i} \\ & + N\nabla y_{0,x_1} + N^{(1)}\nabla y_{0,x_1+1} + \dots \\ & + M_1y_{1,x_1} + M_1^{(1)}y_{1,x_1+1} + \dots + M_1^{(i-1)}y_{1,x_1+i-1} \\ & + N_1\nabla y_{1,x_1} + N_1^{(1)}\nabla y_{1,x_1+1} + \dots \\ & + \dots \\ & + M_{n-1}y_{n-1,x_1} + M_{n-1}^{(1)}y_{n-1,x_1+1} + \dots + M_{n-1}^{(i-n+1)}y_{n-1,x_1+i-n+1} \\ & + N_{n-1}\nabla y_{n-1,x_1} + \dots, \end{aligned}$$

$y_{0,x_1}, y_{1,x_1}, \dots, y_{n-1,x_1}, \nabla y_{0,x_1}, \nabla y_{1,x_1}, \dots, \nabla y_{n-1,x_1}$  being the  $2n$  arbitrary functions of the integral of the equation  $\nabla^2 y_{i,x_1} = 0$ ; we will have, in the same manner, the integrals of the equations  $\nabla^3 y_{i,x_1} = 0, \nabla^4 y_{i,x_1} = 0, \dots$

I have named elsewhere (*see* Volumes VI and VII of the *Mémoires des Savants étrangers*<sup>2</sup>) the series formed according to the equation  $\nabla^r y_{i,x} = 0$  *récurro-récurrentes series*; they differ from recurrent series, in that in those the terms are functions only one variable alone: thus, all their terms in Table (Q) are either in one same vertical rank, or in one same horizontal rank, or on one same straight line inclined to the horizon in any manner, instead that the terms of a *récurro-récurrente* series, being functions of two variables, fill all the extent of Table (Q) and form a surface, such that the arbitrary quantities, which, in the case of recurrent series, are determined by as many points of the line on which their terms are disposed, are determined here by the straight lines or by some polygons placed arbitrarily in the preceding Table. The equation which expresses the law of a recurrent series is in finite differences; that which expresses the law of a *récurro-récurrente* series is in partial finite differences, and its integral contains a number of arbitrary functions equal to the degree of that equation.

## XVII.

The value of  $y_{i,x_1}$  in formula ( $\lambda$ ) of the preceding article, depending on the knowledge of  $M^{(r)}, M_1^{(r)}, \dots$ , it is clear that these quantities will be known when we have the coefficient of  $\frac{1}{t_1^r}$  in the expansion of  $Z_{i-n+1}^{(0)}$ ; all is reduced therefore to determining

<sup>2</sup>*Oeuvres de Laplace*, T. VIII, p. 5 and p. 69, "Mémoire sur les suites récurro-récurrentes et sur leurs usages dans la théorie des hasards" and "Recherches sur l'intégration des équations différentielles aux différences finies, et sur leur usage dans la théorie des hasards."

this coefficient; now we have, by article V,

$$Z_i^{(0)} = -\frac{1}{a\alpha^{i+1}(\alpha - \alpha_1)(\alpha - \alpha_2)\cdots} \\ -\frac{1}{a\alpha_1^{i+1}(\alpha_1 - \alpha_1)(\alpha_1 - \alpha_2)\cdots} \\ -\frac{1}{a\alpha_2^{i+1}(\alpha_2 - \alpha_1)(\alpha_2 - \alpha_1)\cdots} \\ - \dots$$

$\alpha, \alpha_1, \alpha_2, \dots$  being functions of  $\frac{1}{t_1}$ . If we make  $\frac{1}{t_1} = s$ , and if we differentiate the preceding expression of  $Z_i^{(0)}$ ,  $n$  times in sequence with respect to  $s$ , we will have  $n + 1$  equations, by means of which, by eliminating the  $n$  quantities  $\alpha^i, \alpha_1^i, \alpha_2^i, \dots$ , we will arrive to an equation among  $Z_i^{(0)}, \frac{dZ_i^{(0)}}{ds}, \frac{d^2Z_i^{(0)}}{ds^2}, \dots$ , of which the coefficients will be functions of  $\alpha_1, \alpha_2, \dots$  and of their differences; now it is clear that  $\alpha, \alpha_1, \alpha_2, \dots$  must enter in the same manner in these coefficients; we can therefore, by the known methods, determine them as rational functions of the coefficients of the equation which gives the values of  $\alpha, \alpha_1, \dots$  and of the differences of these coefficients, and, consequently, as rational functions of  $s$ ; by making next the denominators of these functions disappear, we will have a linear equation between  $Z_i^{(0)}$  and its differentials, of which the coefficients will be some rational and entire functions of  $s$ , or, that which returns to the same, of  $\frac{1}{t_1}$ . This put, we will consider any term of this equation, such as  $\frac{K}{t_1^m} \frac{d^\mu Z_i^{(0)}}{ds^\mu}$ , and name  $\lambda_r$  the coefficient of  $\frac{1}{t_1^r}$  in the expansion of  $Z_i^{(0)}$ ; this coefficient in the expansion of  $\frac{K}{t_1^m} \frac{d^\mu Z_i^{(0)}}{ds^\mu}$  will be

$$K(r - m + \mu)(r - m + \mu - 1)(r - m + \mu - 2)\cdots(r - m)\lambda_{r-m+\mu}.$$

By thus passing thus from the generating functions to their corresponding variables, the preceding equation between  $Z_{i-n+1}^{(0)}$  and its differences will give an equation among  $\lambda_r, \lambda_{r+1}, \dots$  of which the coefficients are variables, and, by integrating it, we will have the value of  $\lambda_r$ .

It follows thence that integration of every linear equation in finite partial differences, of which the coefficients are constants, depend: 1° on the integration of a linear equation in finite differences of which the coefficients are variables; 2° of a *definite* integral; I name thus any integral taken from one determined value of the variable to another determined value of the variable. The definite integral on which the value of  $y_{i,x_1}$  in formula ( $\lambda$ ) depends is relative to  $r$  and must be extended from  $r = 0$  to  $r = i$ . Relatively to the differential equations of the first order, we have

$$Z_i^{(0)} = -\frac{1}{a\alpha^{i+1}};$$

we have, moreover

$$a = A + B_1s, \\ \alpha = -\frac{B}{a},$$

that which gives

$$Z_i^{(0)} = -\frac{(A + B_1 s)^i}{(-B)^{i+1}},$$

whence we deduce this differential equation

$$0 = \frac{dZ_i^{(0)}}{ds}(A + B_1 s)^i - iB_1 Z_i^{(0)},$$

that which gives the equation in finite differences

$$0 = A(r + 1)\lambda_{r+1} + B_1 r \lambda_r - iB_1 \lambda_r.$$

We have next, in this case,

$$M^{(r)} = B\lambda_r;$$

formula ( $\lambda$ ) of the preceding article will become therefore

$$y_{i,x_1} = B\Sigma\lambda_r y_{0,x_1+r};$$

this will be the complete integral of the equation in partial differences

$$0 = Ay_{i,x_1} + By_{i+1,x_1} + B_1 y_{i,x_1+1},$$

provided that the integral be taken from  $r = 0$  to  $r = i + 1$ , and that the arbitrary constant of the value of  $\lambda_r$  be such that

$$\lambda_0 = -\frac{A^i}{(-B)^{i+1}}.$$

In passing from the finite to the infinitely small, the preceding method will give the integral of the equations linear in infinitely small partial differences of which the coefficients are constants: 1° by integrating a linear equation in infinitely small differences; 2° by means of definite integrals, that which give the integration of these equations in an infinity of cases which resist the known methods; but, as the passage from the finite to the infinitely small can offer here some difficulties, I have preferred to seek a method directly applicable to equations linear in infinitely small partial differences, and I have found the following, which has the advantage of extending itself to the linear equations of which the coefficients are variables. I will limit myself to consider the differential equations of the second order as being those which present themselves the most frequently in the application of analysis to physical questions.

### XVIII.

All equations linear in the infinitely small partial differences of the second order can be put under this form

$$(S) \quad 0 = \frac{\partial^2 u}{\partial s \partial s_1} + m \frac{\partial u}{\partial s} + n \frac{\partial u}{\partial s_1} + lu,$$



$m$ ,  $n$  and  $l$  being any given functions of  $s$  and of  $s_1$ , and, if we name  $\phi_1(s)$  the integral  $\int ds \phi(s)$ ,  $\phi_2(s)$  the integral  $\int ds \phi_1(s)$ ,  $\phi_3(s)$  the integral  $\int ds \phi_2(s)$ , and thus in sequence; if we name similarly  $\psi_1(s_1)$  the integral  $\int ds_1 \psi(s_1)$ ,  $\psi_2(s_1)$  the integral  $\int ds_1 \psi_1(s_1)$ , and thus in sequence, the value of  $u$  can be expressed by a series of this form

$$u = A\phi_1(s) + A^{(1)}\phi_2(s) + A^{(2)}\phi_3(s) + A^{(3)}\phi_4(s) + \dots \\ + B\psi_1(s_1) + B^{(1)}\psi_2(s_1) + B^{(2)}\psi_3(s_1) + B^{(3)}\psi_4(s_1) + \dots$$

$\phi(s)$  and  $\psi(s_1)$  being two arbitrary functions, the one of  $s$  and the other of  $s_1$  (*see on this the Mémoires de l'Académie for the year 1773, p. 355 and following.*<sup>3</sup>) This put, if we substitute this value of  $u$  into equation (S) and if we compare separately the terms multiplied by  $\phi(s)$ ,  $\phi_1(s)$ ,  $\phi_2(s)$ , ...,  $\psi(s_1)$ ,  $\psi_1(s_1)$ ,  $\psi_2(s_1)$ , ..., we will have, in order to determine  $A$ ,  $A^{(1)}$ ,  $A^{(2)}$ , ...,  $B$ ,  $B^{(1)}$ ,  $B^{(2)}$ , ..., the following equations:

$$(\gamma) \quad \begin{cases} 0 = \frac{\partial A}{\partial s_1} + mA, \\ 0 = \frac{\partial A^{(1)}}{\partial s_1} + mA^{(1)} + \frac{\partial^2 A}{\partial s \partial s_1} + m \frac{\partial A}{\partial s} + n \frac{\partial A}{\partial s_1} + lA, \\ 0 = \frac{\partial A^{(2)}}{\partial s_1} + mA^{(2)} + \frac{\partial^2 A^{(1)}}{\partial s \partial s_1} + m \frac{\partial A^{(1)}}{\partial s} + n \frac{\partial A^{(1)}}{\partial s_1} + lA^{(1)}, \\ \dots \end{cases}$$

$$(\gamma') \quad \begin{cases} 0 = \frac{\partial B}{\partial s} + nB, \\ 0 = \frac{\partial B^{(1)}}{\partial s} + nB^{(1)} + \frac{\partial^2 B}{\partial s \partial s_1} + m \frac{\partial B}{\partial s} + n \frac{\partial B}{\partial s_1} + lB, \\ 0 = \frac{\partial B^{(2)}}{\partial s} + nB^{(2)} + \frac{\partial^2 B^{(1)}}{\partial s \partial s_1} + m \frac{\partial B^{(1)}}{\partial s} + n \frac{\partial B^{(1)}}{\partial s_1} + lB^{(1)}, \\ \dots \end{cases}$$

When, in satisfying these equations, we succeed to find  $A^{(\mu)} = 0$  or  $B^{(\mu)} = 0$ ,  $\mu$  being a positive whole number, then  $u$  can always be expressed in finite terms, by having regard only to the variables  $s$  and  $s_1$  alone of the equation. I have given in the Mémoires cited a general and quite simple method to have in this case the complete integral of this equation; but, if one or the other of the two equations  $A^{(\mu)} = 0$  and  $B^{(\mu)} = 0$  cannot hold, there must be necessarily, in order to have the expression of  $u$  in finite terms, introduced a new variable in the following manner.

For this, we will observe that, if we make the integral  $\int ds \phi(s)$  begin when  $s = 0$ , we will have

$$\int ds \phi(s) = ds\{\phi(0) + \phi(ds) + \phi(2ds) + \phi(3ds) + \dots \\ + \phi(r ds) + \phi[(r+1) ds] + \dots + \phi(s)\};$$

<sup>3</sup>Oeuvres de Laplace, T. IX, p. 21

therefore, if we name  $T$  the series

$$\begin{aligned} & \phi(0) + t\phi(ds) + t^2\phi(2ds) + t^3\phi(3ds) + \dots \\ & + t^r\phi(r ds) + t^{r+1}\phi[(r+1) ds] + \dots + t^{\frac{s}{ds}}\phi(s), \end{aligned}$$

$\int ds \phi(s)$  or  $\phi_1(s)$  will be equal to the coefficient of  $t^{\frac{s}{ds}}$  in the expansion of the function  $\frac{T ds}{1-t}$ . It is easy to conclude that  $\phi_2(s)$  will be equal to the coefficient of  $t^{\frac{s}{ds}}$  in the expansion of  $\frac{T ds^2}{(1-t)^2}$ , and, generally, that  $\phi_\mu(s)$  will be equal to the coefficient of  $t^{\frac{s}{ds}}$  in the expansion of  $\frac{T ds^\mu}{(1-t)^\mu}$ ; moreover, it is clear that the coefficient of  $\phi(r ds)$  in  $\phi_\mu(s)$  is equal to the coefficient of  $t^{\frac{s}{ds}-r}$  in the expansion of  $\frac{ds^\mu}{(1-t)^\mu}$ , and consequently equal to

$$\frac{\left(\frac{s}{ds} - r + 1\right) \left(\frac{s}{ds} - r + 2\right) \left(\frac{s}{ds} - r + 3\right) \dots \left(\frac{s}{ds} - r + \mu - 1\right) ds^\mu}{1.2.3 \dots (\mu - 1)}.$$

We suppose  $r$  infinite and equal to  $\frac{z}{ds}$ , we will have  $\frac{(s-z)^{\mu-1} ds}{1.2.3 \dots (\mu-1)}$  for this coefficient; whence it follows that the coefficient of  $\phi(r ds)$  or  $\phi(z)$  in the expression of  $u$  will be

$$ds \left[ A + A^{(1)}(s-z) + \frac{A^{(2)}}{1.2} (s-z)^2 + \frac{A^{(3)}}{1.2.3} (s-z)^3 + \frac{A^{(4)}}{1.2.3.4} (s-z)^4 + \dots \right];$$

therefore, if we name  $\Gamma(s-z)$  the sum of the series

$$A + A^{(1)}(s-z) + \frac{A^{(2)}}{1.2} (s-z)^2 + \frac{A^{(3)}}{1.2.3} (s-z)^3 + \dots$$

and if we suppose  $ds = dz$ , we will have  $\int dz \Gamma(s-z)\phi(z)$  equal to the series

$$A\phi_1(s) + A^{(1)}\phi_2(s) + A^{(2)}\phi_3(s) + A^{(3)}\phi_4(s) + \dots,$$

provided that the integral is taken from  $z = 0$  to  $z = s$ .

If we name similarly  $\Pi(s_1 - z)$  the sum of the series

$$B + B^{(1)}(s_1 - z) + \frac{B^{(2)}}{1.2} (s_1 - z)^2 + \frac{B^{(3)}}{1.2.3} (s_1 - z)^3 + \dots$$

we will find, by the same process, that  $\int dz \Pi(s_1 - z)\psi(z)$  is equal to the series

$$B\psi_1(s_1) + B^{(1)}\psi_2(s_1) + B^{(2)}\psi_3(s_1) + \dots,$$

provided that the integral be taken from  $z = 0$  to  $z = s_1$ ; we will have therefore

$$u = \int dz \Gamma(s-z)\phi(z) + \int dz \Pi(s_1 - z)\psi(z),$$

the integral of the first term being taken from  $z = 0$  to  $z = s$ , and that of the second term being taken from  $z = 0$  to  $z = s_1$ . We can observe here that the functions

$\Gamma(s - z)$  and  $\Pi(s_1 - z)$  are as well particular values which satisfy for  $u$  in the proposed equation in partial differences. Indeed, it is clear, by the nature of the values of  $A, A^{(1)}, A^{(2)}, \dots$ , that, if we substitute into this equation, in place of  $u$ , the series

$$A + A^{(1)}(s - z) + \frac{A^{(2)}}{1.2}(s - z)^2 + \dots,$$

$z$  being regarded as constant, it will be satisfied. But, among all the particular values of  $u$  which contain an arbitrary constant  $z$ , we must choose for  $\Gamma(s - z)$  that which gives  $0 = \frac{\partial u}{\partial s_1} + mu$  when  $z = s$ , because then  $u$  is reduced to  $A$ , and that we must have  $0 = \frac{\partial A}{\partial s_1} + mA$ ; it is necessary similarly to choose for  $\Pi(s_1 - z)$  a particular value of  $u$  which contains an arbitrary constant  $z$ , and in which we have  $0 = \frac{\partial u}{\partial s} + nu$  when  $z = s'$ , because in this case  $u$  is reduced to  $B$  and that we must have  $0 = \frac{\partial B}{\partial s} + nB$ . We can arrive directly to these results in the following manner:

We suppose that the integral  $\int p dz \phi(z)$ , taken from  $z$  equal to any constant to  $z = s$ , is a particular value of  $u$ ; we will have, in this case,

$$\begin{aligned} \frac{\partial u}{\partial s_1} &= \int \frac{\partial p}{\partial s_1} dz \phi(z), \\ \frac{\partial u}{\partial s} &= \int \frac{\partial p}{\partial s} dz \phi(z) + P\phi(s), \end{aligned}$$

$P$  being that which  $p$  becomes when we make  $z = s$ ; thence we will deduce

$$\frac{\partial^2 u}{\partial s \partial s_1} = \int \frac{\partial^2 p}{\partial s \partial s_1} dz \phi(z) + \frac{\partial P}{\partial s_1} \phi(s).$$

By substituting these values into equation (S) in partial differences, we will have

$$0 = \left( \frac{\partial P}{\partial s_1} + mP \right) \phi(s) + \int dz \phi(z) \left( \frac{\partial^2 p}{\partial s \partial s_1} + m \frac{\partial p}{\partial s} + n \frac{\partial p}{\partial s_1} + lp \right),$$

that which gives, by equating separately to zero the terms affected by the integral sign,

$$\begin{aligned} 0 &= \frac{\partial P}{\partial s_1} + mP, \\ 0 &= \frac{\partial^2 p}{\partial s \partial s_1} + m \frac{\partial p}{\partial s} + n \frac{\partial p}{\partial s_1} + lp. \end{aligned}$$

We see thus that, if we have two particular values of  $u$  represented by  $p$  and  $p_1$ , which contains an arbitrary constant  $z$ , and which are such that we have

$$\begin{aligned} 0 &= \frac{\partial P}{\partial s_1} + mP, \\ 0 &= \frac{\partial P_1}{\partial s} + nP_1, \end{aligned}$$

$P$  being that which  $p$  becomes when we make  $z = s$ , and  $P_1$  being that which  $p_1$  becomes when we make  $z = s_1$ , we will have, for the complete expression of  $u$ ,

$$u = \int p dz \phi(z) + \int p_1 dz \psi(z),$$

$\phi(z)$  and  $\psi(z)$  being two arbitrary functions of  $z$ , and the integral of the first term being taken from  $z$  equal to any constant, which we will suppose zero, to  $z = s$ , that of the second term being taken from  $z = 0$  to  $z = s$ , that of the second term being taken from  $z = 0$  to  $z = s_1$ .

If we change  $z$  into  $st$  in the term  $\int p dz \phi(z)$ , and if we name  $q$  that which  $p$  becomes by this change, we will have

$$\int p dz \phi(z) = \int qs dt \phi(st),$$

and, as the integral relative to  $z$  must be taken from  $z = 0$  to  $z = s$ , it is clear that the integral relative to  $t$  must be taken from  $t = 0$  to  $t = 1$ . If we name similarly  $q_1$  that which  $p_1$  becomes when we change  $z$  into  $s_1t$ , we will have

$$\int p_1 dz \psi(z) = \int q_1 s_1 dt \psi(s_1t),$$

the integral relative to  $t$  being taken again from  $t = 0$  to  $t = 1$ ; we can consequently give to  $u$  this form

$$u = \int dt [sq \phi(st) + s_1 q_1 \psi(s_1t)],$$

the integral being taken from  $t = 0$  to  $t = 1$ .

If we name  $K$  the integral  $\int p dz \phi(z)$  taken from  $z = 0$  to  $z = \infty$ ; this integral, taken from  $z = 0$  to  $z = s$ , will be clearly equal to  $K - \int p dz \phi(z)$ , this last integral being taken from  $z = s$  to  $z = \infty$ ; therefore, if we make  $z = s + z_1$ , and if we name  $r$  that which  $p$  becomes by this change, we will have

$$\int p dz \phi(z) = K - \int r dz_1 \phi(s + z_1),$$

the integral relative to  $z$  being taken from  $z = 0$  to  $z = s$ , and the integral relative to  $z_1$  being taken from  $z_1 = 0$  to  $z_1 = \infty$ . If we name similarly  $K_1$  the integral  $\int p_1 dz \psi(z)$  taken from  $z = 0$  to  $z = \infty$ , if we make  $z = s_1 + z_1$ , and if we name  $r_1$  that which  $p_1$  becomes by this change, we will have

$$\int p_1 dz \psi(z) = K_1 - \int r_1 dz_1 \psi(s_1 + z_1),$$

the integral relative to  $z$  being taken from  $z = 0$  to  $z = s_1$ , and the integral relative to  $z_1$  being taken from  $z_1 = 0$  to  $z_1 = \infty$ ; we will have therefore

$$u = K + K_1 - \int dz_1 [r \phi(s + z_1) + r_1 \psi(s_1 + z_1)].$$

The functions  $\phi(s + z_1)$  and  $\psi(s_1 + z_1)$  being arbitrary and even being able to be supposed discontinuous, we can, without harm to the generality of this value of  $u$ , suppose them so that we have  $K + K_1 = 0$ ; we will have therefore, by changing the sign of these functions,

$$u = \int dz_1 [r \phi(s + z_1) + r_1 \psi(s_1 + z_1)],$$

the integral being taken from  $z_1 = 0$  to  $z_1 = \infty$ .

These different forms which we can give to  $u$  have each some particular advantages, according to the different problems which we are proposed to solve. We will see below (art. XX) a use of the last in the theory of sound; but we must observe that they are all dependent on definite integrals and that they can be restored to some indefinite integrals only in the case where one or the other of the quantities  $p$  and  $p_1$  is a rational and entire function of  $z$ .

Every difficulty of the integration of equations linear in the partial differences of the second order is reduced thus to determine these quantities; it is that which seems very difficult in general: we will limit ourselves to consider some particular cases which are relative to many interesting problems which we have been able to solve yet only in a particular manner.

### XIX.

We suppose first  $m, n$  constants in equation (S); we will satisfy equations ( $\gamma$ ) and ( $\gamma'$ ) of the preceding article by making

$$\begin{aligned}
 A &= e^{-ms_1 - ns}, \\
 A^{(1)} &= e^{-ms_1 - ns} (mn - l) s_1, \\
 A^{(2)} &= e^{-ms_1 - ns} \frac{(mn - l)^2}{1.2} s_1^2, \\
 &\dots, \\
 A^{(\mu)} &= e^{-ms_1 - ns} \frac{(mn - l)^\mu}{1.2.3 \dots \mu} s_1^\mu, \\
 &\dots, \\
 B &= e^{-ms_1 - ns}, \\
 B^{(1)} &= e^{-ms_1 - ns} (mn - l) s, \\
 B^{(2)} &= e^{-ms_1 - ns} \frac{(mn - l)^2}{1.2} s^2, \\
 &\dots, \\
 B^{(\mu)} &= e^{-ms_1 - ns} \frac{(mn - l)^\mu}{1.2.3 \dots \mu} s^\mu, \\
 &\dots,
 \end{aligned}$$

$e$  being the number of which the hyperbolic logarithm is unity; we will have thus

$$\Gamma(s - z) = e^{-ms_1 - ns} \left[ 1 + (mn - l) s_1 (s - z) + \frac{(mn - l)^2}{1.2} s_1^2 (s - z)^2 + \frac{(mn - l)^3}{1.2.3} s_1^3 (s - z)^3 + \dots \right],$$

so that  $\Gamma(s - z)$  is equal to a function of  $s_1(s - z)$  multiplied by  $e^{-ms_1 - ns}$ . Let  $y$  be this function and we name  $\theta$  the quantity  $s_1(s - z)$ ;  $e^{-ms_1 - ns} y$  will be, by that which

precedes, a particular integral of the proposed equation in partial differences. We will substitute it therefore for  $u$  in this equation and we will observe that, in this case,

$$\frac{\partial u}{\partial s} = -ne^{-ms_1 - ns}y + e^{-ms_1 - ns}\frac{\partial y}{\partial \theta}\frac{\partial \theta}{\partial s};$$

now we have

$$\frac{\partial \theta}{\partial s} = s_1,$$

hence

$$\frac{\partial u}{\partial s} = e^{-ms_1 - ns} \left( -ny + s_1 \frac{\partial y}{\partial \theta} \right).$$

We will have similarly

$$\begin{aligned} \frac{\partial u}{\partial s_1} &= e^{-ms_1 - ns} \left[ -my + (s - z) \frac{\partial y}{\partial \theta} \right], \\ \frac{\partial^2 u}{\partial s \partial s_1} &= e^{-ms_1 - ns} \left[ mny - n(s - z) \frac{\partial y}{\partial \theta} - ms_1 \frac{\partial y}{\partial \theta} + \frac{\partial y}{\partial \theta} + \theta \frac{\partial^2 y}{\partial \theta^2} \right]. \end{aligned}$$

If we substitute these values into equation (S), we will have this in the ordinary differences

$$0 = (l - mn)y + \frac{\partial y}{\partial \theta} + \theta \frac{\partial^2 y}{\partial \theta^2},$$

and it will be necessary to determine the two arbitrary constants of its integral in a manner that we have  $y = 1$  and  $\frac{\partial y}{\partial \theta} = mn - l$  when  $\theta = 0$ . Let  $\mathcal{I}(\theta)$  be that which this integral becomes, we will have

$$\Gamma(s - z) = e^{-ms_1 - ns} \mathcal{I}[s_1(s - z)];$$

it is easy to see that we will have similarly

$$\Pi(s_1 - z) = e^{-ms_1 - ns} \mathcal{I}[s(s_1 - z)],$$

hence

$$u = e^{-ms_1 - ns} \left\{ \int dz \mathcal{I}[s_1(s - z)] \phi(z) + \int dz \mathcal{I}[s(s_1 - z)] \psi(z) \right\},$$

the integral of the first term being taken from  $z = 0$  to  $z = s$ , and the integral of the second term being taken from  $z = 0$  to  $z = s_1$ . Indeed, if we substitute this value of  $u$  into the proposed equation in partial differences, we will be assured easily that it satisfies it; but, in order to make this substitution, we must observe in general that, the integral  $\int u dz$  must be taken from  $z = 0$  to  $z = s$ , its difference taken with respect to  $s$  is  $ds \int \frac{du}{ds} dz + U ds$ ,  $U$  being that which  $u$  becomes when we suppose  $z = s$ .

If,  $l$ ,  $m$  and  $n$  being always supposed constants, we have  $l - mn = 0$ , we will have  $y = 1$ , and the expression of  $u$  will become

$$u = e^{-ms_1 - ns} \left[ \int dz \phi(z) + \int dz \psi(z) \right] = e^{-ms_1 - ns} [\phi_1(s_1) + \psi_1(s_1)],$$

so that the value of  $u$  is then independent of any definite integral. But this case is the sole where this can take place, and it is that which results similarly from that which has been demonstrated in the *Mémoires de l'Académie*, year 1773, page 369.<sup>4</sup>

The equation of the vibrating strings in a medium resistant as the speed is

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + b \frac{\partial u}{\partial t},$$

$u$  being the ordinate of the vibrating string of which the abscissa is  $x$ ,  $t$  representing the time, and  $a$  and  $b$  being two dependent constants, the one of the size and of the tension of the string, and the other of the intensity of the resistance. If we make  $at + x = s$  and  $at - x = s_1$ , the preceding equation will become

$$0 = a^2 \frac{\partial^2 u}{\partial s \partial s_1} + \frac{b}{4a} \frac{\partial u}{\partial s} + \frac{b}{4a} \frac{\partial u}{\partial s_1};$$

the preceding expression of  $u$  will become then, by substituting in the place of  $s$  and of  $s_1$ , their values  $at + x$  and  $at - x$ ,

$$u = e^{\frac{bt}{2}} \left\{ \begin{array}{l} \int dz \mathcal{I}[(at - x)(at + x - z)]\phi(z) \\ + \int dz \mathcal{I}[(at + x)(at - x - z)]\psi(z) \end{array} \right\},$$

the first integral being taken from  $z = 0$  to  $z = at + x$ , and the second integral being taken from  $z = 0$  to  $z = at - x$ . We see thence that the problem of the vibrating strings depends then on the integration of the differential equation

$$0 = -\frac{b^2}{16a^2}y + \frac{dy}{d\theta} + \theta \frac{\partial^2 y}{\partial \theta^2};$$

we see moreover that, because of the factor  $e^{-\frac{bt}{2}}$ , the ordinate  $u$  of the vibrating string diminishes without ceasing and becomes null after an infinite time, that which besides is clear *a priori*.

## XX.

We suppose next, in the general equation (S) of article XVIII,  $m = \frac{f}{s+s_1}$ ,  $n = \frac{g}{s+s_1}$ , and  $l = \frac{h}{(s+s_1)^2}$ , so that we have to integrate this equation in partial differences

$$(T) \quad 0 = \frac{\partial^2 u}{\partial s \partial s_1} + \frac{f}{s+s_1} \frac{\partial u}{\partial s} + \frac{g}{s+s_1} \frac{\partial u}{\partial s_1} + \frac{hu}{(s+s_1)^2};$$

we will be assured easily that the following values satisfy equations ( $\gamma$ ) and ( $\gamma'$ ), of the

<sup>4</sup>*Oeuvres de Laplace*, T. IX, p. 35.

article cited

$$\begin{aligned}
A &= (s + s_1)^{-f}, \\
A^{(1)} &= [f(1 - g) + h] \frac{A}{s + s_1}, \\
2A^{(2)} &= [(f + 1)(2 - g) + h] \frac{A^{(1)}}{s + s_1}, \\
3A^{(3)} &= [(f + 2)(3 - g) + h] \frac{A^{(2)}}{s + s_1}, \\
&\dots \\
\mu A^{(\mu)} &= [(f + \mu - 1)(\mu - g) + h] \frac{A^{(\mu-1)}}{s + s_1}, \\
&\dots \\
B &= (s + s_1)^{-g}, \\
B^{(1)} &= [g(1 - f) + h] \frac{B}{s + s_1}, \\
2B^{(2)} &= [(g + 1)(2 - f) + h] \frac{B^{(1)}}{s + s_1}, \\
3B^{(3)} &= [(g + 2)(3 - f) + h] \frac{B^{(2)}}{s + s_1}, \\
&\dots \\
\mu B^{(\mu)} &= [(g + \mu - 1)(\mu - f) + h] \frac{B^{(\mu-1)}}{s + s_1}, \\
&\dots
\end{aligned}$$

We will have thus

$$\Gamma(s - z) = (s + s_1)^{-f} \left\{ 1 + [f(1 - g) + h] \frac{s - z}{s + s_1} + [(f + 1)(2 - g) + h] \left( \frac{s - z}{s + s_1} \right)^2 + \dots \right\};$$

therefore, if we make  $\frac{s - z}{s + s_1} = \theta$ ,  $\Gamma(s - z)$  will be equal to a function of  $\theta$ , multiplied by  $(s + s_1)^{-f}$ . We name this function  $y$ , so that

$$\Gamma(s - z) = (s + s_1)^{-f} y,$$

$(s + s_1)^{-f} y$  will be a particular value of  $u$ , and we will have in this case

$$\frac{\partial u}{\partial s} = -f(s + s_1)^{-f-1} y + (s + s_1)^{-f} \frac{dy}{d\theta} \frac{\partial \theta}{\partial s};$$

now we will have

$$\frac{\partial \theta}{\partial s} = \frac{1}{s + s_1} - \frac{s - z}{(s + s_1)^2} = \frac{1}{s + s_1} (1 - \theta),$$



hence

$$\frac{\partial u}{\partial s} = (s + s_1)^{-f-1} \left[ \frac{dy}{d\theta} (1 - \theta) - fy \right];$$

we will find similarly

$$\begin{aligned} \frac{\partial u}{\partial s_1} &= -(s + s_1)^{-f-1} \left( fy + \theta \frac{dy}{d\theta} \right), \\ \frac{\partial^2 u}{\partial s \partial s_1} &= (s + s_1)^{-f-2} \left[ f(f+1)y + \frac{dy}{d\theta} (2f\theta + 2\theta - f - 1) - \theta(1-\theta) \frac{d^2 y}{d\theta^2} \right]. \end{aligned}$$

By substituting these values into the proposed equation in partial differences, we will have the following equation in ordinary differences

$$(a_1) \quad 0 = \theta(1 - \theta) \frac{d^2 y}{d\theta^2} + [\theta(g - f - 2) + 1] \frac{dy}{d\theta} + (fg - f - h)y;$$

it will be necessary to determine the two arbitrary constants of its integral, in a way so that we have  $y = 1$  and  $\frac{dy}{d\theta} = f(1 - g) + h$  when  $\theta = 0$ ; by naming therefore  $\mathcal{J}(\theta)$  that which  $y$  becomes then, we will have

$$\Gamma(s - z) = \frac{\mathcal{J}\left(\frac{s-z}{s+s_1}\right)}{(s + s_1)^f}.$$

If we change  $g$  into  $f$ , and reciprocally  $f$  into  $g$  in equation  $(a_1)$ , we will have

$$(b_1) \quad 0 = \theta(1 - \theta) \frac{d^2 y}{d\theta^2} + [\theta(f - g - 2) + 1] \frac{dy}{d\theta} + (fg - g - h)y;$$

and if we determine the two arbitrary constants, in a way that we have  $y = 1$  and  $\frac{dy}{d\theta} = g(1 - f) + h$  when  $\theta = 0$ , by naming  $\mathcal{Q}(\theta)$  that which  $y$  becomes then, we will have

$$\Pi(s_1 - z) = \frac{\mathcal{Q}\left(\frac{s_1-z}{s+s_1}\right)}{(s + s_1)^g}.$$

The two functions  $\mathcal{J}(\theta)$  and  $\mathcal{Q}(\theta)$  have between them a very simple relation, by means of which, when the one of the two will be known, the other will be similarly: indeed, if, in equation  $(b_1)$ , we make

$$y_1 = (1 - \theta)^{f-g} y,$$

we will have

$$0 = \theta(1 - \theta) \frac{d^2 y_1}{d\theta^2} + [\theta(g - f - 2) + 1] \frac{dy_1}{d\theta} + (fg - f - h)y_1;$$

an equation which is the same as equation  $(a_1)$ . Moreover, as we must have, relatively to equation  $(b_1)$ ,  $y = 1$  and  $\frac{dy}{d\theta} = g - fg + h$  when  $\theta = 0$ , we will have, in this same case,  $y_1 = 1$  and

$$\frac{dy}{d\theta} = \frac{dy_1}{d\theta} - (f - g)y_1 = g - fg + h,$$

that which gives

$$\frac{dy_1}{d\theta} = f - fg + h;$$

thus the two arbitrary constants of the integral of the equation in  $y_1$  are the same as those of the integral of equation  $(a_1)$ , that which gives

$$y_1 = \mathfrak{I}(\theta),$$

hence

$$\square(\theta) = (1 - \theta)^{f-g} \mathfrak{I}(\theta).$$

We have besides, relatively to equation  $(b_1)$ ,

$$\theta = \frac{s_1 - z}{s + s_1};$$

therefore

$$\square\left(\frac{s_1 - z}{s + s_1}\right) = \frac{(s + z)^{f-g} \mathfrak{I}\left(\frac{s_1 - z}{s + s_1}\right)}{(s + s_1)^{f-g}}$$

and

$$\Pi(s_1 - z) = \frac{(s + z)^{f-g} \mathfrak{I}\left(\frac{s_1 - z}{s + s_1}\right)}{(s + s_1)^f}.$$

We will have consequently, by article XVIII,

(V)

$$u = \frac{1}{(s + s_1)^f} \left[ \int dz \mathfrak{I}\left(\frac{s - z}{s + s_1}\right) \phi(z) + \int dz (s + s_1)^{f-g} \mathfrak{I}\left(\frac{s_1 - z}{s + s_1}\right) \psi(z) \right];$$

the first integral being taken from  $z = 0$  to  $z = s$ , and the second being taken from  $z = 0$  to  $z = s_1$ .

If either of these two quantities  $\mathfrak{I}\left(\frac{s-z}{s+s_1}\right)$  and  $\square\left(\frac{s_1-z}{s+s_1}\right)$ , this one for example,  $\mathfrak{I}\left(\frac{s-z}{s+s_1}\right)$ , is a rational and entire function of  $z$ , then the expression of  $u$ , considered relatively to the corresponding arbitrary function which, in this case, is  $\phi(z)$ , will be expressed by a finite series of terms multiplied by the successive integrals of  $\phi(s)$ ; because it is clear that then  $\int dz \mathfrak{I}\left(\frac{s-z}{s+s_1}\right) \phi(z)$  will be composed of terms of the form  $H \int z^\mu dz \phi(z)$ ,  $\mu$  being a positive whole number; now we have, by integrating by parts,

$$\begin{aligned} \int z^\mu dz \phi(z) &= z^\mu \phi_1(z) - \mu z^{\mu-1} \phi_2(z) \\ &+ \mu(\mu-1) z^{\mu-2} \phi_3(z) - \dots \pm 1.2.3 \dots \mu \phi_{\mu+1}(z) + C, \end{aligned}$$

an expression delivered with the  $\int$  sign, and in which we must make  $z = s$ . We see thus that the part of the expression of  $u$  relative to the arbitrary function  $\phi(z)$  is independent, not only of every definite integral, but further of every kind of integral; now there results from this what I have demonstrated, in the Mémoires cited in 1773, that the complete

expression of  $u$  is then entirely independent of every definite integral, that is that it can be expressed by some indefinite integrals, uniquely relative to the variables  $s$  and  $s_1$ , of the proposed equation. We can be assured of it yet very easily by means of formula (V), because it is clear that the integral

$$\int dz (s+z)^{f-g} \mathfrak{I} \left( \frac{s_1-z}{s+s_1} \right) \psi(z)$$

will be in this case reducible to some terms of this form

$$H \int z^\mu dz (s+z)^{f-g} \psi(z)$$

$\mu$  being a positive whole number or zero; now we can, by some integrations by parts, reduce the integral

$$\int z^\mu dz (s+z)^{f-g} \psi(z)$$

to some terms delivered with the  $\int$  sign and to some integrals of this form

$$\int dz (s+z)^r \psi_i(z);$$

this last integral, necessarily being taken from  $z = 0$  to  $z = s_1$ , is evidently equal to this one

$$\int ds_1 (s+s_1)^r \psi_i(z)$$

and, consequently, independent of every definite integral; we see thence how the integral

$$\int dz (s+z)^{f-g} \mathfrak{I} \left( \frac{s_1-z}{s+s_1} \right) \psi(z)$$

can be reduced to some indefinite integrals, although the factor

$$(s+z)^{f-g} \mathfrak{I} \left( \frac{s_1-z}{s+s_1} \right)$$

may not be a rational and entire function of  $z$ .

Now, the condition necessary in order that the expression of  $\mathfrak{I} \left( \frac{s_1-z}{s+s_1} \right)$ , reduced to series, is terminated, is that we have  $A^{(\mu)} = 0$ ,  $\mu$  being a positive number, that which gives

$$0 = (f + \mu - 1)(\mu - g) + h,$$

whence we deduce

$$\mu = \frac{1 + g - f \pm \sqrt{(f + g - 1)^2 - 4h}}{2}.$$

When either of these two values of  $\mu$  is zero or a positive whole number, then  $\mathfrak{I} \left( \frac{s-z}{s+s_1} \right)$  is a rational and entire function of  $z$ ; by changing  $f$  into  $g$  and reciprocally, we will have

$$\mu = \frac{1 + f - g \pm \sqrt{(f + g - 1)^2 - 4h}}{2},$$

and, if one or the other of these values of  $\mu$  is zero or a positive whole number, the value of  $\square\left(\frac{s_1-z}{s+s_1}\right)$  will be a rational and entire function of  $z$ ; in all these cases, the expression of  $u$  will not depend on any definite integral; otherwise it will be necessarily dependent.

If we name  $x$  the distance from one molecule of air to the origin of the sound in a state of equilibrium;  $x + u$  its distance after time  $t$ , we will have

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + \frac{ma^2}{x} \frac{\partial u}{\partial x} - \frac{ma^2 u}{x^2},$$

$a^2$  being a constant coefficient depending on the elasticity and on the density of the air, and  $m$  being 0, or 1, or 2, according as we consider the air either with one alone, or with two, or with three dimensions (*see*, on this object, the learned researches of Mr. de la Grange on sound, inserted in Volume II of *Mémoires de la Société royale de Turin*). Let  $x + at = s$ ,  $x - at = s_1$ ; the preceding equation will become

$$0 = \frac{\partial^2 u}{\partial s \partial s_1} + \frac{m}{2(s+s_1)} \left( \frac{\partial u}{\partial s} + \frac{\partial u}{\partial s_1} \right) - \frac{mu}{(s+s_1)^2};$$

formula (V) will become therefore

$$u = \frac{1}{2^{\frac{m}{2}} x^{\frac{m}{2}}} \left[ \int dz \mathfrak{I} \left( \frac{x+at-z}{2x} \right) \phi(z) + \int dz \mathfrak{I} \left( \frac{x-at-z}{2x} \right) \psi(z) \right],$$

the first integral being taken from  $z = 0$  to  $z = x + at$ , and the second being taken from  $z = 0$  to  $z = x - at$ . The function  $\mathfrak{I} \left( \frac{x \pm at - z}{2x} \right)$  is the value of  $y$  in the differential equation

$$0 = \theta(1-\theta) \frac{d^2 y}{d\theta^2} + \theta(1-2\theta) \frac{dy}{d\theta} + \frac{m^2 + 2m}{4} y,$$

in which  $\theta = \frac{x \pm at - z}{2x}$ , the two arbitrary constants of its integral being necessary to determine, so that we have

$$y = 1 \quad \text{and} \quad \frac{dy}{d\theta} = -\frac{m}{4}(2+m).$$

If we have  $m = 0$  or  $m = 2$ , the value of  $y$  ordered according to the powers of  $\theta$  is terminated, and then the value of  $u$  is independent of every definite integral; but, when  $m = 1$ , that which takes place when we have considered the air only with two dimensions, the expression of  $u$  is necessarily dependent on a definite integral.

If we change in  $\mathfrak{I} \left( \frac{x \pm at - z}{2x} \right)$ ,  $z$  into  $x \pm at - z_1$ , we will have, by article XVIII,

$$u = \frac{1}{2^{\frac{m}{2}} x^{\frac{m}{2}}} \int dz_1 \mathfrak{I} \left( -\frac{z_1}{2x} \right) [\phi(z + at + z_1) + \psi(z - at + z_1)],$$

the integral being taken from  $z_1 = 0$  to  $z_1 = \infty$ . There results evidently from this value of  $u$  that the molecule of air of which it expresses the derangement begins to be shaken only when  $x - at + z_1$  is equal or less than the radius of the small sphere agitated at the beginning; whence it follows that, in the three cases where the air has one, or two, or three dimensions, the speed of the sound is the same and is determined

by the equation  $t = \frac{x}{a}$ ; we see thus that the preceding forms of the integrals of the equations in partial differences have the same advantage in the physical questions as the forms known at present.

We could still apply the preceding method to the research on the vibrations of unequally thick strings, to the theory of sound in some tubes of any figure and to many other important questions; but these discussions would divert us too much from our principal object.

## XXI.

We return presently to the equations linear in the partial differences; although the formulas which we have given in article XVI, in order to integrate them, have the greatest generality, there are however some cases where they cannot serve: these cases have place when the equation  $z = 0$  gives the expression of  $\frac{1}{t^i}$  in  $\frac{1}{t_1}$  by an infinite series, that which arrives every time that, in the function  $z$ , the highest power of  $\frac{1}{t}$  is multiplied by a rational and entire function of  $\frac{1}{t_1}$ . In order to have then the expression of  $y_{x,x_1}$  in finite terms, it is necessary to resort to some artifices of analysis which we are going to exhibit, by applying them to the following equation

$$\frac{1}{tt_1} - \frac{1}{t_1} - \frac{b}{t} - c = 0;$$

this equation gives

$$\frac{1}{t} = \frac{c + \frac{a}{t_1}}{\frac{1}{t_1} - b},$$

hence

$$\frac{1}{t^x t_1^{x_1}} = \frac{\left(c + \frac{a}{t_1}\right)^x}{t_1^{x_1} \left(\frac{1}{t_1} - b\right)^x}.$$

By expanding, with respect to the powers of  $\frac{1}{t_1}$  the second member of this equation, we will have an infinite series, that which will give  $y_{x,x_1}$  in an infinite series; in order to prevent this disadvantage, we will put the preceding equation under this form

$$\frac{1}{t^x t_1^{x_1}} = \frac{\left(\frac{1}{t_1} - b + b\right)^{x_1} \left[c + ab + a\left(\frac{1}{t_1} - b\right)\right]^x}{\left(\frac{1}{t_1} - b\right)^x}.$$

If we expand the second member of this equation, with respect to the powers of  $\frac{1}{t_1} - b$ , we will have

$$\begin{aligned} \frac{1}{t^x t_1^{x_1}} = & \left[ \left(\frac{1}{t_1} - b\right)^{x_1} + x_1 b \left(\frac{1}{t_1} - b\right)^{x_1-1} + \frac{x_1(x_1-1)}{1.2} b^2 \left(\frac{1}{t_1} - b\right)^{x_1-2} + \dots \right] \\ & \times \left[ a^x + x(c+ab) \frac{a^{x-1}}{\frac{1}{t_1} - b} + \frac{x(x-1)}{1.2} (c+ab)^2 \frac{a^{x-2}}{\left(\frac{1}{t_1} - b\right)^2} + \dots \right] \end{aligned}$$

Let

$$\begin{aligned}
V &= a^x, \\
V^{(1)} &= x_1 b a^x + x(c + ab)a^{x-1}, \\
V^{(2)} &= \frac{x_1(x_1 - 1)}{1.2} b^2 a^x + x_1 x b(c + ab)a^{x-1} + \frac{x(x-1)}{1.2} (c + ab)^2 a^{x-2}, \\
V^{(3)} &= \frac{x_1(x_1 - 1)(x_1 - 2)}{1.2.3} b^3 a^x + \frac{x_1(x_1 - 1)}{1.2} x b^2 (c + ab)a^{x-1} \\
&\quad + x_1 \frac{x(x-1)}{1.2} b(c + ab)^2 a^{x-2} + \frac{x(x-1)(x-2)}{1.2.3} (c + ab)^3 a^{x-3}, \\
&\quad \dots
\end{aligned}$$

we will have

$$\frac{u}{t^x t_1^{x_1}} = u \left\{ \begin{aligned} &V \left( \frac{1}{t_1} - b \right)^{x_1} + V^{(1)} \left( \frac{1}{t_1} - b \right)^{x_1-1} + V^{(2)} \left( \frac{1}{t_1} - b \right)^{x_1-2} + \dots \\ &+ V^{(x_1)} + \frac{V^{(x_1+1)}}{\frac{1}{t_1} - b} + \frac{V^{(x_1+2)}}{\left( \frac{1}{t_1} - b \right)^2} + \dots + \frac{V^{(x+x_1)}}{\left( \frac{1}{t_1} - b \right)^x} \end{aligned} \right\};$$

now the equation

$$\frac{1}{t t_1} - \frac{1}{t_1} - \frac{b}{t} - c = 0 \quad \text{gives} \quad \frac{1}{\frac{1}{t_1} - b} = \frac{\frac{1}{t} - a}{c + ab},$$

hence

$$\frac{u}{t^x t_1^{x_1}} = u \left\{ \begin{aligned} &V \left( \frac{1}{t_1} - b \right)^{x_1} + V^{(1)} \left( \frac{1}{t_1} - b \right)^{x_1-1} + \dots \\ &+ V^{(x_1)} + \frac{V^{(x_1+1)}}{c + ab} \left( \frac{1}{t} - a \right) + \frac{V^{(x_1+2)}}{(c + ab)^2} \left( \frac{1}{t} - a \right)^2 + \dots \\ &+ \frac{V^{(x+x_1)}}{(c + ab)^x} \left( \frac{1}{t} - a \right)^x \end{aligned} \right\}.$$

In order to pass again now from the generating functions to their corresponding variables, we will observe: 1° that the coefficient of  $t^0 t_1^0$  in  $\frac{u}{t^x t_1^{x_1}}$  is  $y_{x, x_1}$ ; 2° that this same coefficient in any term such as  $u \left( \frac{1}{t_1} - b \right)^r$  or  $u b^r \left( \frac{1}{b t_1} - 1 \right)^r$  is equal to

$$b^r \left[ \frac{y_{0, r}}{b^r} - r \frac{y_{0, r-1}}{b^{r-1}} + \frac{r(r-1)}{1.2} \frac{y_{0, r-2}}{b^{r-2}} - \dots \right]$$

and, consequently, equal to  $b^{r-1} \Delta^r \frac{y_{0, x_1}}{b^{x_1}}$ , the differential characteristic  $^1 \Delta$  corresponding to the variability of  $x_1$ , and this variable being necessarily supposed null after the differentiations; 3° that this coefficient, in  $u \left( \frac{1}{t} - a \right)^r$ , is  $a^r \Delta^r \frac{y_{x, 0}}{a^x}$ , the characteristic

$\Delta$  corresponding to the variability of  $x$ , and this variable being necessarily supposed null after the differentiations; we will have therefore with this condition

$$\begin{aligned} y_{x,x_1} = & Vb^{x_1-1} \Delta^{x_1} \frac{y_{0,x_1}}{b^{x_1}} + V^{(1)}b^{x_1-1} \Delta^{x_1-1} \frac{y_{0,x_1}}{b^{x_1}} + V^{(2)}b^{x_1-2} \Delta^{x_1-2} \frac{y_{0,x_1}}{b^{x_1}} + \dots \\ & + V^{(x_1)}y_{0,0} + \frac{V^{(x_1+1)}}{c+ab} a \Delta \frac{y_{x,0}}{a^x} + \frac{V^{(x_1+2)}}{(c+ab)^2} a^2 \Delta^2 \frac{y_{x,0}}{a^x} + \dots \\ & + \frac{V^{(x+x_1)}}{(c+ab)^x} a^x \Delta^x \frac{y_{x,0}}{a^x}; \end{aligned}$$

this will be the complete integral of the equation

$$y_{x+1,x_1+1} - ay_{x,x_1+1} - by_{x+1,x_1} - cy_{x,x_1} = 0,$$

and it is clear that this integral supposes that we know the first horizontal rank and the first vertical rank of Table (Q) of article XVI.

## XXII.

In order to clarify by an example the method which we have given previously in order to integrate the equations in finite partial differences, we suppose that we have the equation

$$0 = t \left( \frac{1}{t} - 1 \right)^2 - t_1 \left( \frac{1}{t_1} - 1 \right)^2,$$

we will have

$$\frac{1}{t} = \frac{1}{2}t_1 + \frac{1}{2t_1} \pm \frac{1}{2} \left( \frac{1}{t_1} - t_1 \right).$$

Let

$$\frac{1}{t^x} = Z + \frac{1}{t} Z^{(1)},$$

$Z$  and  $Z^{(1)}$  being some function of  $t_1$  and of  $x$ ; we will determine these functions by substituting successively into the preceding equation, in the place of  $\frac{1}{t}$ , its two values; that which gives

$$\begin{aligned} \left[ \frac{1}{2}t_1 + \frac{1}{2t_1} + \frac{1}{2} \left( \frac{1}{t_1} - t_1 \right) \right]^x &= Z + Z^{(1)} \left[ \frac{1}{2t_1} + \frac{1}{2}t_1 + \frac{1}{2} \left( \frac{1}{t_1} - t_1 \right) \right], \\ \left[ \frac{1}{2}t_1 + \frac{1}{2t_1} - \frac{1}{2} \left( \frac{1}{t_1} - t_1 \right) \right]^x &= Z + Z^{(1)} \left[ \frac{1}{2t_1} + \frac{1}{2}t_1 - \frac{1}{2} \left( \frac{1}{t_1} - t_1 \right) \right], \end{aligned}$$

whence it is easy to conclude

$$\begin{aligned} Z &= \frac{\frac{1}{t_1^{x-2}} - t_1^x}{t_1^2 - 1}, \\ Z^{(1)} &= \frac{\frac{1}{t_1^x} - t_1^x}{\frac{1}{t_1} - t_1}, \end{aligned}$$

hence

$$\frac{u}{t^x} = u \frac{\frac{1}{t_1^{x-2}} - t_1^x}{t_1^2 - 1} + \frac{u \frac{1}{t_1^x} - t_1^x}{t \frac{1}{t_1} - t_1}.$$

Presently, the coefficient of  $t^0 t_1^{r_1}$  in  $\frac{u}{t^x}$  is  $y_{x,x_1}$ , and, if we designate by  $\mathbb{T}\lambda_x$  and  $\mathbb{I}\lambda_x$  the coefficients of  $t^x$  in the expansion of the functions  $\frac{\nu}{t^2-1}$  and  $\frac{\nu}{\frac{1}{t}-1}$ ,  $\nu$  being equal to the infinite series  $\lambda_0 + \lambda_1 t + \lambda_2 t^2 + \dots$ , we will have

$$\begin{array}{llll} 1^\circ & \mathbb{T}y_{0,x+x_1-2} & \text{for the coefficient of } t^0 t_1^{r_1} \text{ in} & \frac{u}{t_1^{x-2}(t_1^2-1)}; \\ 2^\circ & \mathbb{T}y_{0,x+x_1} & \text{for this coefficient in} & \frac{u t_1^x}{t_1^2-1}; \\ 3^\circ & \mathbb{I}y_{1,x+x_1} & \text{''} & \frac{\frac{u}{t} \frac{1}{t_1^x}}{\frac{1}{t_1} - t_1}; \\ 4^\circ & \mathbb{I}y_{1,x_1-x} & \text{''} & \frac{\frac{u}{t} t_1^x}{\frac{1}{t_1} - t_1}. \end{array}$$

We will have therefore

$$y_{x,x_1} = \mathbb{T}y_{0,x+x_1-2} - \mathbb{T}y_{0,x_1-x} + \mathbb{I}y_{1,x+x_1} - \mathbb{I}y_{1,x_1-x},$$

and, if we represent

$$\mathbb{T}y_{0,x+x_1-2} + \mathbb{I}y_{1,x+x_1} \quad \text{by} \quad \phi(x+x_1)$$

and

$$-\mathbb{T}y_{0,x_1-x} - \mathbb{I}y_{1,x_1-x} \quad \text{by} \quad \psi(x-x_1).$$

$\phi(x)$  and  $\psi(x)$  being two arbitrary functions of  $x$ , we will have

$$y_{x,x_1} = \phi(x+x_1) + \psi(x_1-x).$$

This put, if we multiply the equation

$$0 = t \left( \frac{1}{t} - 1 \right)^2 - t_1 \left( \frac{1}{t_1} - 1 \right)^2$$

by  $u$  and if we pass again from the generating functions to their corresponding variables, we will have the equation in partial differences

$$y_{x+1,x_1} - 2y_{x,x_1} + y_{x-1,x_1} = y_{x,x_1+1} - 2y_{x,x_1} + y_{x,x_1-1};$$

its complete integral will be consequently

$$y_{x,x_1} = \phi(x+x_1) + \psi(x_1-x),$$

that which is clear moreover by the simple substitution, but I have belief that one would not be angry to see how this integral is deduced from the preceding methods.



We suppose now that, in the following Table

$$(Z) \quad \begin{cases} y_{0,0}, & y_{1,0}, & y_{2,0}, & y_{3,0}, & y_{4,0}, & \dots, & y_{n-1,0}, & y_{n,0}, \\ y_{0,1}, & y_{1,1}, & y_{2,1}, & y_{3,1}, & y_{4,1}, & \dots, & y_{n-1,1}, & y_{n,1}, \\ y_{0,2}, & y_{1,2}, & y_{2,2}, & y_{3,2}, & y_{4,2}, & \dots, & y_{n-1,2}, & y_{n,2}, \\ y_{0,3}, & y_{1,3}, & y_{2,3}, & y_{3,3}, & y_{4,3}, & \dots, & y_{n-1,3}, & y_{n,3}, \\ \dots, & \dots, & \dots, & \dots, & \dots, & \dots, & \dots, & \dots, \\ y_{0,\infty}, & y_{1,\infty}, & y_{2,\infty}, & y_{3,\infty}, & y_{4,\infty}, & \dots, & y_{n-1,\infty}, & y_{n,\infty}, \end{cases}$$

we know the first two horizontal ranks contained between the two extreme vertical columns

$$\begin{array}{ccccccc} y_{0,0}, & y_{0,1}, & y_{0,2}, & \dots, & y_{0,\infty}, \\ y_{n,0}, & y_{n,1}, & y_{n,2}, & \dots, & y_{n,\infty}, \end{array}$$

and that we know moreover all the terms of these two columns; we could determine all the values of  $y_{x,x_1}$  which fall between these two columns, because, if we wish to form the third horizontal rank, we will resume the equation

$$y_{x+1,x_1} - 2y_{x,x_1} + y_{x-1,x_1} = y_{x,x_1+1} - 2y_{x,x_1} + y_{x,x_1-1},$$

which is reduced to

$$y_{x,x_1+1} = y_{x+1,x_1} + y_{x-1,x_1} - y_{x,x_1-1};$$

by making  $x_1 = 1$ , and successively  $x = 1, x = 2, x = 3, \dots, x = n - 1$ , we will have

$$\begin{array}{l} y_{1,2} = y_{2,1} + y_{0,1} - y_{1,0}, \\ y_{2,2} = y_{3,1} + y_{1,1} - y_{2,0}, \\ y_{3,2} = y_{4,1} + y_{2,1} - y_{3,0}, \\ \dots \\ y_{n-1,2} = y_{n,1} + y_{n-2,1} - y_{n-1,0}. \end{array}$$

We will form in the same manner the fourth horizontal rank, and thus in sequence to infinity; but, if we wished to determine the values of  $y_{x,x_1}$  which fall outside of Table (Z), the preceding conditions would not suffice, and it would be necessary to join them to others.

We seek presently the expression of  $y_{x,x_1}$ ; for this, we resume the integral

$$y_{x,x_1} = \phi(x + x_1) + \psi(x_1 - x);$$

and we suppose that the second horizontal rank which determines one of the two arbitrary functions is such that we have  $\psi(x_1 - x) = \phi(x - x_1)$ , we will have

$$y_{x,x_1} = \phi(x_1 + x) + \phi(x - x_1);$$

by making  $x_1 = 0$ , we will have  $\phi(x) = \frac{1}{2}y_{x,0}$ , hence

$$y_{x,x_1} = \frac{1}{2}y_{x+x_1,0} + \frac{1}{2}y_{x-x_1,0}.$$

It is easy to see that this equation satisfies the proposed equation in partial differences; but it is only a particular integral which corresponds to the case where the second horizontal rank is formed from the first, by means of the equation

$$y_{x,1} = \frac{1}{2}y_{x+1,0} + \frac{1}{2}y_{x-1,0}.$$

As much as  $x + x_1$  will be equal or less than  $n$ , and as  $x - x_1$  will be positive or null, we will have the value of  $y_{x,x_1}$ , by means of the first horizontal rank; but, when  $x_1$  increasing,  $x + x_1$  will become greater than  $n$  and if  $x - x_1$  will become negative, we must determine the values of  $y_{x+x_1,0}$  and of  $y_{x-x_1,0}$  by means of the extreme vertical columns. We suppose that all the terms of these two columns are zero and that if we have  $y_{0,x_1} = 0$  and  $y_{n,x_1} = 0$ ; by making  $x = 0$  in the equation

$$y_{x,x_1} = \frac{1}{2}y_{x+x_1,0} + \frac{1}{2}y_{x-x_1,0}.$$

we will have

$$y_{-x_1,0} = -y_{x_1,0};$$

by making next  $x = n$ , we will have

$$y_{n+x_1,0} = -y_{n-x_1,0}.$$

If we change, in this last equation,  $x_1$  into  $n + x_1$ , we will have

$$y_{2n+x_1,0} = -y_{-x_1,0} = y_{x_1,0};$$

by changing next  $x_1$  into  $n + x_1$ , we will have

$$y_{3n+x_1,0} = y_{n+x_1,0} = -y_{n-x_1,0},$$

whence we deduce generally

$$y_{2rn+x_1,0} = y_{x_1,0}$$

and

$$y_{(2r+1)n+x_1,0} = -y_{n-x_1,0}.$$

We may thus, by means of these two equations, continue the values of  $y_{x,0}$  to infinity, on the side of the positive values of  $x$ , and we will conclude from it those which correspond to  $x$  negative, by means of the equation  $y_{-x_1,0} = -y_{x_1,0}$ ; thence results the following construction.

If we represent the values of  $y_{x,0}$  from  $x = 0$  to  $x = n$ , by the ordinates of the angles of a polygon of which the abscissa is  $x$  and of which the two extremities  $A$  and  $B$  lead to the points where  $x = 0$  and  $x = n$ , we will carry this polygon from  $x = n$  to  $x = 2n$ , giving a position to it contrary to the one which it had from  $x = 0$  to  $x = n$ , that is a position such that the parts which were above the axis of the abscissas is found below, the point  $B$  of the polygon remaining moreover, in this second position, in the same place as in the first, and the point  $A$  corresponding thus to the abscissa  $x = 2n$ ; we will place next this same polygon from  $x = 2n$  to  $x = 3n$ , by giving it a position

contrary to the second and consequently like the first, in a manner that the point  $A$  conserves, in this third position, the same place as in the second, and that thus the point  $B$  corresponds to the abscissa  $x = 3n$ . By continuing to place thus this polygon alternately above and below the axis of the abscissas, the ordinates drawn at the angles of these polygons will be the values of  $y_{x,0}$  which correspond to  $x$  positive.

Similarly, we will place this polygon from  $x = 0$  to  $x = -n$ , by giving it a position contrary to that which it had from  $x = 0$  to  $x = n$ , the point  $A$  remaining moreover, in this second position, in the same place as in the first; we will place next this polygon from  $x = -n$  to  $x = -2n$ , by giving it a position contrary to the second, the point  $B$  remaining moreover in the same place, and thus in sequence to infinity. The ordinates of these polygons will represent the values of  $y_{x,0}$  which correspond to  $x$  negative; we will have next the value of  $y_{x,x_1}$  by taking the mean of the sum of the two ordinates which correspond to the abscissas  $x + x_1$  and  $x - x_1$ .

This geometric construction is general, whatever be the nature of the polygon which we just considered; it will serve to determine all the values of  $y_{x,x_1}$  contained from  $x = 0$  to  $x = n$  and from  $x_1 = 0$  to  $x_1 = \infty$ , provided that we have  $y_{0,x_1} = 0$  and  $y_{n,x_1} = 0$ , and that moreover the second horizontal rank of Table (Z) is such that we have

$$y_{x,1} = \frac{1}{2}y_{x+1,0} + \frac{1}{2}y_{x-1,0}$$

or, that which returns to the same,

$$y_{x,1} - y_{x,0} = \frac{1}{2}(y_{x+1,0} - 2y_{x,0} + y_{x-1,0}).$$

We can, besides, be assured easily of the truth of the preceding results in some particular examples, by giving to  $n$  some particular values, by taking next some numbers at will to form the first horizontal rank of Table (Z) and by forming the second rank by means of the equation

$$y_{x,1} = \frac{1}{2}y_{x+x_1,0} + \frac{1}{2}y_{x-x_1,0};$$

finally by supposing generally  $y_{0,x_1} = 0$  and  $y_{n,x_1} = 0$ ; because, if by means of these conditions and from the proposed equation in partial differences

$$y_{x,x_1+1} = y_{x+1,x_1} + y_{x-1,x_1} - y_{x,x_1-1},$$

we form the other horizontal ranks of Table (Z), we will find that they will be the same as those which result from the preceding construction.

We have, by that which precedes,

$$y_{x,x_1+n} = \frac{1}{2}y_{x+x_1+n,0} + \frac{1}{2}y_{x-n-x_1,0};$$

moreover,

$$y_{x+x_1+n,0} = -y_{x-n-x_1,0}$$

and

$$y_{x-n-x_1,0} = -y_{n+x_1-x,0};$$

therefore

$$y_{x,x_1+n} = -\frac{1}{2}y_{x-n-x_1,0} - \frac{1}{2}y_{n-x+x_1,0} = -y_{n-x,x_1}.$$

It follows thence that, in Table (Z), the  $(x_1 + n)^{\text{th}}$  horizontal rank is the  $x_1^{\text{st}}$  horizontal rank taken with a contrary sign and in a reverse order, that is that the  $r^{\text{th}}$  term of the  $(x_1 + n)^{\text{th}}$  rank is the  $(n - r)^{\text{th}}$  term of the  $x_1^{\text{th}}$  rank taken with a contrary sign.

We have next

$$y_{x,x_1+2n} = \frac{1}{2}y_{2n+x+x_1,0} + \frac{1}{2}y_{x-x_1-2n,0};$$

we have besides

$$y_{2n+x+x_1,0} = y_{x+x_1,0}$$

and

$$y_{x-x_1-2n,0} = -y_{2n+x_1-x,0} = -y_{x_1-x,0} = y_{x-x_1,0},$$

hence

$$y_{x,x_1+2n} = \frac{1}{2}y_{x+x_1,0} + \frac{1}{2}y_{x-x_1,0} = y_{x,x_1};$$

whence it follows that the  $(x_1 + 2n)^{\text{th}}$  horizontal rank is exactly equal to the  $x_1^{\text{st}}$  rank.

We will consider presently the vibrations of a string of which the initial figure is anything, but very little elongated from the axis of the abscissas; we name  $x$  the abscissa,  $t$  the time,  $y_{x,t}$  the ordinate of any point of the cord after time  $t$ ; we imagine moreover the abscissa  $x$  divided into an infinity of small parts equal to  $dx$  and which we take for unity. This put, we will have, by the known principles of Dynamics,

$$\frac{\partial^2 y_{x,t}}{\partial t^2} = \frac{a^2}{dx^2}(y_{x+1,t} - 2y_{x,t} + y_{x-1,t}),$$

$a$  being a constant coefficient depending on the tension and on the thickness of the string. If we make  $t = \frac{x_1}{a}$ , we will have  $dt = \frac{dx_1}{a}$ , and  $y_{x,t}$  will become a function of  $x$  and of  $x_1$ , which we will designate by  $y_{x,x_1}$ ; now, the magnitude of  $dt$  being arbitrary, we can suppose it such that the variation of  $x_1$  is equal to that of  $x$ , which we have taken for unity. The preceding equation will become thus

$$y_{x,x_1+1} - 2y_{x,x_1} + y_{x,x_1-1} = y_{x+1,x_1} - 2y_{x,x_1} + y_{x-1,x_1};$$

$x$  and  $x_1$  being some infinite numbers. This equation is the same as we just considered; thus the geometric construction which we have given, by means of the polygon which represents the value of  $y_{x,0}$  from  $x = 0$  to  $x = n$ , can be used in this case: the polygon will be here the initial curve of the string; but, for this, we must suppose  $n$  equal to the length of the string and to imagine it divided into an infinity of parts; it is necessary, moreover, that the string be fixed at its two extremities, finally that we have  $y_{0,x_1} = 0$  and  $y_{n,x_1} = 0$ ; moreover the equation of condition

$$y_{x,1} - y_{x,0} = \frac{1}{2}(y_{x+1,0} - 2y_{x,0} + y_{x-1,0})$$

is changed into this one

$$dt \frac{\partial y_{x,0}}{\partial t} = \frac{1}{2} dx^2 \frac{\partial^2 y_{x,0}}{\partial x^2},$$

that which gives

$$\frac{\partial y_{x,0}}{\partial t} = 0;$$

now  $\frac{\partial y_{x,0}}{\partial t}$  is the initial velocity of the string; this velocity must therefore be null at the origin of the movement. Every time that these conditions will hold, the preceding construction will give always the movement of the string, whatever be moreover its initial figure, provided however that, in all its points,  $y_{x+2,0} - 2y_{x+1,0} + y_{x,0}$  is infinitely small of the second order, that is that two contiguous sides of the curve do not form at all between them a finite angle. This condition is necessary in order that the differential equation of the problem can subsist, and in order that this

$$\frac{\partial y_{x,0}}{\partial t} dt = \frac{1}{2} (y_{x+1,0} - 2y_{x,0} + y_{x-1,0})$$

gives

$$\frac{\partial y_{x,0}}{\partial t} = 0;$$

but besides it is evident, by that which precedes, that the initial figure of the string can be discontinuous and composed of any number of arcs of a circle or of portions of string which touch themselves.

We see easily that all the different situations of the string correspond to the horizontal ranks of Table (Z), and, as the ranks which correspond to the values of  $x_1, x_1 + 2n, x_1 + 4n, \dots$  are the same, by that which precedes, there results from it that the string will return to the same situation after time  $t, t + \frac{2n}{a}, t + \frac{4n}{a}, \dots, n$  being always the total length of the string.

This analysis of the vibrating strings establishes, if I do not deceive myself, in an incontestable manner the possibility of admitting some discontinuous functions into this problem, and it seems to me that we can generally conclude that these functions can be employed in all the problems which correspond to the partial differences, provided that they can subsist with the differential equations and with the conditions of the problem. We can consider, indeed, any equation in infinitely small partial differences as a particular case of an equation in partial finite differences, in which we suppose that the variables become infinities: now, nothing being neglected in the theory of equations in the finite differences, it is clear that the arbitrary functions of their integrals are not at all subject to the law of continuity, and that the constructions of these equations by means of the polygons have place whatever be the nature of these polygons. Now, when we pass from the finite to the infinitely small, these polygons change themselves into curves which, consequently, can be discontinuous: thus the law of continuity appears unnecessary, neither in the arbitrary functions of the integrals of the equations in the infinitely small partial differences, nor in the geometric constructions which represent these integrals; we must observe only that, if the differential equation is of order  $n$ , and if we name  $u$  its principal variable,  $x$  and  $t$  being the two other variables, we must not at all have a jump between two consecutive values of  $\frac{\partial^{n-r} u}{\partial x^s \partial t^{n-r-s}}$ , that is that the

difference of this quantity must be infinitely small with respect to this quantity itself. This condition is necessary in order that the proposed differential equation can subsist, because every differential equation supposes that the differences of  $u$  of which it is composed, divided by the respective powers of  $dx$  and of  $dt$ , are some finite quantities and comparables among themselves; but nothing obliges to admit the preceding condition relatively to the differences of  $u$  of the order  $n$  or of a superior order; we must therefore subject the arbitrary functions of the integral to this that there is no jump between two consecutive values of a difference of these functions less than  $n$ , and the curves which represent them must be subject to a similar condition, such that it must not at all have a jump between two consecutive tangents if the equation is differentiable of second order, or between two consecutive osculating radii if it is differentiable of the third order, and thus in sequence. For example, in the problem of the vibrating strings which we just analyzed, and which lead to a differential equation of the second order, it is necessary that the curves of which we make use in order to construct it are such that two contiguous sides do not form between them a finite angle: now, this is that which will take place in the construction which we have given if the initial figure of the string is such that this condition is fulfilled; because, by putting it alternately above and below the axis of the abscissas, as we have prescribed, the infinite curve which results from it satisfies in all its extent the same condition.

The sole case which seems to make exception to that which we just said is the one in which the integral contains arbitrary functions and their differences; because, by substituting it into the differential equation in order to satisfy it, we introduce the differences of the arbitrary functions of an order superior to  $n$ , that which supposes that the law of continuity extends beyond the differences of order  $n - 1$ ; but we must then consider as the true arbitrary functions of the integral the most elevated differences of these functions, and to regard all the inferior differences as their successive integrals, in consideration of which the previously given rule on the continuity of some arbitrary functions and of their differences will subsist in its entirety. We can even present it in a simpler manner, by observing that there is no jump at all between two consecutive values of the integral of any arbitrary and discontinuous function; because, by naming  $\phi(s)$  this function, two consecutive values of its integral  $\int ds \phi(s)$  differs between them only by the quantity  $ds\phi(s)$ , which could be always infinitely small, when even there could be a jump between two consecutive values of  $\phi(s)$ . The preceding rule can therefore be reduced to the following:

*If the integral of an equation in partial differences of order  $n$  contains the  $r^{\text{th}}$  difference of an arbitrary function of  $s$ , we can, in place of the  $(n + r)^{\text{th}}$  difference of this function, divided by  $ds^{n+r}$ , employ any function discontinuous in  $s$ .*

When, in the problem of the vibrating strings, the initial figure of the string is such that two of these contiguous sides form a finite angle, for example when it is formed by the joining of two straight lines, it seems to me that geometrically the preceding solution cannot be admitted; but, if we consider physically this problem and all others of this type, such as that of the sound, it appears that we can apply the construction which we have given, even in the case where the string would be formed by a system of many straight lines: because we see, *a priori*, that its movement must differ very little from the one which it takes by supposing that, at the points where these lines meet themselves, there were some small curves which permit using this construction.

XXIII.

We can still apply the calculus of the generating functions to the integration of equations in partial differences, by finite parts and by infinitely small parts; for this, we will consider the equation

$$0 = ay_{x,x_1} + b\Delta y_{x,x_1} - \frac{\partial y_{x,x_1}}{\partial x_1},$$

the finite characteristic  $\Delta$  corresponding to the variable  $x$ , of which the difference is unity, and the characteristic  $d$  corresponding to the variable  $x_1$ , of which the difference is consequently  $dx_1$ .

The generating equation of the preceding is

$$0 = a + b \left( \frac{1}{t} - 1 \right) - \frac{1}{dx_1} \left( \frac{1}{t_1^{dx_1}} - 1 \right),$$

whence we deduce, to the infinitely small nearly,

$$\frac{1}{t^x} = \frac{1}{(b dx_1)^x} \left[ \frac{1}{t_1^{dx_1} (1 + a dx_1 - b dx_1)} - 1 \right]^x.$$

Now, if we name  $y_{x,x_1}$  the coefficient of  $t^x t_1^x$  in  $u$ , the coefficient of  $t^0 t_1^{x_1}$  in  $\frac{u}{t^x}$  will be  $y_{x,x_1}$ ; this same coefficient in

$$u \left[ \frac{1}{t_1^{dx_1} (1 + a dx_1 - b dx_1)} - 1 \right]^x$$

will be

$$(1 + a dx_1 - b dx_1)^{\frac{x_1}{dx_1}} \left\{ \begin{array}{l} \frac{y_{0,x_1} + x dx_1}{(1 + a dx_1 - b dx_1)^{\frac{x_1}{dx_1} + x}} - x \frac{y_{0,x_1} + (x-1) dx_1}{(1 + a dx_1 - b dx_1)^{\frac{x_1}{dx_1} + x - 1}} \\ + \frac{x(x-1)}{1.2} \frac{y_{0,x_1} + (x-2) dx_1}{(1 + a dx_1 - b dx_1)^{\frac{x_1}{dx_1} + x - 2}} - \dots \end{array} \right\}$$

$$= (1 + a dx_1 - b dx_1)^{\frac{x_1}{dx_1}} d^x \frac{y_{0,x_1}}{(1 + a dx_1 - b dx_1)^{\frac{x_1}{dx_1}}}.$$

Now we have

$$(1 + a dx_1 - b dx_1)^{\frac{x_1}{dx_1}} = e^{(a-b)x_1},$$

$e$  being the number of which the hyperbolic logarithm is unity; the coefficient of  $t^0 t_1^x$

in  $u \left[ \frac{1}{t_1^{dx_1} (1 + a dx_1 - b dx_1)} - 1 \right]^x$  will be therefore

$$e^{(a-b)x_1} d^x \left( y_{0,x_1} e^{(a-b)x_1} \right);$$

hence, we will have

$$y_{x,x_1} = \frac{e^{(a-b)x_1}}{b^x} \frac{d^x (e^{(a-b)x_1} y_{0,x_1})}{dx_1^x}$$

or, more simply,

$$y_{x,x_1} = \frac{e^{(a-b)x_1}}{b^x} \frac{d^x \phi(x_1)}{dx_1^x}$$

$\phi(x_1)$  being an arbitrary function of  $x_1$ .

We can integrate, by the same process, the general equation

$$0 = \Delta^n y_{x,x_1} + a \Delta^{n-1} \frac{\partial y_{x,x_1}}{\partial x_1} + b \partial^{n-2} \frac{\partial^2 y_{x,x_1}}{\partial x_1^2} + \dots;$$

its generating equation is

$$0 = \left(\frac{1}{t} - 1\right)^n + \frac{a}{dx_1} \left(\frac{1}{t} - 1\right)^{n-1} \left(\frac{1}{t^{dx_1}} - 1\right) + \frac{b}{dx_1^2} \left(\frac{1}{t} - 1\right)^{n-2} \left(\frac{1}{t^{dx_1}} - 1\right)^2 + \dots$$

By naming therefore  $\alpha, \alpha_1, \alpha_2$  the  $n$  roots of the equation

$$0 = v^n + av^{n-1} + bv^{n-2} + cv^{n-3} + \dots,$$

we will have the  $n$  partial equations

$$\begin{aligned} \frac{1}{t} &= 1 + \frac{\alpha}{dx_1} \left(\frac{1}{t^{dx_1}} - 1\right), \\ \frac{1}{t} &= 1 + \frac{\alpha_1}{dx_1} \left(\frac{1}{t^{dx_1}} - 1\right), \\ &\dots; \end{aligned}$$

the first gives

$$\frac{1}{t^x} = \frac{\alpha^x}{(dx_1)^x} \left[ \frac{1}{t^{dx_1} \left(1 - \frac{dx_1}{\alpha}\right)} - 1 \right]^x.$$

Now the coefficient of  $t^0 t_1^{dx_1}$  in  $\frac{u}{t^x}$  is  $y_{x,x_1}$ ; this same coefficient in  $u \left[ \frac{1}{t^{dx_1} \left(1 - \frac{dx_1}{\alpha}\right)} - 1 \right]^x$  is

$$\begin{aligned} &\left(1 - \frac{dx_1}{\alpha}\right)^{\frac{x_1}{dx_1}} \left\{ \begin{aligned} &\frac{y_{0,x_1} + x dx_1}{\left(1 - \frac{dx_1}{\alpha}\right)^{\frac{x_1}{dx_1} + x}} - x \frac{y_{0,x_1} + (x-1) dx_1}{\left(1 - \frac{dx_1}{\alpha}\right)^{\frac{x_1}{dx_1} + x - 1}} \\ &+ \frac{x(x-1) y_{0,x_1} + (x-2) dx_1}{1.2 \left(1 - \frac{dx_1}{\alpha}\right)^{\frac{x_1}{dx_1} + x - 2}} - \dots \end{aligned} \right\} \\ &= \left(1 - \frac{dx_1}{\alpha}\right)^{\frac{x_1}{dx_1}} d^x \frac{y_{0,x_1}}{\left(1 - \frac{dx_1}{\alpha}\right)^{\frac{x_1}{dx_1}}} = e^{-\frac{x_1}{\alpha}} d^x y_{0,x_1} e^{\frac{x_1}{\alpha}}, \end{aligned}$$



since

$$\left(1 - \frac{dx_1}{\alpha}\right)^{\frac{x_1}{dx_1}} = e^{-\frac{x_1}{\alpha}};$$

we will have therefore

$$y_{x,x_1} = \alpha^x e^{-\frac{x_1}{\alpha}} \frac{d^x \left( y_{0,x_1} e^{\frac{x_1}{\alpha}} \right)}{dx_1^x},$$

or, more simply,

$$y_{x,x_1} = \alpha^x e^{-\frac{x_1}{\alpha}} \frac{d^x \phi(x_1)}{dx_1^x},$$

$\phi(x_1)$  being an arbitrary function of  $x_1$ .

It follows thence that, if we designate by  $\phi_1(x_1)$ ,  $\phi_2(x_1)$ ,  $\phi_3(x_1)$ , ... some other arbitrary functions of  $x_1$ , the complete expression  $y_x$  will be

$$y_{x,x_1} = \alpha^x e^{-\frac{x_1}{\alpha}} \frac{d^x \phi(x_1)}{dx_1^x} + \alpha_1^x e^{-\frac{x_1}{\alpha_1}} \frac{d^x \phi_1(x_1)}{dx_1^x} + \alpha_2^x e^{-\frac{x_1}{\alpha_2}} \frac{d^x \phi_2(x_1)}{dx_1^x} + \dots$$

#### XXIV.

*Theorems on the expansion of functions in two variables into series.*

If we apply to the functions in two variables the method exhibited in articles X and XI, we will have, in the expansion of these functions into series, some theorems analogous to those in which we are arrived in these two articles. We suppose that  $u$  is equal to the infinite series

$$\begin{aligned} & y_{0,0} + y_{1,0}t + y_{2,0}t^2 + y_{3,0}t^3 + \dots \\ & + y_{0,1}t_1 + y_{1,1}t_1t + y_{2,1}t_1t^2 + \dots \\ & + \dots \end{aligned}$$

and if we designate by the characteristic  $\Delta$  the finite difference of  $y_{x,x_1}$ , taken by making  $x$  and  $x_1$  vary at the same time, the generating function of  $\Delta y_{x,x_1}$  will be  $u \left( \frac{1}{tt_1} - 1 \right)$ ; whence it follows that the function  $\Delta^n y_{x,x_1}$  will be  $u \left( \frac{1}{tt_1} - 1 \right)^n$ . Now we have

$$\frac{1}{tt_1} - 1 = \left(1 + \frac{1}{t} - 1\right) \left(1 + \frac{1}{t_1} - 1\right) - 1,$$

that which gives

$$u \left( \frac{1}{tt_1} - 1 \right)^n = u \left[ \left(1 + \frac{1}{t} - 1\right) \left(1 + \frac{1}{t_1} - 1\right) - 1 \right]^n;$$

hence, if we designate by the characteristic  $\Delta_1$  the finite difference of  $y_{x,x_1}$ , taken by making only  $x$  vary, and by the characteristic  $\Delta_2$  that difference taken by making only  $x_1$  vary, we will have, by passing again from the generating functions to the corresponding variables,

$$\Delta^n y_{x,x_1} = [(1 + \Delta_1 y_{x,x_1})(1 + \Delta_2 y_{x,x_1}) - 1]^n,$$

provided that, in the expansion of the second member of this equation, we apply to the characteristics  $\Delta_1$  and  $\Delta_2$  the exponents of the powers of  $\Delta_1 y_{x,x_1}$  and  $\Delta_2 y_{x,x_1}$ .

By changing  $n$  into  $-n$ , we will be assured easily, by reasoning analogous to that of article X, that the preceding equation will become

$$\Sigma^n y_{x,x_1} = \frac{1}{[(1 + \Delta_1 y_{x,x_1})(1 + \Delta_2 y_{x,x_1}) - 1]^n},$$

provided that, in the expansion of the second member of this equation, we change the negative differences into integrals.

It is clear that  $u \left( \frac{1}{t^i t_1^{i_1}} - 1 \right)^n$  is the generating function of the  $n$ th finite difference of  $y_{x,x_1}$ , when  $x$  varies with  $i$ , and when  $x_1$  varies with  $i_1$ ; now we have

$$u \left( \frac{1}{t^i t_1^{i_1}} - 1 \right)^n = u \left[ \left( 1 + \frac{1}{t} - 1 \right)^i \left( 1 + \frac{1}{t_1} - 1 \right)^{i_1} - 1 \right]^n;$$

therefore, if we designate by the characteristic  ${}^1\Delta$  the finite differences, and by the characteristic  ${}^1\Sigma$  the finite integrals, when  $x$  varies with  $i$  and when  $x_1$  varies with  $i_1$ , we will have, by passing again from the generating functions to the corresponding variables,

$$\begin{aligned} {}^1\Delta^n y_{x,x_1} &= [(1 + \Delta_1 y_{x,x_1})^i (1 + \Delta_2 y_{x,x_1})^{i_1} - 1]^n, \\ {}^1\Sigma^n y_{x,x_1} &= \frac{1}{[(1 + \Delta_1 y_{x,x_1})^i (1 + \Delta_2 y_{x,x_1})^{i_1} - 1]^n}, \end{aligned}$$

provided that, in the expansion of the second members of these equations, we apply to the characteristics  $\Delta_1$  and  $\Delta_2$  the exponents of the powers of  $\Delta_1 y_{x,x_1}$  and  $\Delta_2 y_{x,x_1}$ , and that we change the negative differences into integrals.

The two preceding equations yet hold, by supposing that, in the differences  $\Delta_1 y_{x,x_1}$  and  $\Delta_2 y_{x,x_1}$ ,  $x$  and  $x_1$  instead of varying from unity, vary from any quantity  $\varpi$ ; we must solely observe that, in the difference  ${}^1\Delta y_{x,x_1}$ ,  $x$  will vary from  $i\varpi$  and  $x_1$  will vary from  $i_1\varpi$ ; now, if we suppose  $\varpi$  infinitely small, the differences  $\Delta_1 y_{x,x_1}$  and  $\Delta_2 y_{x,x_1}$  will be changed: the first into  $dx \frac{\partial y_{x,x_1}}{\partial x}$  and the second into  $dx_1 \frac{\partial y_{x,x_1}}{\partial x_1}$ . Moreover, if we make  $i$  and  $i_1$  infinitely great and if we suppose  $i dx = \alpha$  and  $i_1 dx_1 = \alpha_1$ , we will have

$$(1 + \Delta_1 y_{x,x_1})^i = \left( 1 + dx \frac{\partial y_{x,x_1}}{\partial x} \right)^{\frac{\alpha}{dx}} = e^{\alpha \frac{\partial y_{x,x_1}}{\partial x}},$$

$e$  being the number of which the hyperbolic logarithm is unity; we will have similarly

$$(1 + \Delta_2 y_{x,x_1})^{i_1} = e^{\alpha_1 \frac{\partial y_{x,x_1}}{\partial x_1}},$$

hence

$$\begin{aligned} \Delta^n y_{x,x_1} &= \left( e^{\alpha \frac{\partial y_{x,x_1}}{\partial x} + \alpha_1 \frac{\partial y_{x,x_1}}{\partial x_1}} - 1 \right)^n, \\ \Sigma^n y_{x,x_1} &= \frac{1}{\left( e^{\alpha \frac{\partial y_{x,x_1}}{\partial x} + \alpha_1 \frac{\partial y_{x,x_1}}{\partial x_1}} - 1 \right)^n}, \end{aligned}$$

$x$  varying from  $\alpha$  and  $x_1$  varying from  $\alpha_1$  in the two first members of these equations.

If, instead of supposing  $\varpi$  infinitely small, we suppose it finite and  $i$  infinitely small and equal to  $dx$ ; if we suppose, moreover,  $i_1$  infinitely small and equal to  $dx_1$ , we will have

$$(1 + \Delta_1 y_{x,x_1})^i = (1 + \Delta_1 y_{x,x_1})^{dx} = 1 + dx \log(1 + \Delta_1 y_{x,x_1}).$$

We will have similarly

$$(1 + \Delta_2 y_{x,x_1})^{i_1} = 1 + dx_1 \log(1 + \Delta_2 y_{x,x_1});$$

moreover  $\Delta^n y_{x,x_1}$  is changed into  $d^n y_{x,x_1}$ ; hence

$$d^n y_{x,x_1} = \{[1 + dx \log(1 + \Delta_1 y_{x,x_1})][1 + dx_1 \log(1 + \Delta_2 y_{x,x_1})] - 1\}^n$$

or, more simply,

$$d^n y_{x,x_1} = [dx \log(1 + \Delta_1 y_{x,x_1}) + dx_1 \log(1 + \Delta_2 y_{x,x_1})]^n.$$

We can obtain in this manner an infinity of other similar formulas; but it suffices to have exhibited the method for arriving to them.

All that which we have said on the functions of two variables can be applied equally to those of three or of a greater number of variables, we will not insist further on this object.