MÉMOIRE

SUR

LES INTÉGRALES DÉFINIES

ET

LEUR APPLICATION AUX PROBABILITÉS

ET SPÉCIALEMENT A LA RECHERCHE DU MILIEU QU'IL FAUT CHOISIR ENTRE LES RÉSULTATS DES OBSERVATIONS.

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Mémoires de l'Académie des Sciences, I^{re} Série, T. XI (I^{re} Partie); 1810; 1811
Œuvres complètes T. X, pp. 357–412

I have given, it has been around thirty years ago, in the *Mémoires de l'Académie des Sciences*[1], the theory of generating functions and that of the approximation of formulas which are functions of large numbers. The first has for object the relations of the coefficients of the powers of an undetermined variable in the development of a function of this variable to the function itself. From the simple consideration of these relations there results, with an extreme facility, the interpolation of the series, their transformation, integration of equations in the ordinary and partial differences, the analogy of the powers and of the differences, and generally the transport of the exponents of the powers to the characteristics which express the manner of being of the variables. The theory of the approximations by formulas functions of very great numbers is based on the expression of the variables given by some equations in the differences, by means of definite integrals which we integrate by some very convergent approximations; and there is this of the quite remarkable, namely that the quantity under the integral sign is the generating function of the variable expressed by the definite integral, so that the theories of the generating functions and of the approximations by formulas functions of very great numbers can be considered as the two branches of the same calculus, which I designate by the name of *calculus of the generating functions*. This which Arbogast[2] has named *Method of separation of the scale of operations* is contained in the first part of the calculus of generating functions, which gives simultaneously the demon-

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[1] *Oeuvres de Laplace*, T. X. "Mémoire sur les suites."

[2] *Translator's note*: Louis Arbogaste (1759-1803). Professor of mathematics at Alsace. His chief work is *Calcul des Dérivations*, 1800.

stration and the metaphysical truth of this method. This which Kramp[3] and others have named *numerical faculties*, and this which Euler has named *inexplicable functions*, is connected to the second part, with this advantage, that these faculties and these inexplicable functions, put under the form of definite integrals, present then some clear ideas, and are susceptible to all the operations of Analysis.

The calculus of generating functions extends to the infinitely small differences; because, if we develop all the terms of an equation in the differences with respect to the powers of the difference supposed indeterminate, but infinitely small, and if we ignore the infinitely smalls of an order superior relatively to those of an inferior order, we will have an equation in the infinitely small differences, of which the integral is that of the equation in the finite differences, in which we ignore likewise the infinitely small with respect to the finite quantities.

The quantities which we ignore in these passages from the finite to the infinitely small seem to deprive from the infinitesimal Calculus the rigor of the geometric results; but, in order to render rigor to it, it suffices to consider the quantities which we conserve in the development of an equation in the finite differences and of its integral, with respect to the powers of the indeterminate difference, as having all for factor the smallest power of which we compare the coefficients among them. This comparison being rigorous, the differential Calculus, which is evidently only this same comparison, has all the rigor of the other algebraic operations. But the consideration of the infinitely small of different orders, the facility to identify them, *a priori*, by the sole inspection of the magnitudes, and the omission of the infinitely small of an order superior to the one which we conserve, in measure as they present themselves, extremely simplify the calculus, and are one of the principal advantages of the infinitesimal Analysis, which besides, by realizing the infinitely small and attributing to them of very small values, gives, by a first approximation, the differences and the sums of the quantities.

The passage from the finite to the infinitely small has the advantage of clarifying many points of the infinitesimal Analysis, which have been the object of great disputes among the geometers. It is thus that, in the *Mémoires de l'Académie des Sciences* for the year 1779[4], I have shown that the arbitrary functions which the integration of the partial differential equations introduces could be discontinuous, and I have determined the conditions to which this discontinuity must be subject. The transcendent results of Analysis are, as all the abstractions of understanding, some general signs of which we can determine the true extent only by going up again by metaphysical Analysis to the elementary ideas which have led to it, that which presents often great difficulties; because the human mind tests it yet less by carrying to the future than by retreating to itself.

It appears that Fermat, the true inventor of the differential Calculus, has considered this calculus as a derivation of the one of finite differences, by neglecting the infinitely small of a superior order with respect to those of an inferior order: this is at least that which he has done in his method *De maximis* and in that of the tangents, which he has extended to the transcendent curves. We see still by his beautiful solution of the

[3] *Translator's note*: Christian Kramp (1760-1826) of Strassburg. He is now known for introducing $n!$ for factorial n. More importantly, the first published table of the normal probability integral appeared in *Analyse des réfractions, astronomiques et terrestres*, 1799.

[4] *Oeuvres de Laplace*, T. X. "Mémoire sur les suites."

problem of refraction of light, by supposing that it arrive from one point to another in the shortest time, and by imagining that it moves in diverse transparent mediums with different speeds, we see, I say, that he knew to extend his calculus to the irrational functions, by getting rid of the irrationalities by raising radicals to powers. Newton has, since, rendered this calculus more analytic in his *Method of Fluxions*, and he has simplified it and generalized the processes by the invention of his theorem of the binomial; finally, nearly at the same time, Leibnitz has enriched the differential Calculus by a very fortunate notation, and which is adapted from itself to the extension that the differential Calculus has received by the consideration of the partial differentials. The language of Analysis, most perfect of all, being by itself a powerful instrument of discovery, its notations, when they are necessities and fortunately imagined, are the germs of new calculations. Thus the simple idea which Descartes had to indicate the powers of quantities, represented by some letters, by writing towards the height of these letters the numbers which express the degree of these powers, has given birth to the exponential Calculus; and Leibnitz has been led, by his notation, to the singular analogy of powers and differences. The calculus of generating functions, which gives the true origin of this analogy, offers so many examples of this transport from the exponents of the powers to the characteristics, that it can again be considered as the exponential calculus of the characteristics.

The calculus of generating functions is the foundation of a theory which I myself propose to publish soon on the probabilities. The questions related to the events due to chance are restored most often with facility to some equations in the differences: the first branch of this calculus furnishes the most general and most simple solutions. But, when the events which we consider are in very great number, the formulas to which we are led are composed of a so great multitude of terms and of factors, that their numerical calculation becomes impractical. It is then indispensable to have a method which transforms these formulas into convergent series. This is that which the second branch of the calculus of the generating functions makes with so much more advantage as the method becomes more necessary. By this means, we can determine with facility the limits of the probability of the results and of the causes, indicated by the events considered in great number, and the laws according to which this probability approaches its limits, in measure as the events are multiplied. This research, the most delicate of the theory of chances, merits the attention of geometers by the analysis that it requires, and that of the philosophers, by showing how the regularity ends by being established in the same things which appear to us delivered entirely by chance, and by unfolding to us the hidden, but constant, causes on which this regularity depends.

The consideration of the definite integrals by which the quantities are represented in the theory of the approximation by formulas functions of very great numbers has led me to the values of many definite integrals as I have given in the *Mémoires de l'Académie des Sciences* for the year 1782[5], and which offer this of the remarkable, namely that they depend all at once on these two transcendents: the ratio of the circumference to the diameter and the number of which the hyperbolic logarithm is unity. I have obtained these values by a singular analogy, founded on the passages from the real to the

[5] *Oeuvres de Laplace*, T. X. "*Mémoire sur les approximations des formules qui sont fontions de très grands nombres.*"

imaginary, passages which can be considered as means of discovery, of which the first applications have appeared, if I do not deceive myself, in the *Mémoires* cited and which have led to the values of diverse definite integrals depending on sines and cosines. But these means, as the one of induction, although employed with much precaution and reserve, leave always to desire some direct demonstrations of their results. Mr. Poisson just gave many in the *Bulletin de la Société philomathique* of the month of March of this year 1811. I propose myself here to find directly all these values, and those of definite integrals, more general still, and which seems to me might be interesting to geometers.

The research of these values in not at all a simple game of Analysis: it is of great utility in the theory of probabilities. I make here an application of it to three problems of this kind, which it would be very difficult to resolve by other methods. The second of these problems is remarkable in this that its solution offers the first example of the use of the calculus in the infinitely small partial differences, in the questions of probabilities. The third problem is related to the mean which it is necessary to choose among the results given by diverse observations: this is one of the most useful of all the analysis of chances and, for this reason, I treat it with extent; I dare to believe that my analysis will interest the geometers.

When we wish to correct by the assembly of a great number of observations many elements already known nearly, we start in the following manner. Each observation being a function of the elements, we substitute into this function their approximate values, augmented respectively by small corrections which it is the question to know. By developing next the function into series, with respect to these corrections, and neglecting their squares and their products, we equate the series to observation, this which gives a first equation of condition among the correction of the elements. A second observation furnishes a similar equation of condition, and thus of the rest. If the observations were rigorous, it would suffice to employ as many as there are elements; but, seeing the errors of which they are susceptible, we consider a great number of them, in order to compensate these errors by one another, in the values of the corrections which we deduce from their assembly. But in what manner is it necessary to combine among them the equations of condition in order to have the most precise corrections? It is here that the analysis of probabilities can be of a great relief. All the ways to combine these equations are reduced to multiplying each of them by a particular factor and making a sum of all these products: we form thus a first final equation among the corrections of the elements. A second system of factors gives a second final equation, and thus in sequence until we have as many final equations as elements from which we will determine the corrections by resolving these equations. Now it is clear that it is necessary to choose the systems of factors, so that the mean error to fear to *more* or to *less* on each element is a minimum. I intend by *mean error* the sum of the products of each error by its probability. By determining the factors by this condition, the analysis leads to this remarkable result, namely that, if we prepare each equation of condition in a way that its second member is zero, the sum of the squares of the first members is a minimum, by making vary successively each correction. Thus this method, which Messrs. Legendre and Gauss have proposed, and which, until the present, presented only the advantage to furnish, without any groping, the final equations necessary to correct the elements, gives at the same time the most precise corrections.

I.

On the definite integrals.

We consider the definite integral

$$\int_0^\infty \frac{dx\,e^{-ax}}{x^\omega}(\cos rx - \sqrt{-1}\sin rx) \quad \text{or} \quad \int_0^\infty \frac{dx\,e^{-ax-rx\sqrt{-1}}}{x^\omega},$$

e being the number of which the hyperbolic logarithm is unity. By reducing $e^{-rx\sqrt{-1}}$ into series, it becomes

$$\int_0^\infty \frac{dx\,e^{-ax}}{x^\omega}\left[1 - \frac{r^2x^2}{1.2} + \frac{r^4x^4}{1.2.3.4} - \frac{r^6x^6}{1.2.3.4.5.6} + \cdots \right.$$
$$\left. - rx\sqrt{-1}\left(1 - \frac{r^2x^2}{1.2.3} + \frac{r^4x^4}{1.2.3.4.5} + \cdots\right)\right].$$

Now we have generally

$$\int_0^\infty x^{i-\omega}dx\,e^{-ax} = \frac{(1-\omega)(2-\omega)\cdots(i-\omega)}{a^i}\int_0^\infty \frac{dx\,e^{-ax}}{x^\omega};$$

by making next $ax = s$, we have

$$\int_0^\infty \frac{dx}{x^\omega}e^{-ax} = \frac{1}{a^{1-\omega}}\int_0^\infty \frac{ds}{s^\omega}e^{-s}.$$

By naming therefore this last integral k, we will have

$$\int_0^\infty x^{i-\omega}dx\,e^{-ax} = \frac{(1-\omega)(2-\omega)\cdots(i-\omega)}{a^{i+1-\omega}}k,$$

whence we deduce

$$\int \frac{dx\,e^{-ax}}{x^\omega}(\cos rx - \sqrt{-1}\sin rx)$$
$$= \frac{k}{a^{1-\omega}}\left\{1 - \frac{(1-\omega)(2-\omega)}{1.2}\frac{r^2}{a^2} + \frac{(1-\omega)(2-\omega)(3-\omega)(4-\omega)}{1.2.3.4}\frac{r^4}{a^4} - \cdots \right.$$
$$\left. - \sqrt{-1}\left[\frac{1-\omega}{1}\frac{r}{a} - \frac{(1-\omega)(2-\omega)(3-\omega)}{1.2.3}\frac{r^3}{a^3} + \cdots\right]\right\}.$$

If we make $\frac{r}{a} = t$, the second member of this equation becomes

$$\frac{k}{a^{1-\omega}(1+t\sqrt{-1})^{1-\omega}}.$$

Let A be an angle of which t is the tangent, we will have

$$\sin A = \frac{t}{\sqrt{1+t^2}}, \quad \cos A = \frac{1}{\sqrt{1+t^2}},$$

consequently

$$\cos A - \sqrt{-1}\sin A = \frac{1 - t\sqrt{-1}}{\sqrt{1+t^2}} = \frac{\sqrt{1+t^2}}{1+t\sqrt{-1}},$$

this which gives, by the known theorem,

$$\cos(1-\omega)A - \sqrt{-1}\sin(1-\omega)A = \frac{(1+t^2)^{\frac{1-\omega}{2}}}{(1+t\sqrt{-1})^{1-\omega}};$$

the tangent t is not only the tangent of A, but yet that of the angle increased by any multiple of the semi-circumference; but, the first member of this equation must be reduced to unity when t is null, it is clear that we must take for A the smallest positive angles of which t is the tangent.

Now this equation gives, by restoring $\frac{r}{a}$ in the place of t,

$$\frac{k}{a^{1-\omega}(1+t\sqrt{-1})^{1-\omega}} = \frac{k}{(a^2+r^2)^{\frac{1-\omega}{2}}}[\cos(1-\omega)A - \sqrt{-1}\sin(1-\omega)A];$$

we have therefore

$$\int_0^\infty \frac{dx\, e^{-ax}}{x^\omega}(\cos rx - \sqrt{-1}\sin rx)$$
$$= \frac{k}{(a^2+r^2)^{\frac{1-\omega}{2}}}[\cos(1-\omega)A - \sqrt{-1}\sin(1-\omega)A].$$

By comparing separately the real and the imaginary quantities, we have these two equations

$$\int_0^\infty \frac{dx\cos rx\, e^{-ax}}{x^\omega} = \frac{k}{(a^2+r^2)^{\frac{1-\omega}{2}}}\cos(1-\omega)A,$$
$$\int_0^\infty \frac{dx\sin rx\, e^{-ax}}{x^\omega} = \frac{k}{(a^2+r^2)^{\frac{1-\omega}{2}}}\sin(1-\omega)A.$$

If a is null, $\frac{r}{a}$ will be infinity, and the smallest angle of which it is the tangent will be $\frac{\pi}{2}$, π being the semi-circumference of which the radius is unity; we have therefore

$$\int_0^\infty \frac{dx\cos rx}{x^\omega} = \frac{k}{r^{1-\omega}}\cos(\frac{1-\omega)\pi}{2},$$
$$\int_0^\infty \frac{dx\sin rx}{x^\omega} = \frac{k}{r^{1-\omega}}\sin\frac{(1-\omega)\pi}{2}.$$

In the case of $\omega = \frac{1}{2}$, we have, by making $s^{\frac{1}{2}} = t$,

$$k = \int_0^\infty \frac{ds}{s^{\frac{1}{2}}}e^{-s} = 2\int_0^\infty dt\, e^{-t^2}.$$

This last member is $\sqrt{\pi}$; therefore we have $k = \sqrt{\pi}$; if we suppose next $r = 1$, we will have

$$\int_0^\infty \frac{dx\cos x}{\sqrt{x}} = \sqrt{\frac{\pi}{2}} = \int_0^\infty \frac{dx\sin x}{\sqrt{x}}.$$

Euler has attained to all these equations in Book IV of his *Calcul intégral*, published in 1794, by the consideration of the passage from the real to the imaginary.

II.

We consider presently the integral $\int_0^\infty dx \cos rx\, e^{-a^2x^2}$. If we name this integral y, we will have

$$\begin{aligned}\frac{dy}{dr} &= -\int_0^\infty x\,dx \sin rx\, e^{-a^2x^2}\\ &= \left(\frac{1}{2a^2}\sin rx\, e^{-a^2x^2}\right)_0^\infty - \frac{r}{2a^2}\int_0^\infty dx \cos rx\, e^{-a^2x^2};\end{aligned}$$

we will have therefore

$$\frac{dy}{dr} + \frac{r}{2a^2}y = 0.$$

The integral of this equation is

$$y = B\, e^{-\frac{r^2}{4a^2}},$$

B being an arbitrary constant; in order to determine it, we will observe that, if we make r null, $\cos rx$ becomes unity, and we have

$$v = \int_0^\infty dx\, e^{-a^2x^2};$$

this last integral is, as we know, equal to $\frac{\sqrt{\pi}}{2a}$; therefore

$$B = \frac{\sqrt{\pi}}{2a};$$

we have therefore

$$\int_0^\infty dx \cos rx\, e^{-a^2x^2} = \frac{\sqrt{\pi}}{2a}\, e^{-\frac{r^2}{4a^2}}.$$

Thence we deduce

$$\int_0^\infty x^{2n}\, dx \cos rx\, e^{-a^2x^2} = \pm\frac{\sqrt{\pi}}{2a}\,\frac{d^{2n}}{dr^{2n}} e^{-\frac{r^2}{4a^2}},$$

the $+$ sign having place if n is even, and the $-$ sign if n is odd; we will have similarly, by differentiating with respect to r,

$$\int_0^\infty x^{2n+1}\, dx \sin rx\, e^{-a^2x^2} = \mp\frac{\sqrt{\pi}}{2a}\,\frac{d^{2n+1}}{dr^{2n+1}} e^{-\frac{r^2}{4a^2}}.$$

By integrating one time with respect to r the expression of

$$\int_0^\infty dx \cos rx\, e^{-a^2x^2},$$

we will have

$$\int_0^\infty \frac{dx \sin rx}{x} e^{-a^2x^2} = \frac{\sqrt{\pi}}{2a}\int_0^\infty dr\, e^{-\frac{r^2}{4a^2}}.$$

III.

We consider next the double integral

$$\int_0^\infty \int_0^\infty 2dx\, y\, dy\, e^{-y^2(1+x^2)} \cos rx.$$

By integrating it first with respect to y, it becomes

$$\int_0^\infty \frac{dx \cos rx}{1+x^2}.$$

We integrate it now with respect to x; we have, by the preceding article,

$$\int_0^\infty dx \cos rx\, e^{-y^2x^2} = \frac{\sqrt{\pi}}{2y} e^{-\frac{r^2}{4y^2}},$$

this which gives

$$\int_0^\infty \int_0^\infty 2y\, dy\, dx \cos rx\, e^{-y^2(1+x^2)} = \sqrt{\pi} \int_0^\infty dy\, e^{-y^2-\frac{r^2}{4y^2}} = \sqrt{\pi}\, e^r \int_0^\infty dy\, e^{-\left(\frac{2y^2+r}{2y}\right)^2}.$$

r being supposed positive, the quantity $\left(\frac{2y^2+r}{2y}\right)^2$ has a minimum which corresponds to $y = \sqrt{\frac{r}{2}}$, this which gives $2r$ for this minimum. Let therefore

$$y = \frac{1}{2}z + \frac{1}{2}\sqrt{z^2+2r};$$

y needing to be extended from $y = 0$ to $y = \infty$, z must extend from $z = -\infty$ to $z = \infty$. This value of y gives

$$dy = \frac{1}{2}dz + \frac{1}{2}\frac{zdz}{\sqrt{z^2+2r}}.$$

By taking the integrals from $z = -\infty$ to $z = \infty$, we have

$$\int_{-\infty}^\infty dz\, e^{-z^2} = \sqrt{\pi}, \qquad \int_{-\infty}^\infty \frac{z\, dz\, e^{-z^2}}{\sqrt{z^2+2r}} = 0;$$

we have therefore

$$\sqrt{\pi}\, e^r \int_0^\infty dy\, e^{-\left(\frac{2y^2+r}{2y}\right)^2} = \sqrt{\pi}\, e^r \int_{-\infty}^\infty dz\, e^{-z^2-2r} = \frac{\pi e^{-r}}{2},$$

hence

$$\int_0^\infty \frac{dx \cos rx}{1+x^2} = \frac{\pi}{2e^r}.$$

By differentiating with respect to r, we have

$$\int_0^\infty \frac{x\, dx \sin rx}{1+x^2} = \frac{\pi}{2e^r},$$

this which gives

$$\int_0^\infty \frac{dx\,(\cos rx + x\sin rx)}{1+x^2} = \frac{\pi}{2e^r};$$

by making $r = 1$, we will have the theorem which I have given in the *Mémoires de l'Académie des Sciences*, year 1782, page 59[6].

If we make $rx = t$, we will have

$$\int_0^\infty \frac{dx\cos rx}{1+x^2} = \int_0^\infty \frac{r\,dt\cos t}{r^2+t^2},$$

hence

$$\int_0^\infty \frac{dt\cos t}{r^2+t^2} = \frac{\pi e^{-r}}{2r}.$$

Let $r^2 = q$, we will have by differentiating $i-1$ times with respect to q the preceding equation, and restoring for t its value rx,

$$\int_0^\infty \frac{dx\cos rx}{(1+x^2)^i} = \frac{(-1)^{i-1}q^{i-\frac{1}{2}}}{1.2.3\ldots(i-1)}\frac{\pi}{2}\frac{d^{i-1}}{dq^{i-1}}\frac{e^{-\sqrt{q}}}{\sqrt{q}};$$

we could integrate therefore generally, from x null to x infinity, the differential

$$\frac{(A+Bx^2+Cx^4+\cdots Hx^{2i-2})\,dx\cos rx}{(1+x^2)^i},$$

because, by putting any term, such as Fx^{2n}, under the from $F(1+x^2-1)^n$, and by developing it according to the powers of $1+x^2$, we will reduce the preceding differential to a sequence of differentials of the form $\frac{k\,dx\cos rx}{(1+x^2)^n}$, which will be integrable by that which precedes; we will have therefore as function of r the integral of the preceding differential. We could likewise, instead of $\cos rx$, substitute in it a whole and positive power of this cosine, because this power is decomposed into cosine of the angle and of its multiples. We name R the function of r of which there is question, we will have, by differentiating with respect to r this integral,

$$\int_0^\infty x\,dx\sin rx\,\frac{A+Bx^2+Cx^4+\cdots Hx^{2i-2}}{(1+x^2)^i} = -\frac{dR}{dr}.$$

By integrating with respect to r, after having multiplied by dr, we will have

$$\int_0^\infty dx\sin rx\,\frac{A+Bx^2+Cx^4+\cdots Hx^{2i-2}}{x(1+x^2)^i} = \int_0^r R\,dr.$$

We can, by means of passages from the real to the imaginary, easily conclude from the value of the integral $\int_0^\infty \frac{dx\cos rx}{1+x^2}$ the value of the integral $\int_0^\infty \frac{M}{N}dx\cos rx$, M and N being some rational and entire functions of x^2, such that the denominator N is of a higher degree than M, and has no real factor in x of the first degree. In this case, the

[6] *Oeuvres de Laplace*, T. X. p. 264. "*Mémoire sur les approximations des formules qui sont fontions de très grands nombres.*" *Section XXI.*

fraction $\frac{M}{N}$ is decomposable into fractions of the form $\frac{A}{1+\beta^2x^2}$, A and β being reals or imaginaries. Now we have, by making $\beta x = x'$ and $\frac{r}{\beta} = r'$.

$$A\int_0^\infty \frac{dx\cos rx}{1+\beta^2x^2} = \frac{A}{\beta}\int_0^\infty \frac{dx'\cos r'x'}{1+x'^2};$$

by giving therefore generally to this last integral the value which it has in the case of x' real and which, by that which precedes, is equal to $\frac{\pi}{2}e^{-r'}$, we will have

$$A\int_0^\infty \frac{dx\cos rx}{1+\beta^2x^2} = \frac{A\pi}{2\beta}e^{-\frac{r}{\beta}}.$$

$\frac{1}{\beta}$ is the square root of $\frac{1}{\beta^2}$, and this root is equally $-\frac{1}{\beta}$; but the integral $\int_0^\infty \frac{M}{N}dx\cos rx$ never becoming infinite in the case even of r infinity, and moreover $\cos rx$ changing not at all, in changing the sign of r, it is clear that we must choose the one of the two roots $\frac{1}{\beta}$ and $-\frac{1}{\beta}$, of which the real part is positive. We find thus, for example,

$$\int_0^\infty \frac{dr\cos rx}{1+x^4} = \frac{\pi}{2\sqrt{2}}e^{\frac{-r}{\sqrt{2}}}\left(\cos\frac{r}{\sqrt{2}} + \sin\frac{r}{\sqrt{2}}\right).$$

IV.

Application of the preceding analysis to the probabilities.

We apply the preceding analysis to the theory of probabilities. For this, we consider two players A and B, of whom the skills are equal, and playing together in a way that B has originally r tokens; that, at each trial which he loses, he gives one of his tokens to player A, and that, at each trial which he wins, he receives one of them from player A, who is supposed to have an infinite number. The game continues until player A has won all the tokens from B. This put, r being a great number, we demand how many trials we can wager one against one, or two against one, or three against two, etc., that player A will have won the game.

We are going to establish first that the game must end. For this, let y_r be the probability that it will end. After the first trial, this probability is either y_{r-1} or y_{r+1}, according as player A wins or loses this trial; we have therefore

$$y_r = \frac{1}{2}y_{r+1} + \frac{1}{2}y_{r-1}.$$

The integral of this equation in the differences is

$$y_r = a + br,$$

a and b being some arbitrary constants. I observe first that the constant b must be null, otherwise y_r increases indefinitely, this which cannot be, because it can never overtake unity. Moreover y_r is 1 when $r = 0$, because then, B having no more tokens, the game is ended; therefore

$$y_r = 1.$$

We seek now the probability that the game will end after or at trial x. By naming $y_{r,x}$ this probability, we will have

$$y_{r,x} = \frac{1}{2} y_{r+1,x-1} + \frac{1}{2} y_{r-1,x-1}.$$

It is necessary to integrate this equation in the finite partial differences by fulfilling the following conditions: 1° that $y_{r,x}$ is null when x is less than r; 2° that it is equal to unity when r is null. These two conditions being fulfilled, the preceding equation in the differences gives all the values of $y_{r,x}$ whatever be r and x. Presently, the following expression of $y_{r,x}$ satisfies these conditions and the equation in the partial differences, whence it follows that it expresses the true value of $y_{r,x}$,

$$y_{r,x} = 1 - \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{d\phi \sin r\phi \, (\cos\phi)^{r+2i+1}}{\sin\phi}.$$

x is equal to $r + 2i$; in fact, it can be only r or this number increased by an even number, because the number of games played must be equal to r or surpass it by an even number, since A cannot win the game, unless he wins the number r of tokens of B, plus those which he has lost, and it is necessary on account of it two games for each token, one for the loss and the other for the win. I will give not at all the analysis which has led me to the preceding expression: I will be content to show that it satisfies the equation in the partial differences and the conditions prescribed above. First, by substituting it into the equation in the partial differences,we have

$$\int_0^{\frac{\pi}{2}} \frac{d\phi \sin r\phi \, (\cos\ \phi)^{x+1}}{\sin\phi} = \int_0^{\frac{\pi}{2}} \frac{d\phi \, (\cos\phi)^{x}}{\sin\phi} \left[\frac{1}{2} \sin(r+1)\phi + \frac{1}{2} \sin(r-1)\phi \right],$$

equation evident. Moreover, if we make r null, the preceding expression of $y_{r,x}$ becomes unity. Finally, if we make i negative, it is reduced to zero. In fact, we have

$$\begin{aligned} \frac{\sin r\phi}{\sin\phi} &= \frac{e^{r\phi\sqrt{-1}} - e^{-r\phi\sqrt{-1}}}{e^{\phi\sqrt{-1}} - e^{-\phi\sqrt{-1}}} \\ &= e^{(r-1)\phi\sqrt{-1}} + e^{(r-3)\phi\sqrt{-1}} + \cdots + e^{-(r-3)\phi\sqrt{-1}} + e^{-(r-1)\phi\sqrt{-1}}. \end{aligned}$$

Moreover, $(\cos\phi)^{r-2i+1}$ is equal to

$$\frac{(e^{\phi\sqrt{-1}} - e^{-\phi\sqrt{-1}})^{r-2i+1}}{2^{r-2i+1}}.$$

By developing this function and multiplying this development by the one of $\frac{\sin r\phi}{\sin\phi}$, each term of the first development will give, in the product, a term independent of ϕ; the sum of all these terms will be therefore $\frac{(1+1)^{r-2i+1}}{2^{r-2i+1}}$ or unity, and by multiplying this sum by $\frac{2}{\pi}\int_0^{\frac{\pi}{2}} d\phi$, the product will be unity. The other terms of the product of the two preceding developments will be of the cosines of 2ϕ, 4ϕ, ..., and the integral of their products by $d\phi$ will be null. We have therefore

$$y_{r,x} = 0,$$

when i is negative.

We suppose now that r and i are great numbers. The maximum of the function

$$\frac{\phi(\cos\phi)^{r+2i+1}}{\sin\phi}$$

corresponds to $\phi = 0$, this which gives 1 for this maximum. The function decreases next with a very great rapidity, and in the interval where it has a sensible value, we can suppose

$$\log\sin\phi = \log\phi + \log(1 - \frac{1}{6}\phi^2) = \log\phi - \frac{1}{6}\phi^2,$$

$$\begin{aligned}\log(\cos\phi)^{r+2i+1} &= (r+2i+1)\log(1 - \frac{1}{2}\phi^2 + \frac{1}{24}\phi^4) \\ &= -\frac{r+2i+1}{2}\phi^2 - \frac{r+2i+1}{12}\phi^4,\end{aligned}$$

this which gives, by neglecting the sixth powers of ϕ and its fourth powers which are not multiplied by $r + 2i + 1$,

$$\log\frac{(\cos\phi)^{r+2i+1}}{\sin\phi} = -\log\phi - \frac{r+2i+\frac{2}{3}}{2}\phi^2 - \frac{r+2i+\frac{2}{3}}{12}\phi^4.$$

By making therefore

$$a^2 = \frac{r+2i+\frac{2}{3}}{2},$$

we will have

$$\frac{(\cos\phi)^{r+2i+1}}{\sin\phi} = \frac{1 - \frac{a^2}{6}\phi^4}{\phi}e^{-a^2\phi^2},$$

consequently,

$$\int_0^{\frac{\pi}{2}} \frac{d\phi\sin r\phi\,(\cos\phi)^{r+2i+1}}{\sin\phi} = \int_0^{\frac{\pi}{2}} \frac{d\phi\left(1 - \frac{a^2}{6}\phi^4\right)}{\phi}\sin r\phi\, e^{-a^2\phi^2}.$$

This last integral can be taken from $\phi = 0$ to ϕ infinity, because it must be taken from $\phi = 0$ to $\phi = \frac{1}{2}\pi$; now, a^2 being a large number, $e^{-a^2\phi^2}$ becomes excessively small when we make $\phi = \frac{1}{2}\pi$, so that we can suppose it null, seeing the extreme rapidity with which this exponential diminishes when ϕ increases. Now we have

$$\frac{d}{dr}\int_0^{\infty} \frac{d\phi\left(1 - \frac{a^2}{6}\phi^4\right)}{\phi}\sin r\phi\, e^{-a^2\phi^2} = \int_0^{\infty} d\phi\,\left(1 - \frac{a^2}{6}\phi^4\right)e^{-a^2\phi^2}\cos r\phi;$$

we have besides, by article II,

$$\int_0^{\infty} d\phi\cos r\phi\, e^{-a^2\phi^2} = \frac{\sqrt{\pi}}{2a}e^{\frac{-r^2}{4a^2}},$$

$$\int_0^{\infty} \phi^4\, d\phi\cos r\phi\, e^{-a^2\phi^2} = \frac{\sqrt{\pi}}{2a}\frac{d^4}{dr^4}\, e^{\frac{-r^2}{4a^2}} = \frac{3\sqrt{\pi}}{8a^3}\, e^{\frac{-r^2}{4a^2}}\left(1 - \frac{r^2}{a^2} + \frac{r^4}{12a^4}\right);$$

whence we deduce, by supposing $\frac{r^2}{4a^2} = t^2$,

$$\int_0^\infty \frac{d\phi \sin r\phi\,(\cos\phi)^{r+2i+1}}{\sin\phi} = \sqrt{\pi}\left[\int_0^T dt\, e^{-t^2} - \frac{te^{-t^2}}{8a^2}(1 - \frac{2}{3}t^2)\right];$$

thus the probability that A will win the game in a number $r + 2i$ of trials is

$$1 - \frac{2}{\sqrt{\pi}}\left[\int_0^T dt\, e^{-t^2} - \frac{Te^{-T^2}}{8a^2}(1 - \frac{2}{3}T^2)\right],$$

T^2 being equal to $\frac{r^2}{4a^2}$,

If we seek the number of trials in which we can wager one against one that this will take place, we make this probability equal to $\frac{1}{2}$, this which gives

$$\int_0^T dt\, e^{-t^2} = \frac{\sqrt{\pi}}{4} + \frac{Te^{-T^2}}{8a^2}(1 - \frac{2}{3}T^2).$$

We name T' the value of t which corresponds to the equation

$$\int_0^t dt\, e^{-t^2} = \frac{\sqrt{\pi}}{4},$$

and we suppose $T = T' + q$, q being of the order $\frac{1}{a^2}$. The integral $\int_0^{T'} dt\, e^{-t^2}$ will be augmented very nearly by $qe^{-T'^2}$, this which gives

$$qe^{-T'^2} = \frac{T'e^{-T'^2}}{8a^2}(1 - \frac{2}{3}T'^2);$$

we will have

$$T^2 = T'^2 + \frac{T'^2}{4a^2}(1 - \frac{2}{3}T'^2).$$

Having thus T^2 to the quantities nearly of order $\frac{1}{a^4}$, we will have, to the quantities nearly of order $\frac{1}{a^2}$, by virtue of the equation

$$2a^2 = r + 2i + \frac{2}{3} = \frac{r^2}{2T^2},$$

the following

$$r + 2i = \frac{r^2}{2T'^2} - \frac{7}{6} + \frac{2}{3}T'^2.$$

In order to determine the value of T'^2, we will observe that here T' is smaller than $\frac{1}{2}$; thus the transcendent and integral equation

$$\int dt\, e^{-t^2} = \frac{\sqrt{\pi}}{4}$$

can be transformed into the following:

$$T' - \frac{1}{3}T'^3 + \frac{1}{1.2}\frac{1}{5}T'^5 - \frac{1}{1.2.3}\frac{1}{7}T'^7 + \cdots = \frac{\sqrt{\pi}}{4}.$$

By resolving this equation, we find

$$T'^2 = 0.2102497.$$

Thus, by supposing $r = 100$, we will have

$$r + 2i = 23780.14;$$

it is therefore then disadvantage to wager one against one that A will win the game in 23780 trials; but there is advantage to wager that he will win it in 23781 trials.

V.

We consider two urns A and B, each containing the same number n of balls; and we suppose that, in the total number $2n$ of balls, there are as many of them white as of black. We imagine that we draw at the same time a ball from each urn, and that next we put into one urn the ball extracted from the other. We suppose that we repeat this operation any number r times, by agitating at each time the urns in order to well mix the balls; and we seek the probability that after this number r of operations there will be x white balls in urn A.

Let $z_{x,r}$ be this probability; n^{2r} is the number of possible combinations in the r operations, because, at each operation, the balls of urn A can be combined with each of the n balls of urn B, this which produces n^2 combinations; $n^{2r}z_{x,r}$ is therefore the number of combinations in which there can be x white balls in urn A after these operations. Now it can happen that the $(r+1)^{\text{st}}$ operation makes a white ball exit from urn A, and makes a white ball reenter it: the number of the cases in which this can happen is the product of $n^{2r}z_{x,r}$ by the number x of white balls in urn A, and by the number $n-x$ of white balls which must be then in urn B, since the total number of white balls in the two urns is n; in all these cases, there remain x white balls in urn A; the product $x(n-x)n^{2r}z_{x,r}$ is therefore one of the parts of $n^{2r+2}z_{x,r+1}$.

It can happen yet that the $(r+1)^{\text{st}}$ operation makes a black ball exit and reenter into urn A, this which conserves in this urn x white balls; thus $n-x$ being after the r^{th} operation the number of black balls in urn A, and x being that of the same balls in urn B, we see, by the preceding reasoning, that $(n-x)xn^{2r}z_{x,r}$ is again a part of $n^{2r+2}z_{x,r+1}$.

If there are $x-1$ white balls in urn A after the r^{th} operation, and if the following operation makes a black ball exit and makes a white ball reenter, there will be x white balls in urn A at the $(r+1)^{\text{st}}$ operation. The number of cases in which this can take place is the product of $n^{2r}z_{x-1,r}$ by the number $n-x+1$ of black balls in urn A, after the r^{th} operation, and the number $n-x+1$ white balls in urn B after the same operation; $(n-x+1)^2 n^{2r}z_{x-1,r}$ is therefore further a part of $n^{2r+2}z_{x,r+1}$.

Finally, if there are $x+1$ white balls in urn A after the r^{th} operation, and if the following operation makes a white ball exit and makes a black ball reenter it, there will be again, after this last operation, x white balls in the urn. The number of cases in which this can happen is the product of $n^{2r} z_{x+1,r}$ by the number $x+1$ of white balls in urn A, and by the number $x+1$ of black balls in urn B; $(x+1)^2 n^{2r} z_{x+1,r}$ is therefore further a part of $n^{2r+2} z_{x,r+1}$.

By reuniting all these parts and by equating their sum to $n^{2r+2} z_{x,r+1}$, we will have the equation in the finite partial differences

$$z_{x,r+1} = \left(\frac{x+1}{n}\right)^2 z_{x+1,r} + \frac{2x}{n}\left(1 - \frac{x}{n}\right) z_{x,r} + \left(1 - \frac{x-1}{n}\right)^2 z_{x-1,r}.$$

Although this equation is differential of second order with respect to the variable x, its integral contains only one arbitrary function which depends on the probability of the diverse values of x in the initial state of the urns. In fact, it is clear that if we know the values of $z_{x,0}$ corresponding to all the values of x, from $x = 0$ to $x = n$, the preceding equation gives all the values of $z_{x,1},\ z_{x,2}, \ldots$, by observing that, the negative values of x being impossibles, $z_{x,r}$ is null when x is negative.

When x is a very great number, this equation is transformed into an equation in the infinitely small partial differences that we obtain thus; we have then, very nearly,

$$\begin{aligned} z_{x+1,r} &= z_{x,r} + \frac{\partial z_{x,r}}{\partial x} + \frac{1}{2}\frac{\partial^2 z_{x,r}}{\partial x^2}, \\ z_{x-1,r} &= z_{x,r} - \frac{\partial z_{x,r}}{\partial x} + \frac{1}{2}\frac{\partial^2 z_{x,r}}{\partial x^2}, \\ z_{x,r+1} &= z_{x,r} + \frac{\partial z_{x,r}}{\partial r}. \end{aligned}$$

Let

$$x = \frac{n + \mu\sqrt{n}}{2}, \qquad r = nr', \qquad z_{x,r} = U;$$

the preceding equation in the partial differences will become, by neglecting the terms of order $\frac{1}{n^2}$,

$$\frac{\partial U}{\partial r'} = 2U + 2\mu\frac{\partial U}{\partial \mu} + \frac{\partial^2 U}{\partial \mu^2}.$$

In order to integrate this equation, which, as we can be assured by the method which I have given for this object, in the *Mémoires de l'Académie des Sciences* of the year 1773[7], it is integrable, in finite terms, only by means of definite integrals, we make

$$U = \int \phi\, dt\, e^{-\mu t};$$

ϕ being a function of t and of r', we will have

$$2\mu\frac{\partial U}{\partial \mu} = 2e^{-\mu t} t\phi - 2\int e^{-\mu t}(\phi\, dt + t\, d\phi), \qquad \frac{\partial^2 U}{\partial \mu^2} = \int e^{-\mu t}\, t^2 \phi\, dt;$$

[7] *Oeuvres de Laplace*, T. VIII. "Recherches sur l'integration des équations differentielles aux différences finie, & sur leur usage dan la théorie des hasards."

the equation in the partial differences in U becomes thus

$$\int e^{-\mu t}\frac{\partial\phi}{\partial r'}dt = 2e^{-\mu t}t\phi + \int e^{-\mu t}dt\left(t^2\phi - 2t\frac{\partial\phi}{\partial t}\right).$$

By equating among them the terms affected with the $\int$ sign, conforming to the method which I have given in the *Mémoires de l'Académie des Sciences* of 1782[8], we will have the equation in the partial differences

$$\frac{\partial\phi}{\partial r'} = t^2\phi - 2t\frac{\partial\phi}{\partial t},$$

and the term out of the $\int$ sign, equal to zero, will give, for the equation in the limits of the integral,

$$0 = t\phi\, e^{-\mu t}.$$

The integral of the preceding equation in the partial differences is

$$\phi = e^{\frac{1}{4}t^2}\psi\left(\frac{t}{e^{2r'}}\right),$$

$\psi\left(\frac{t}{e^{2r'}}\right)$ being an arbitrary function of $\frac{t}{e^{2r'}}$; we have therefore

$$U = \int dt\, e^{-\mu t+\frac{1}{4}t^2}\psi\left(\frac{t}{e^{2r'}}\right).$$

Let

$$t = 2\mu + 2s\sqrt{-1};$$

the preceding equation will take this form

$$\text{(A)} \qquad U = e^{-\mu^2}\int ds\, e^{-s^2}\Gamma\left(\frac{s-\mu\sqrt{-1}}{e^{2r'}}\right).$$

It is easy to see that the equation to the limits of the integral, given above, requires that the limits of the integral relative to s be taken from $s = -\infty$ to $s = \infty$. By taking the radical $\sqrt{-1}$ with the $-$ sign, we will have for U an expression of this form

$$U = e^{-\mu^2}\int ds\, e^{-s^2}\Pi\left(\frac{s+\mu\sqrt{-1}}{e^{2r'}}\right),$$

the arbitrary function $\Pi(s)$ can be other than the function $\Gamma(s)$. The sum of these two expressions of U will be the entire value of U; but it is easy to be assured that the integrals, being taken from $s = -\infty$ to $s = \infty$, the addition of this new expression of U adds nothing to the generality of the first in which it is contained.

[8] *Oeurvres de Laplace*, T. X. "*Mémoire sur les approximations des formules qui sont fontions de très grands nombres*."

We develop now the second member of equation (A), according to the powers of $\frac{1}{e^{2r'}}$, and we consider one of the terms of this development, such as

$$\frac{H^{(i)}e^{-\mu^2}}{e^{4ir'}}\int ds\, e^{-s^2}(s-\mu\sqrt{-1})^{2i};$$

this term becomes, after the integrations,

$$\frac{1.3.5\ldots(2i-1)}{2^i}\sqrt{\pi}\frac{H^{(i)}e^{-\mu^2}}{4ir'}\left[1-\frac{i}{1.2}(2\mu)^2+\frac{i(i-1)}{1.2.3.4}(2\mu)^4-\frac{i(i-1)(i-2)}{1.2.3.4.5.6}(2\mu)^6+\cdots\right].$$

We consider next a term of the development of the expression of U, such as

$$\frac{L^{(i)}\sqrt{-1}\,e^{-\mu^2}}{e^{(4i+2)r'}}\int ds\, e^{-s^2}(s-\mu\sqrt{-1})^{2i+1};$$

this term becomes, after the integrations,

$$\frac{1.3.5\ldots(2i+1)L^{(i)}\mu e^{-\mu^2}\sqrt{\pi}}{2^i e^{(4i+1)r'}}\left[1-\frac{i}{1.2.3}(2\mu)^2+\frac{i(i-1)}{1.2.3.4.5}(2\mu)^4-\frac{i(i-1)(i-2)}{1.2.3.4.5.6.7}(2\mu)^6+\cdots\right].$$

We will have therefore thus the general expression of the probability U, developed into a series ordered according to the powers of $\frac{1}{e^{2r'}}$, a series which becomes very convergent, when r' is very considerable. This expression must be such that $\int U dx$ or $\frac{1}{2}\int U d\mu\sqrt{n}$ is equal to unity, the integrals being extended to all the values of which x and μ are susceptibles, that is to say from x null to $x=n$ and from $\mu=-\sqrt{n}$ to $\mu=\sqrt{n}$; because it is certain that the urn must or not contain white balls. By taking the integral $\int e^{-\mu^2}d\mu$ within these limits, and generally in the limits $\pm n^{\frac{1}{r}}$, we have the same result to very nearly as by taking it from $\mu=-\infty$ to $\mu=\infty$; the difference is here only of order $\frac{e^{-n}}{\sqrt{n}}$, and, seeing the extreme rapidity with which e^{-n} diminishes in measure as n increases, we see that this difference is entirely insensible when n is a great number. This put, we consider in the integral $\frac{1}{2}\int U d\mu\sqrt{n}$ the term

$$\frac{1.3.5\ldots(2i-1)\frac{1}{2}H^{(i)}\sqrt{n\pi}}{2^i e^{4ir'}}\int e^{-\mu^2}d\mu\left[1-\frac{i}{1.2}(2\mu)^2+\frac{i(i-1)}{1.2.3.4}(2\mu)^4-\cdots\right].$$

By extending the integral from $\mu=-\infty$ to $\mu=\infty$, this term becomes

$$\frac{1.3.5\ldots(2i-1)\frac{1}{2}H^{(i)}\pi\sqrt{n}}{2^i e^{4ir'}}\left[1-i+\frac{i(i-1)}{1.2}-\frac{i(i-1)(i-2)}{1.2.3}+\cdots\right];$$

the last factor $1-i+\frac{i(i-1)}{1.2}-\cdots$ is equal to $(i-1)^i$; it is therefore null, except in the case of $i=0$, where it is reduced to unity. It is clear that the terms of the expression of U which contain some odd powers of μ give a null result in the integral $\frac{1}{2}\int U d\mu\sqrt{n}$, extended from $\mu=-\infty$ to $\mu=\infty$; because these terms have for factor $e^{-\mu^2}$, and we have generally in these limits

$$\int_{-\infty}^{\infty} d\mu\,\mu^{2i+1}e^{-\mu^2}=0;$$

there is therefore only the first term of the expression of U, a term that we represent by $He^{-\mu^2}$, which can give a result in the integral $\frac{1}{2}\int_{-\infty}^{\infty} U d\mu\,\sqrt{n}$, this result is $\frac{1}{2}Hn\sqrt{\pi}$; we have therefore

$$\frac{1}{2}Hn\sqrt{\pi} = 1,$$

consequently

$$H = \frac{2}{n\sqrt{\pi}}.$$

The general expression of U has thus the following form

$$U = \frac{2e^{-\mu^2}}{\sqrt{n\pi}}\left[1 + \frac{Q^{(1)}(1-2\mu^2)}{e^{4r'}} + \frac{Q^{(2)}(1-4\mu^2+\frac{4}{3}\mu^4)}{e^{4r'}} + \cdots \right.$$
$$\left. + \frac{L^{(0)}\mu}{e^{2r'}} + \frac{L^{(1)}\mu(1-\frac{2}{3}\mu^2)}{e^{6r'}} + \frac{L^{(2)}\mu(1-\frac{4}{3}\mu^2+\frac{4}{15}\mu^4)}{e^{10r'}} + \cdots\right],$$

$Q^{(1)}$, $Q^{(2)}, \ldots,$ $L^{(0)}, L^{(1)}, \ldots$ being some indeterminate constants which depend on the initial value of U.

We suppose that U becomes X when r' is null, X being a given function of μ. We have generally these two theorems

$$0 = Q^{(i)}\int \mu^{2q}\,d\mu\,U_i\,e^{-\mu^2},$$
$$0 = L^{(i)}\int \mu^{2q+1}\,d\mu\,U_i'\,e^{-\mu^2},$$

when q is less than i, U_i and U_i' being the functions of μ by which $\frac{2Q^{(i)}e^{-\mu^2}}{\sqrt{n\pi}\,e^{4ir'}}$ and $\frac{2L^{(i)}e^{-\mu^2}}{\sqrt{n\pi}\,e^{(4i+2)r'}}$ are multiplied in the expression of U. By that which precedes, the term $\frac{2Q^{(i)}Ue^{-\mu^2}}{e^{4ir'}\sqrt{n\pi}}$ is equal to

$$\frac{H^{(i)}(\sqrt{-1})^{2i}}{e^{4ir'}}e^{-\mu^2}\int ds\,e^{-s^2}(\mu+s\sqrt{-1})^{2i};$$

it is necessary therefore to demonstrate that we have

$$0 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mu^{2q}\,ds\,d\mu\,e^{-\mu^2-s^2}(\mu+s\sqrt{-1})^{2i}.$$

By integrating first respect to μ, this term becomes

$$\frac{2q-1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mu^{2q-2}\,d\mu\,ds\,e^{-\mu^2-s^2}(\mu+s\sqrt{-1})^{2i}$$
$$+ i\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mu^{2q-1}\,d\mu\,ds\,e^{-\mu^2-s^2}(\mu+s\sqrt{-1})^{2i-1}.$$

By continuing to integrate thus by parts, we arrive to some terms of the form

$$K\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\mu\, ds\, e^{-\mu^2-s^2}(\mu+s\sqrt{-1})^{2c},$$

c being not zero; and, by that which precedes, this term is null. We will prove in the same manner that we have

$$0 = L^{(i)}\int_{-\infty}^{\infty}\mu^{2q+1}\, d\mu\, U_i'\, e^{-\mu^2}.$$

Thence it follows that we have generally

$$0 = \int_{-\infty}^{\infty} U_i\, U_{i'}\, d\mu\, e^{-\mu^2},$$
$$0 = \int_{-\infty}^{\infty} U_i'\, U_{i'}'\, d\mu\, e^{-\mu^2},$$

i and i' being some different numbers; because, if, for example, i' is greater than i, all the powers of μ in U_i will be less than $2i'$; each of the terms of U will give therefore, by the preceding theorems, a result null in the integral $\int_{-\infty}^{\infty} U_i U_{i'} d\mu\; e^{-\mu^2}$. The same reason holds for the integral $\int_{-\infty}^{\infty} U_i'\, U_{i'}'\, d\mu\, e^{-\mu^2}$.

But these integrals are not null when $i' = i$; we will obtain them, in this case, in this manner. We have, by that which precedes,

$$U_i = \frac{2^i(\sqrt{-1})^{2i}\int_{-\infty}^{\infty} ds\, e^{-s^2}(\mu+s\sqrt{-1})^{2i}}{1.3.5\ldots(2i-1)\sqrt{\pi}}.$$

The term which has for factor μ^{2i} in this expression is

$$\frac{2^i(\sqrt{-1})^{2i}\mu^{2i}}{1.3.5\ldots(2i-1)};$$

now we can consider only this term in the first factor U_i of the integral $\int_{-\infty}^{\infty} U_i U_i d\mu\, e^{-\mu^2}$; because the inferior powers of μ in this factor give a result null in the integral; we have therefore

$$\int_{-\infty}^{\infty} U_i\, U_i\, d\mu\, e^{-\mu^2} = \frac{2^{2i}}{[1.2.3\ldots(2i-1)]^2\sqrt{\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mu^{2i}\, d\mu\, ds\, e^{-\mu^2-s^2}(\mu+s\sqrt{-1})^{2i}.$$

We have, by integrating with respect to μ, from $\mu = -\infty$ to $\mu = \infty$,

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mu^{2i}\, d\mu\, ds\, e^{-\mu^2-s^2}(\mu+s\sqrt{-1})^{2i}\\ &= \frac{2i-1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mu^{2i-2}\, d\mu\, ds\, e^{-\mu^2-s^2}(\mu+s\sqrt{-1})^{2i}\\ &\quad+\frac{2i}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mu^{2i-1}\, d\mu\, ds\, e^{-\mu^2-s^2}(\mu+s\sqrt{-1})^{2i-1}.\end{aligned}$$

The first term of the second member of this equation is null by that which precedes; this member is reduced therefore to its second term; we find in the same manner that we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mu^{2i-1}\,d\mu\,ds\,e^{-\mu^2-s^2}(\mu+s\sqrt{-1})^{2i-1}$$
$$=\frac{2i-1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mu^{2i-2}\,d\mu\,ds\,e^{-\mu^2-s^2}(\mu+s\sqrt{-1})^{2i-2}$$

and thus in sequence; we have therefore

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mu^{2i}\,d\mu\,ds\,e^{-\mu^2-s^2}(\mu+s\sqrt{-1})^{2i}=\frac{1.2.3\ldots 2i\pi}{2^{2i}},$$

consequently

$$\int_{-\infty}^{\infty}U_i\,U_i\,d\mu\,e^{-\mu^2}=\frac{2.4.6\ldots 2i\sqrt{\pi}}{1.3.5\ldots(2i-1)}.$$

We will find in the same manner

$$\int_{-\infty}^{\infty}U_i'\,U_i'\,d\mu\,e^{-\mu^2}=\frac{1}{2}\frac{2.4.6\ldots 2i\sqrt{\pi}}{1.3.5\ldots(2i+1)};$$

we have evidently

$$\int_{-\infty}^{\infty}U_i\,U_{i'}'\,d\mu\,e^{-\mu^2}=0$$

in the case where i and i' are not different, because the product $U_i\,U_{i'}'$ contains only some odd powers of μ.

This put, the general expression of U gives, for its initial value, what we have designated by X,

$$X=\frac{2e^{-\mu^2}}{\sqrt{n\pi}}[1+Q^{(1)}(1-2\mu^2)+\cdots+L^{(0)}\mu+L^{(1)}\mu(1-\frac{2}{3}\mu^2)+\cdots].$$

If we multiply this equation by $U_i d\mu$, and if we take the integrals from $\mu=-\infty$ to $\mu=\infty$, we will have, by virtue of the preceding theorems,

$$\int_{-\infty}^{\infty}XU_i d\mu=\frac{2}{\sqrt{n\pi}}Q^{(i)}\int U_iU_i d\mu\,e^{-\mu^2};$$

whence we deduce

$$Q^{(i)}=\frac{1.3.5\ldots(2i-1)\frac{1}{2}\sqrt{n}}{2.4.6\ldots 2i}\int_{-\infty}^{\infty}XU_i d\mu;$$

we will find in the same manner

$$L^{(i)}=\frac{1.3.5\ldots(2i+1)\sqrt{n}}{2.4.6\ldots 2i}\int_{-\infty}^{\infty}XU_i' d\mu;$$

We will have therefore thus the successive values of $Q^{(1)}$, $Q^{(2)}, \ldots,$ $L^{(0)}, L^{(1)}, \ldots$by means of the definite integrals, when X or the initial value of U will be given.

In the case where X is equal to $\frac{2i}{\sqrt{n\pi}} e^{-i^2\mu^2}$, the general expression of U takes a very simple form. The arbitrary function $\Gamma\left(\frac{s-\mu\sqrt{-1}}{e^{2r'}}\right)$ of formula (A) is of the form $ke^{-\beta\left(\frac{s-\mu\sqrt{-1}}{e^{2r'}}\right)^2}$. In order to determine the constants β and k, we will observe that, by supposing

$$\beta' = \frac{\beta}{e^{4r'}},$$

we will have

$$U = ke^{\frac{-\mu^2}{1+\beta'}} \int_{-\infty}^{\infty} ds\, e^{-(1+\beta')\left(s-\frac{\beta'\mu\sqrt{-1}}{1+\beta'}\right)^2};$$

by making next

$$\sqrt{1+\beta'}\left(s - \frac{\beta'\mu\sqrt{-1}}{1+\beta'}\right) = s',$$

and observing that the integral relative to s necessarily being taken from $s = -\infty$ to $s = +\infty$, the integral relative to s' must be taken within the same limits, we will have

$$U = \frac{k\sqrt{\pi}}{\sqrt{1+\beta'}} e^{\frac{-\mu^2}{1+\beta'}}.$$

By comparing this expression to the initial value of U, which is

$$U = \frac{2i}{\sqrt{n\pi}} e^{-i^2\mu^2},$$

and observing that β is the initial value of β', we will have

$$i^2 = \frac{1}{1+\beta},$$

whence we deduce

$$\beta = \frac{1-i^2}{i^2}, \qquad \beta' = \frac{1-i^2}{i^2 e^{4r'}}.$$

We must have next

$$\frac{k\sqrt{\pi}}{\sqrt{1+\beta}} = \frac{2i}{\sqrt{n\pi}},$$

that which gives

$$k\sqrt{\pi} = \frac{2}{\sqrt{n\pi}},$$

a value that we obtain further by the condition that

$$\frac{1}{2}\int_{-\infty}^{\infty} U d\mu\sqrt{n} = 1;$$

we will have therefore for the expression of U, whatever be r',

$$U = \frac{2}{\sqrt{n\pi(1+\beta')}} e^{\frac{-\mu^2}{1+\beta'}}.$$

We find, in fact, that this value of U, substituted into the equation in the partial differentials in U, satisfies it. β' diminishes without ceasing when r' increases, the value of U varies without ceasing and becomes to its limit, when r' is infinity,

$$U = \frac{2}{\sqrt{n\pi}} e^{-\mu^2}.$$

In order to give an application of these formulas, we imagine in an urn C a very great number m of white balls and a like number of black balls. These balls have been mixed, we suppose that we draw from the urn n balls which we put into urn A. We suppose next that we put into urn B as many white balls as there are black balls in urn A, and as many black balls as there are white balls in the same urn. It is clear that the number of cases in which we will have x white balls, and consequently $n-x$ black balls in urn A, is equal to the product of the number of combinations of m white balls of urn C, taken x by x, by the number of combinations of m black balls of the same urn, taken $n-x$ by $n-x$. This product is equal to

$$\frac{m(m-1)(m-2)\cdots(m-x+1)}{1.2.3\ldots x} \, \frac{m(m-1)(m-2)\cdots(m-n+x+1)}{1.2.3\ldots(n-x)}$$

or to

$$\frac{(1.2.3\ldots m)^2}{1.2.3\ldots x.1.2.3\ldots(m-x).1.2.3\ldots(n-x).1.2.3\ldots(m-n+x)}.$$

The number of all the possible cases is the number of combinations of $2m$ balls taken n by n; this number is

$$\frac{1.2.3\ldots 2m}{1.2.3\ldots n.1.2.3\ldots(2m-n)};$$

by dividing therefore the preceding fraction by this one, we will have for the probability of x or for the initial value of U

$$\frac{(1.2.3\ldots m)^2.1.2.3\ldots n.1.2.3\ldots(2m-n)}{1.2.3\ldots x.1.2.3\ldots(m-x).1.2.3\ldots(n-x).1.2.3\ldots(m-n+x).1.2.3\ldots 2m}.$$

Now, if we observe that we have very nearly, when s is a very great number,

$$1.2.3\ldots s = s^{s+\frac{1}{2}} e^{-s} \sqrt{2\pi},$$

we will find after all the reductions, by making

$$x = \frac{n+\mu\sqrt{n}}{2}$$

and neglecting the quantities of order $\frac{1}{n}$,

$$U = \frac{2}{\sqrt{n\pi}}\sqrt{\frac{m}{2m-n}}e^{\frac{-m\mu^2}{e^{2m-n}}};$$

by making therefore

$$i^2 = \frac{m}{2m-n},$$

we will have

$$U = \frac{2i}{\sqrt{n\pi}}e^{-i^2\mu^2}.$$

If the number m is infinity, then $i^2 = \frac{1}{2}$, and the initial value of U is

$$U = \frac{\sqrt{2}}{\sqrt{m\pi}}e^{-\frac{1}{2}\mu^2}.$$

Its value, after any number of drawings, is

$$U = \frac{2}{\sqrt{n\pi\left(1+e^{\frac{-4r}{n}}\right)}}e^{-\frac{\mu^2}{1+e^{-\frac{4r}{n}}}}.$$

The case of m infinity returns to the one in which urn A will be replenished, by projecting n times a coin which would bring forth indifferently *heads* or *tails*, and by putting into urn A a white ball each time that *heads* would arrive, and a black ball each time that *tails* would arrive; because it is clear that the probability of drawing a white or black ball from urn C is $\frac{1}{2}$ as that of bringing forth *heads* or *tails*. By taking the integral $\int U dx$ or $\frac{1}{2}\int U\, d\mu\sqrt{n}$ from $\mu = -a$ to $\mu = a$, we will have the probability that the number of white balls in urn A will be contained within the limits $\pm a\sqrt{n}$.

VI.

On the mean which it is necessary to choose among the results of observations.

When we wish to correct an element already known very nearly, by the whole of a great number of observations, we form some equations of condition in the following manner. Let z be the correction of the element and β the observation; the analytic expression of this will be a function of the element. By substituting into it, in the place of the element, its approximated value, plus the correction z; by reducing into series with respect to z and neglecting the square of z, this function will take the form $m+pz$, to which we equate the observed quantity β, this which gives

$$\beta = m + pz;$$

z will be therefore thus determined, if the observation was rigorous; but, as it is susceptible of error, by naming ϵ this error, we have rigorously

$$\beta + \epsilon = m + pz$$

or, by making $\beta - m = \phi$, we have

$$\epsilon = pz - \phi.$$

Each observation furnishes a similar equation of condition, that we can represent for the $(i+1)^{\text{st}}$ by this

$$\epsilon^{(i)} = p^{(i)}z - \phi^{(i)}.$$

By reuniting all these equations, we have

$$\mathrm{S}\epsilon^{(i)} = z\mathrm{S}p^{(i)} - \mathrm{S}\phi^{(i)},$$

the sign S corresponds to all the values of i, from $i = 0$ to $i = s - 1$, s being the total number of observations. By supposing null the sum of the errors, this equation gives

$$z = \frac{\mathrm{S}\phi^{(i)}}{\mathrm{S}p^{(i)}};$$

this is that which we name ordinarily *mean result* of the observations.

I have given in the preceding Volume[9] the law of the probability of errors of this result; but, instead of supposing null the sum of the errors, we can suppose null any linear function of these errors which we will represent thus

$$(m) \qquad q\epsilon + q^{(1)}\epsilon^{(1)} + q^{(2)}\epsilon^{(2)} + \cdots + q^{(s-1)}\epsilon^{(s-1)},$$

q, $q^{(1)}$, $q^{(2)}$, ... being some positive or negative numbers which we suppose will be entire. By substituting into the function (m), in the place of ϵ, $\epsilon^{(1)}, \ldots$, their values given by the equations of condition, it becomes

$$z\mathrm{S}Sp^{(i)}q^{(i)} - \mathrm{S}q^{(i)}\phi^{(i)}.$$

By equating therefore to zero the function (m), we have

$$z = \frac{\mathrm{S}q^{(i)}\phi^{(i)}}{\mathrm{S}p^{(i)}q^{(i)}}.$$

Let z' be the error of this result, so that we have

$$z = \frac{\mathrm{S}q^{(i)}\phi^{(i)}}{\mathrm{S}p^{(i)}q^{(i)}} + z',$$

this which gives for the expression of the function (m)

$$z'\mathrm{S}p^{(i)}q^{(i)};$$

and we determine the law of probability of the error z' of the result, when the observations are in great number. For this, we consider the product

$$\mathrm{S}\Psi\left(\frac{x}{a}\right)e^{qx\varpi\sqrt{-1}} \times \mathrm{S}\Psi\left(\frac{x}{a}\right)e^{q^{(1)}x\varpi\sqrt{-1}} \times \cdots \times \mathrm{S}\Psi\left(\frac{x}{a}\right)e^{q^{(s-1)}x\varpi\sqrt{-1}},$$

[9] See, *Oeuvres de Laplace*, T. XII, p. 322 ff. “Mémoire sur les approximations des formules qui sont fonctions de très grands nombres et sur leur application aux probabilités.” Section VI.

the sign S extending here from the extreme negative value of x to its extreme positive value: $\Psi\left(\frac{x}{a}\right)$ is the probability of an error x in each observation, x being supposed, thus as a, formed of an infinity of parts taken for unity. It is clear that the coefficient of any exponential $e^{l\varpi\sqrt{-1}}$ in this product will be the probability that the sum of the errors of each observation, multiplied respectively by q, $q^{(1)}$, ..., that is to say the function (m), will be equal to l; by multiplying therefore the preceding product by $e^{-l\varpi\sqrt{-1}}$, the term independent of ϖ, in this new product, will express this probability. If we suppose, as we make it here, the probability of the errors of each observation the same for the errors, either positives or negatives, we can, in the sum $\mathrm{S}\Psi\left(\frac{x}{a}\right)e^{qx\varpi\sqrt{-1}}$, reunite the two multiplied terms, the one by $e^{qx\varpi\sqrt{-1}}$ and the other by $e^{-qx\varpi\sqrt{-1}}$, then this sum takes the form $2\mathrm{S}\Psi\left(\frac{x}{a}\right)\cos qx\varpi$. It is likewise of the other similar sums. Thence it follows that the probability that the function (m) will be equal to l is the term independent of ϖ in the function

$$e^{-l\varpi\sqrt{-1}} \times 2\mathrm{S}\Psi\left(\frac{x}{a}\right)\cos qx\varpi \times 2\mathrm{S}\Psi\left(\frac{x}{a}\right)\cos q^{(1)}x\varpi \times \cdots \\ \times 2\mathrm{S}\Psi\left(\frac{x}{a}\right)\cos q^{(s-1)}x\varpi.$$

By changing $-l$ into l in it, we will have the probability that the function (m) will be equal to $-l$; by reuniting these two expressions, the term independent of ϖ in the product

$$2\cos l\varpi \times 2\mathrm{S}\Psi\left(\frac{x}{a}\right)\cos qx\varpi \times 2\mathrm{S}\Psi\left(\frac{x}{a}\right)\cos q^{(1)}x\varpi \times \cdots \\ \times 2\mathrm{S}\Psi\left(\frac{x}{a}\right)\cos q^{(s-1)}x\varpi$$

is the probability that the function (m) will be either $+l$ or $-l$; this probability is therefore

$$\frac{2}{\pi}\int_0^\pi d\varpi\cos l\varpi \times 2\mathrm{S}\Psi\left(\frac{x}{a}\right)\cos qx\varpi \times 2\mathrm{S}\Psi\left(\frac{x}{a}\right)\cos q^{(1)}x\varpi \times \cdots \\ \times 2\mathrm{S}\Psi\left(\frac{x}{a}\right)\cos q^{(s-1)}x\varpi$$

We have, by reducing the cosines to series,

$$2\mathrm{S}\Psi\left(\frac{x}{a}\right)\cos qx\varpi = \mathrm{S}\Psi\left(\frac{x}{a}\right) - \frac{1}{2}q^2a^2\varpi^2\mathrm{S}\frac{x^2}{a^2}\Psi\left(\frac{x}{a}\right) + \cdots$$

If we make $\frac{x}{a} = x'$, and if we observe that, the variation of x being unity, we have $dx' = \frac{1}{a}$, we will have

$$\mathrm{S}\Psi\left(\frac{x}{a}\right) = a\int dx'\Psi(x').$$

We name k the integral $2\int dx'\Psi(x')$, taken from x null to its extreme value; we name similarly k' the integral $\int x'^2\,dx'\Psi(x')$ extended within the same limits, and thus in sequence; we will have

$$2\mathrm{S}\Psi\left(\frac{x}{a}\right)\cos qx\varpi = ak\left(1 - \frac{k'}{k}q^2a^2\varpi^2 + \cdots\right);$$

its logarithm is

$$-\frac{k'q^2}{k}a^2\varpi^2 - \cdots + \log ak.$$

ak or $2a\int dx'\Psi(x')$ being equal to $2\mathrm{S}\Psi\left(\frac{x}{a}\right)$, it expresses the probability that the error of each observation will be contained within its limits, this which is certain; we have therefore $ak = 1$, this which reduces the preceding logarithm to

$$-\frac{k'q^2}{k}a^2\varpi^2 - \cdots .$$

Thence it is easy to conclude that the logarithm of the product

$$2\mathrm{S}\Psi\left(\frac{x}{a}\right)\cos qx\varpi \times 2\mathrm{S}\Psi\left(\frac{x}{a}\right)\cos q^{(1)}x\varpi \times \cdots \times 2\mathrm{S}\Psi\left(\frac{x}{a}\right)\cos q^{(s-1)}x\varpi$$

is equal to

$$-\frac{k'}{k}\mathrm{S}q^{(i)^2}a^2\varpi^2 - \cdots ,$$

the sign S extending here from $i = 0$ to $i = s - 1$. When the observations are in very great number, we can conserve only the first term of the series; because it is easy to see that the sum of the squares or of the cubes, ... of q, $q^{(1)}, \ldots$ being of order s, each of the terms of the series has for factor a quantity of this order; but, if we suppose that $sa^2\varpi^2$ is always of order less than $\sqrt{s}$, then the second term of the series being of order $sa^4\varpi^4$, it will be very small and will become null in the case of s infinity; we can therefore neglect vis-à-vis of the first term the second, and for stronger reason the following. Now if we pass again from the logarithms to the numbers, we will have

$$2\mathrm{S}\Psi\left(\frac{x}{a}\right)\cos qx\varpi \times 2\mathrm{S}\Psi\left(\frac{x}{a}\right)\cos q^{(1)}x\varpi \times \cdots = e^{-\frac{k'}{k}a^2\varpi^2\mathrm{S}q^{(i)^2}};$$

the probability that the function (m) will be equal to $+l$ and $-l$ is therefore, by integrating from ϖ null to $\varpi = \pi$,

$$\frac{2}{a\pi}\int_0^\pi a\,d\varpi\cos l\varpi\, e^{-\frac{k'}{k}a^2\varpi^2\mathrm{S}q^{(i)^2}}.$$

If we make $a\varpi = t$, this integral becomes

$$\frac{2}{a\pi}\int_0^{a\pi} dt\cos\frac{l}{a}t\, e^{-\frac{k'}{k}t^2\mathrm{S}q^{(i)^2}}.$$

The integral relative to ϖ must be taken from ϖ null to $\varpi = \pi$, the integral relative to t must be taken from t null to $t = a\pi$ or to infinity, a being supposed of an infinite number of units. In truth, we are arrived to the preceding integral, by supposing $sa^2\varpi^2$ or st^2 of an order smaller than $\sqrt{s}$; but, when st^2 is of order $\sqrt{s}$, the exponential $e^{-\frac{k'}{k}t^2\mathrm{S}q^{(i)^2}}$ becomes so excessively small that we can, without fear of any sensible error, extend the integral to the beyond to infinity. This put, this integral becomes, by article II,

$$\frac{1}{a\sqrt{\pi}}\frac{1}{\sqrt{\frac{k'}{k}\,\mathrm{S}q^{(i)^2}}}\,e^{-\frac{kl^2}{4k'a^2\mathrm{S}q^{(i)^2}}}$$

By making therefore $l = ar$, and observing that, the variation of l being unity, we have $adr = 1$, we will have

$$\frac{1}{\sqrt{\frac{k'\pi}{k}\,\mathrm{S}q^{(i)^2}}}\int dr\, e^{-\frac{kr^2}{4k'a^2\mathrm{S}q^{(i)^2}}}$$

for the probability that the function (m) will be contained within the limits $\pm ar$.

We determine presently the mean value of the error to fear, by adopting for mean result of the observations the correction

$$\frac{\mathrm{S}q^{(i)}\phi^{(i)}}{\mathrm{S}p^{(i)}q^{(i)}},$$

which results, as we have seen, from the equality of the function (m) to zero. z being supposed the correction of this result, the function (m) becomes $z'\mathrm{S}p^{(i)}q^{(i)}$. By making this quantity equal to ar, we will have

$$\frac{dr\, e^{-\frac{kr^2}{4k'\mathrm{S}q^{(i)^2}}}}{2\sqrt{\frac{k'\pi}{k}\,\mathrm{S}q^{(i)^2}}} = \frac{dz'\mathrm{S}p^{(i)}q^{(i)}}{2a\sqrt{\pi}\sqrt{\frac{k'}{k}\,\mathrm{S}q^{(i)^2}}} e^{-\frac{kz'^2(\mathrm{S}p^{(i)}q^{(i)})^2}{4k'\mathrm{S}q^{(i)^2}}};$$

the coefficient dz' in the second member of this equation is therefore the ordinate of the curve of probabilities of the errors z' which represent the abscissas of this curve, which we can extend to the infinity on each side of the ordinate which corresponds to z' null. This put, all error, either positive, or negative, must be considered as a disadvantage or a real loss to any game; now, by the known principles of the Calculus of probabilities, we evaluate this disadvantage by taking the sum of all the products of each disadvantage by its probability; the mean value of the error to fear is therefore the sum of the products of each error, setting aside the sign, by its probability; consequently it is equal to the integral

$$\frac{\int_0^\infty z'dz'\mathrm{S}p^{(i)}q^{(i)}}{2a\sqrt{\frac{k'\pi}{k}\,\mathrm{S}q^{(i)^2}}} e^{-\frac{kz'^2(\mathrm{S}p^{(i)}q^{(i)})^2}{4k'a^2\mathrm{S}q^{(i)^2}}};$$

the mean error to fear is therefore

$$\text{(B)} \qquad 2a\sqrt{\frac{k'}{k\pi}}\frac{\sqrt{\mathrm{S}q^{(i)^2}}}{\mathrm{S}p^{(i)}q^{(i)}}.$$

The values of p, $p^{(i)}$, $\ldots$ are given by the equations of condition; but the values of q, $q^{(i)}$, $\ldots$ are arbitraries and must be determined by the condition that the preceding expression is a minimum. This condition gives, by making only $q^{(i)}$ vary,

$$\frac{q^{(i)}}{\mathrm{S}q^{(i)^2}} = \frac{p^{(i)}}{\mathrm{S}p^{(i)}q^{(i)}}.$$

This equation holds whatever be i; and, as the variation of i changes not at all the fraction $\frac{\mathrm{S}q^{(i)^2}}{\mathrm{S}p^{(i)}q^{(i)}}$, by naming it μ, we will have

$$q = \mu p, \quad q^{(1)} = \mu p^{(1)}, \quad \ldots, \quad q^{(s-1)} = \mu p^{(s-1)},$$

and we can, whatever be $p,\ p^{(1)},\ \ldots,$ take μ such that the numbers $q,\ q^{(1)},\ \ldots$ are whole numbers, as the preceding analysis requires. Then formula (B) gives, for the mean error to fear,

$$\text{(D)} \qquad \sqrt{\frac{k\pi}{k'}}\mathrm{S}p^{(i)^2};$$

it is, in all the assumptions which we can make on the values of $q,\ q^{(1)},\ \ldots$, the smallest mean error possible. The mean result of the observations becomes then

$$z = \frac{\mathrm{S}p^{(i)}q^{(i)}}{\mathrm{S}p^{(i)^2}}.$$

And if we suppose the values of $q,\ q^{(1)},\ \ldots$, equal to ± 1, the mean error to fear will be the smallest when the $\pm$ sign will be determined in a way that $p^{(i)}q^{(i)}$ is positive, this which reverts to supposing $1 = q = q' = \cdots$, and to prepare the equations of condition, so that the coefficient of z in each of them is positive: this is that which we make in the ordinary method. Then the mean result is

$$z = \frac{\mathrm{S}\phi^{(i)}}{\mathrm{S}p^{(i)}},$$

and the mean error to fear is

$$\frac{2a\sqrt{s}}{\mathrm{S}p^{(i)}\sqrt{\frac{k\pi}{k'}}};$$

but this error surpasses the preceding (D), since that one is the smallest possible. We can convince ourselves of it besides in this manner: we have

$$\frac{\sqrt{s}}{\mathrm{S}p^{(i)}} > \frac{1}{\sqrt{\mathrm{S}p^{(i)^2}}} \quad \text{or} \quad s\mathrm{S}p^{(i)^2} > (\mathrm{S}p^{(i)})^2.$$

In fact, $2pp^{(1)}$ is less than $p^{(2)} + p^{(1)^2}$, since $(p^{(1)} - p)^2$ is a positive quantity; we can therefore, in the second member of the preceding inequality, substitute for $2pp^{(1)}$, $p^{(2)} + p^{(1)^2} - f,\ f$ being a positive quantity. By making some similar substitutions for all the similar products, this second member will be equal to $s(p^2+p^{(1)^2}+\cdots+p^{(s-1)^2})$ less a positive quantity.

The result

$$z = \frac{\mathrm{S}p^{(i)}\phi^{(i)}}{\mathrm{S}p^{(i)^2}},$$

to which corresponds the minimum error to fear, is the one that the method of least squares of the errors gives, because the sum of these squares being

$$(pz-\phi)^2 + (p^{(1)}z - \phi^{(1)})^2 + \cdots + (p^{(s-1)}z - \phi^{(s-1)})^2,$$

the condition of the minimum of this function, by making z vary, gives for this variable the preceding expression; this method must therefore be employed in preference, whatever be the law of facility of errors, a law on which the ratio $\frac{k}{k'}$ depends. Although this law is nearly always unknown, however we can suppose $\frac{k}{k'} > 6$. In fact, if we suppose that the limits of the errors of each observation are $\pm a$, then x' being $\frac{x}{a}$, the value of x' will extend from zero to unity so that we will obtain the integrals

$$2\int_0^1 dx'\,\Psi(x')$$

and

$$\int_0^1 x'^2 dx'\,\Psi(x'),$$

that k and k' represent; it is necessary therefore to show that then

$$2\int_0^1 dx'\,\Psi(x') > 6\int_0^1 x'^2 dx'\,\Psi(x').$$

For this, it suffices to prove that we have

$$x'^2\int_0^{x'} dx'\,\Psi(x') > 3\int_0^{x'} x'^2 dx'\,\Psi(x').$$

In fact, if we differentiate this inequality, we will have

$$2x'\int_0^{x'} dx'\,\Psi(x') > 2\,x'^2\,\Psi(x')$$

or

$$\int_0^{x'} dx'\,\Psi(x') > \; x'\,\Psi(x').$$

Differentiating again this inequality, we will have

$$0 > x'\frac{d\Psi(x')}{dx'};$$

now this inequality is correct, if we suppose that the probability $\Psi(x')$ of the error x of each observation is so much smaller as the error is greater, this which it is natural to admit; the differential of $\Psi(x')$ is thus negative, and, consequently, less than zero.

Thence it follows that the function (D) is less than

$$\frac{2a}{\sqrt{6\pi\,\mathrm{S}p^{(i)2}}}.$$

The half of this function is the mean error to fear to more, by adopting the result given by the method of least squares; this half, taken with the $-$ sign, is the mean error to fear to less. We can therefore estimate thence the degree of approximation of this result, by taking for a the deviation of the mean result which makes to reject an observation. In the entire ignorance in which one is most often of the law of the errors, we can equally take all those which satisfy the two conditions to give the same probability for the equal positive and negative errors, and to render the errors so much less probables as they are greater. Then it is necessary to choose the mean law among all these laws, and which I have determined in the *Mémoires de l'Académie des Sciences*, year 1778, page 258[10]. This law given, for the probability of the error $\pm x$,

$$\frac{1}{2a}\log\frac{a}{x};$$

we find then

$$\frac{k}{k'} = 18,$$

this which gives $\frac{2a}{3\sqrt{2\pi\,\mathrm{S}p^{(i)2}}}$ for the mean error to fear.

If we make

$$z' = \frac{2at\sqrt{\frac{k}{k'}}}{\sqrt{\mathrm{S}p^{(i)2}}},$$

we will have by this which precedes, in the method of least squares of the errors, where $q^{(i)} = \mu p^{(i)}$,

$$\frac{2}{\sqrt{\pi}}\int dt\, e^{-t^2}$$

for the probability that the error of the mean result will be contained within the limits

$$\frac{\pm 2at\sqrt{\frac{k'}{k}}}{\sqrt{\mathrm{S}p^{(i)2}}},$$

In the ordinary method where $q^{(i)} = 1$, the preceding integral expresses the probability that the error of the mean result given by this method will be contained within the limits

$$\frac{\pm 2at\sqrt{\frac{k'}{k}}\sqrt{s}}{\mathrm{S}p^{(i)}},$$

The value of t being supposed the same for the results of the two methods, the probability that the error will be contained within the corresponding limits will be the same; but these limits are more narrowed in the first method than in the second. If we suppose that these limits are the same, relatively to the results of the two methods, the value of t will be greater, and consequently the probability that the error of the mean result will not exceed these limits will be more considerable in the first method than in the second; thus, under this new relationship, the method of least squares merits the preference.

VII.

We suppose now that a similar element is given: 1° by the mean result of s observations of a first kind and that it is, by these observations, equal to A; 2° by the mean result of s' observations of a second kind and that it is equal to $A + q$; 3° by the mean result of s'' observations of a third kind and that it is equal to $A + q'$, and

[10] Oeuvres de Laplace, T. IX, p. 412. "Mémoire sur les probabilités." Section XII.

thus of the rest. If we represent by $A+x$ the true element, the error of the result of the observations s will be $-x$; by supposing therefore β equal to

$$\sqrt{\frac{k}{k'}}\,\frac{\sqrt{\mathrm{S}p^{(i)2}}}{2a},$$

if we make use of the least squares of the errors in order to determine the mean result, or to

$$\sqrt{\frac{k}{k'}}\,\frac{\mathrm{S}p^{(i)}}{2a\sqrt{s}}$$

if we employ the ordinary method, the probability of this error will be, by the preceding article, by supposing s a great number,

$$\frac{\beta}{\sqrt{\pi}}e^{-\beta^2x^2}.$$

The error of the result of the observations s' will be $q-x$, and by designating by β', for these observations, that which we have named β for the observations s, the probability of this error will be

$$\frac{\beta'}{\sqrt{\pi}}e^{-\beta'^2(x-q)^2}.$$

Similarly, the error of the result of the observations s'' will be $q'-s$, and, by naming for them β'' that which we have named β for the observations s, the probability of this error will be

$$\frac{\beta''}{\sqrt{\pi}}e^{-\beta''^2(x-q')^2}$$

and thus in sequence. The product of all these probabilities will be the probability that $-x$, $q-x$, $q'-x, \ldots$ will be the errors of the mean results of the observations s, s', $s'', \ldots$; this probability is therefore equal to

$$\frac{\beta}{\sqrt{\pi}}\,\frac{\beta'}{\sqrt{\pi}}\,\frac{\beta''}{\sqrt{\pi}}\cdots e^{-\beta^2x^2-\beta'^2(x-q)^2-\beta''^2(x-q')^2-\cdots}.$$

By multiplying it by dx and taking the integral from $x=-\infty$ to $x=\infty$, we will have the probability that the mean results of the observations s', $s'', \ldots$ will surpass respectively q, q', $\ldots$ the mean result of the observations s.

If we take the integral within some determined limits, we will have the probability that, the preceding condition being fulfilled, the error of the first result will be contained within these limits; by dividing this probability by that of the condition itself, we will have the probability that the error of the first result will be contained within the given limits, when it is certain that the condition holds effectively; this probability is therefore

$$\frac{\int dx\, e^{-\beta^2x^2-\beta'^2(x-q)^2-\beta''^2(x-q')^2-\cdots}}{\int dx\, e^{-\beta^2x^2-\beta'^2(x-q)^2-\beta''^2(x-q')^2-\cdots}}$$

the integral of the numerator being taken within the given limits and that of the denominator being taken from $x=-\infty$ to $x=\infty$.

We have

$$\beta^2 x^2 + \beta'^2(x-q)^2 + \beta''^2(x-q')^2 + \cdots$$
$$= (\beta^2 + \beta'^2 + \beta''^2 + \cdots)x^2 - 2x(\beta'^2 q + \beta''^2 q' + \cdots) + \beta'^2 q^2 + \beta''^2 q'^2 + \cdots$$

Let there be

$$x = \frac{\beta'^2 q + \beta''^2 q' + \cdots}{\beta^2 + \beta'^2 + \beta''^2 + \cdots} + t;$$

the preceding probability will become

$$\frac{\int dt\, e^{-(\beta^2+\beta'^2+\beta''^2+\cdots)t^2}}{\int dt\, e^{-(\beta^2+\beta'^2+\beta''^2+\cdots)t^2}},$$

the integral of the numerator being taken within some given limits and that of the denominator being taken from $t = -\infty$ to $t = \infty$. This last integral is

$$\frac{\sqrt{\pi}}{\sqrt{\beta^2 + \beta'^2 + \beta''^2 + \cdots}};$$

by making therefore

$$t' = t\sqrt{\beta^2 + \beta'^2 + \beta''^2 + \cdots},$$

the preceding probability becomes

$$\frac{1}{\sqrt{\pi}} \int dt'\, e^{-t'^2}.$$

The most probable value of t' is that which corresponds to t' null, whence it follows that the most probable value of x is that which corresponds to $t = 0$; thus the correction of the first result which gives with the most probability the set of all the observations $s,\ s', s'', \ldots$ is

$$\frac{\beta'^2 q + \beta''^2 q' + \cdots}{\beta^2 + \beta'^2 + \beta''^2 + \cdots},$$

and we will find, by the preceding article, that the mean error to fear is

$$\frac{1}{\sqrt{\pi(\beta^2 + \beta'^2 + \beta''^2 + \cdots)}};$$

of which the half is the error to fear to more, and the other half, taken with the $-$ sign, is the error to fear to less.

The correction which we just gave is that which renders a minimum the function

$$(\beta x)^2 + [\beta'(x-q)]^2 + [\beta''(x-q')]^2 + \cdots;$$

now the greatest ordinate of the curve of probabilities of the errors of the first result is, by that which precedes, $\frac{\beta}{\sqrt{\pi}}$; that of the curve of the probabilities of the errors of the second result is $\frac{\beta'}{\sqrt{\pi}}$, and thus in sequence. The mean that it is necessary to choose among the diverse results is therefore the one which renders a minimum the sum of the squares of the error of each result multiplied by the greatest ordinate of the curve of its probability. This mean is the first result A, plus its correction, or

$$\frac{A\beta^2 + (A+q)\beta'^2 + (A+q')\beta''^2 + \cdots}{\beta^2 + \beta'^2 + \beta''^2 + \cdots};$$

thus the law of the minimum of the squares of the errors becomes necessary when we must take a mean among some results each given by a great number of observations.

VIII.

The analysis exposed in article VI can be extended to the correction of any number of elements by observations. It leads always to this result: namely that the method of least squares of the errors of the observations is that which gives for the correction of the elements the smallest mean error to fear.

When we wish to correct one or many elements already known, to quite nearly, by the assembly of a great number of observations, we form some equations of condition in a manner analogous to the one which we have given in article VI, relatively to a single element.

We consider two elements, and name z the correction of the first and z' that of the second. Let β be the observation; its analytic expression will be a function of the two elements: by substituting into it their approximate values, increased respectively by the corrections z and z', by reducing it next into series and neglecting the product and the squares of z and z', this function will take the form $A + pz + qz'$, and by equating to it the observed quantity β, we will have

$$\beta = A + pz + qz'.$$

A second observation will give a similar equation, and we will have, by resolving these two equations, the values of z and z'. These values will be exact if the observations were rigorous; but, as they are susceptibles of error, we consider a great number of them. By combining next the equations of condition which each of them furnish, in a way to reduce them to two, we obtain the corrections of the elements with so much more exactitude as we employ more observations and as they are better combined. The research on the most advantageous combination is one of the most useful in the theory of the probabilities and merits at the same time the attention of the geometers and of the observers.

If in the preceding equation of condition we make $\beta - A = \alpha$, and if we name ϵ the error of the first observation, we will have

$$\epsilon = pz + qz' - \alpha.$$

The $(i+1)^{\text{st}}$ observation will give a similar equation, which we will represent by this one

$$\epsilon^{(i)} = p^{(i)}z + q^{(i)}z' - \alpha^{(i)},$$

$\epsilon^{(i)}$ being the error of this observation and s being the number of observations, such that i can be extended from $i = 0$ to $i = s - 1$.

Presently, all the ways to combine together these equations are reduced to multiply them respectively by some constants and to add them next. By multiplying them first respectively by m, $m^{(1)}$, $m^{(2),\cdots}$ and adding them, we will have the final equation

$$\mathrm{S}m^{(i)}\epsilon^{(i)} = z\mathrm{S}m^{(i)}p^{(i)} + z'\mathrm{S}m^{(i)}q^{(i)} - \mathrm{S}m^{(i)}\alpha^{(i)}.$$

By multiplying again the same equations respectively by n, $n^{(1)}, \ldots$ and adding these products, we will have a second final equation

$$\mathrm{S}n^{(i)}\epsilon^{(i)} = z\mathrm{S}n^{(i)}p^{(i)} + z'\mathrm{S}n^{(i)}q^{(i)} - \mathrm{S}n^{(i)}\alpha^{(i)},$$

the sign S extending in these two equations to all the values of i, from $i = 0$ to $i = s-1$.

If we suppose null the two functions $\mathrm{S}m^{(i)}\epsilon^{(i)}$ and $\mathrm{S}n^{(i)}\epsilon^{(i)}$, sums which we will designate respectively by (m) and (n), the two preceding final equations will give the corrections z and z' of the two elements. But these corrections are susceptibles of errors, relative to that of which the supposition that we just made is susceptible itself. We imagine therefore that the functions (m) and (n), instead of being nulls, are respectively l and l'; and we name u and u' the corresponding errors of the corrections z and z' determined by that which precedes, the two final equations will become

$$l = u\mathrm{S}m^{(i)}p^{(i)} + u'\mathrm{S}m^{(i)}q^{(i)},$$
$$l' = u\mathrm{S}n^{(i)}p^{(i)} + u'\mathrm{S}n^{(i)}q^{(i)}.$$

It is necessary now to determine the factors m, $m^{(1)}, \ldots, n$, $n^{(1)}, \ldots$, so that the mean error to fear is a minimum. For this, we consider the product

$$\int_{-a}^{a} \phi\left(\frac{x}{a}\right) e^{-(m\varpi+n\varpi')x\sqrt{-1}} \int_{-a}^{a} \phi\left(\frac{x}{a}\right) e^{-(m^{(1)}\varpi+n^{(1)}\varpi')x\sqrt{-1}} \cdots \int_{-a}^{a} \phi\left(\frac{x}{a}\right) e^{-(m^{(r-1)}\varpi+n^{(r-1)}\varpi')x\sqrt{-1}},$$

x being any error of an observation, $-a$ and $+a$ being the limits of this error, $\phi\left(\frac{x}{a}\right)$ being the probability of this error, and the probability of a positive error being supposed the same as that of the corresponding negative error; finally e being the number of which the hyperbolic logarithm is unity. The preceding function becomes, by reuniting the two exponentials relative to x and to $-x$,

$$2\int_{0}^{a} \phi\left(\frac{x}{a}\right) \cos(mx\varpi + nx\varpi') \times 2\int_{0}^{a} \phi\left(\frac{x}{a}\right) \cos(m^{(1)}x\varpi + n^{(1)}x\varpi') \times \cdots \times 2\int_{0}^{a} \phi\left(\frac{x}{a}\right) \cos(m^{(s-1)}x\varpi + n^{(s-1)}x\varpi'),$$

x being supposed, thus as a, divided into an infinity of parts taken for unity. Now, it is clear that the term independent of the exponentials, in the product of the preceding function by $e^{-l\varpi\sqrt{-1}-l'\varpi'\sqrt{-1}}$, is the probability that the sum of the errors of each observation, multiplied respectively by m, $m^{(1)}, \ldots$, or the function (m) will be equal to l at the same time as the function (n), sum of the errors of each observation, multiplied respectively by n, $n^{(1)}, \ldots$, will be equal to l'; this probability is therefore, by supposing m, $m^{(1)}, \ldots, n$, $n^{(1)}, \ldots$, some whole numbers,

$$(1)\quad \begin{cases} \dfrac{1}{4\pi^2}\displaystyle\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} d\varpi\, d\varpi' e^{-l\varpi\sqrt{-1}-l'\varpi'\sqrt{-1}} \\ \qquad \times\left[2\displaystyle\int_0^a \phi\left(\frac{x}{a}\right)\cos(mx\varpi+nx\varpi')\times\cdots\times 2\int_0^a \phi\left(\frac{x}{a}\right)\cos(m^{(s-1)}x\varpi+n^{(s-1)}x\varpi')\right], \end{cases}$$

π being the semi-circumference of which the radius is unity.

By reducing the cosines to series, and making

$$\frac{x}{a}=x', \qquad \frac{1}{a}=dx',$$

$$K=2\int_0^1 dx'\phi(x'), \quad K''=\int_0^1 x'^2\,dx'\,\phi(x'), \quad K^{iv}=\int_0^1 x'^4\,dx'\,\phi(x'), \quad \cdots,$$

we have

$$\begin{aligned} 2\int_0^a \phi\left(\frac{x}{a}\right)&\cos(mx\varpi+nx\varpi') \\ &= aK\left[1-\frac{K''a^2}{K}(m\varpi+n\varpi')^2+\frac{K^{iv}}{12K}a^4(m\varpi+n\varpi')^4+\cdots\right]; \end{aligned}$$

aK or $2a\int dx'\,\phi(x')$ expresses the probability that the error of each observation will be contained within its limits, this which is certain; we have therefore $aK=1$. By taking therefore the logarithm of the second member of the preceding equation, we will have

$$-\frac{K''}{K}a^2(m\varpi+n\varpi')^2+\frac{KK^{iv}-6K''^2}{12K^2}a^4(m\varpi+n\varpi')^4-\cdots$$

Thence it is easy to conclude that the logarithm of the product of the factors

$$2\int_0^a \phi\left(\frac{x}{a}\right)\cos(mx\varpi+nx\varpi'), \quad 2\int_0^a \phi\left(\frac{x}{a}\right)\cos(m^{(1)}x\varpi+n^{(1)}x\varpi'), \quad \cdots,$$

is, the sign S is relating to all the values of i,

$$\begin{aligned} -\frac{K''}{K}&a^2(\varpi^2\mathrm{S}m^{(i)^2}+2\varpi\varpi'\mathrm{S}m^{(i)}n^{(i)}+\varpi'^2\mathrm{S}n^{(i)^2}) \\ &+\frac{KK^{iv}-6K''^2}{12K^2}a^4(\varpi^4\mathrm{S}m^{(i)^4}+4\varpi\varpi'\mathrm{S}m^{(i)^2}n^{(i)}+\cdots)+\cdots. \end{aligned}$$

By passing again from the logarithms to the numbers, we will have, for the product itself,

$$\left[1+\frac{KK^{iv}-6K''^2}{12K^2}a^4(\varpi^4\mathrm{S}m^{(i)^4}+\cdots)+\cdots\right]e^{-\frac{K''}{K}a^2(\varpi^2\mathrm{S}m^{(i)^2}+2\varpi\varpi'\mathrm{S}m^{(i)}n^{(i)}+\varpi'^2\mathrm{S}n^{(i)^2})}.$$

By substituting therefore, in place of this product, this value into the integral function (1), it becomes

$$\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} d\varpi\, d\varpi' \left[1 + \frac{KK^{iv} - 6K''^2}{12K^2} a^4(\varpi^4 \mathrm{S}m^{(i)^4} + \cdots) + \cdots\right]$$
$$\times\, e^{-l\varpi\sqrt{-1} - l'\varpi'\sqrt{-1} - \frac{K''}{K}a^2(\varpi^2\mathrm{S}m^{(i)^2} + 2\varpi\varpi'\mathrm{S}m^{(i)}n^{(i)} + \varpi'^2\mathrm{S}n^{(i)^2})}:$$

s being the number of observations which we suppose very great; we make

$$a\varpi\sqrt{s} = t, \qquad a\varpi'\sqrt{s} = t',$$

this integral becomes

$$\frac{1}{4\pi^2a^2s}\iint dt\, dt' \left[1 + \frac{KK^{iv} - 6K''^2}{12K^2}\left(\frac{t^4\mathrm{S}m^{(i)^4}}{s^2} + \cdots\right) + \cdots\right]$$
$$\times\, e^{-\frac{lt\sqrt{-1}}{a\sqrt{s}} - \frac{l't'\sqrt{-1}}{a\sqrt{s}} - \frac{K''}{K}\left(\frac{t^2\mathrm{S}m^{(i)^2}}{s} + \frac{2t\,t'\mathrm{S}m^{(i)}n^{(i)}}{s} + \frac{t'^2\mathrm{S}n^{(i)^2}}{s}\right)}.$$

$\mathrm{S}m^{(i)^2}$, $\mathrm{S}m^{(i)^4}$, $\mathrm{S}m^{(i)}n^{(i)}$,... are evidently some quantities of order s; by neglecting therefore the terms of order $\frac{1}{s}$, vis-à-vis of unity, the preceding integral is reduced to

$$(2) \qquad \frac{1}{4\pi^2a^2s}\iint dt\, dt'\, e^{-\frac{lt\sqrt{-1}}{\sqrt{s}} - \frac{l't'\sqrt{-1}}{a\sqrt{s}} - \frac{K''}{K}\left(\frac{t^2\mathrm{S}m^{(i)^2}}{s} + \frac{2t\,t'\mathrm{S}m^{(i)}n^{(i)}}{s} + \frac{t'^2\mathrm{S}n^{(i)^2}}{s}\right)}.$$

The integral relative to ϖ being taken from $\varpi = -\pi$ to $\varpi = \pi$, the integral relative to t must be taken from $t = -a\pi\sqrt{s}$ to $t = +a\pi\sqrt{s}$; and, in these two cases, the exponential under the $\int$ sign is insensible to these two limits, either because s is a great number, or because a is here supposed divided into an infinity of parts taken for unity; we can therefore take the integral relative to t from $t = -\infty$ to $t = \infty$, and it is likewise of the integral relative to t'. This put, if we make

$$t'' = t + t'\frac{\mathrm{S}m^{(i)}n^{(i)}}{\mathrm{S}m^{(i)2}} + \frac{Kl\sqrt{s}\sqrt{-1}}{2K''a\mathrm{S}m^{(i)2}},$$
$$t''' = t' - \frac{K\sqrt{s}}{2K''a}\,\frac{(l\,\mathrm{S}m^{(i)}n^{(i)} - l'\mathrm{S}m^{(i)^2})\sqrt{-1}}{\mathrm{S}m^{(i)2}n^{(i)2} - (\mathrm{S}m^{(i)}n^{(i)})^2};$$

if we make next

$$E = \mathrm{S}m^{(i)^2}n^{(i)^2} - (\mathrm{S}m^{(i)}n^{(i)})^2,$$

the preceding double integral becomes

$$e^{-\frac{K}{4K''a^2E}(l^2\mathrm{S}n^{(i)^2} - 2ll'\mathrm{S}m^{(i)}n^{(i)} + l'^2\mathrm{S}m^{(i)^2})}\iint \frac{dt''\, dt'''}{4\pi^2a^2s}\, e^{-\frac{K''}{K}t''^2\frac{\mathrm{S}m^{(i)^2}}{s} - \frac{K''}{K}t'''^2\frac{E}{s\mathrm{S}m^{(i)^2}}}.$$

The integrals relative to t'' and t''' must be taken as those which are relative to t and t' between the limits positive and negative infinities; now we have within these limits, by the known theorems,

$$\int_{-\infty}^{\infty} dt\, e^{-\beta^2t^2} = \frac{\sqrt{\pi}}{\beta};$$

the function (2) is reduced therefore thus to

$$(3)\qquad \frac{K}{4K''a^2\sqrt{E}}\, e^{-\frac{K}{4K''a^2E}(l^2\text{S}n^{(i)^2}-2ll'\text{S}m^{(i)}n^{(i)}+l'^2\text{S}m^{(i)^2})}.$$

It is necessary now, in order to have the probability that the values of l and of l' will be contained within the given limits, to multiply this quantity by $dl\,dl'$ and to integrate it next within these limits; we name therefore x this quantity, the probability of which there is question will be $\iint x\,dl\,dl'$; but, in order to have the probability that the errors u and u' of the corrections of the elements will be contained within some given limits, it is necessary to substitute into this integral, in place of l and l', their values in u and u'. If we differentiate these values by supposing l' constant, we have

$$dl = du\,\text{S}m^{(i)}p^{(i)} + du'\,\text{S}m^{(i)}q^{(i)},$$
$$0 = du\,\text{S}n^{(i)}p^{(i)} + du'\,\text{S}n^{(i)}q^{(i)},$$

this which gives, by making

$$I = \text{S}m^{(i)}p^{(i)}\,\text{S}n^{(i)}q^{(i)} - \text{S}n^{(i)}p^{(i)}\text{S}m^{(i)}q^{(i)},$$
$$dl = \frac{I du}{\text{S}n^{(i)}q^{(i)}};$$

If we differentiate next the expression of l' by supposing u constant, we have

$$dl' = du'\,\text{S}n^{(i)}q^{(i)};$$

we will have therefore

$$dl\,dl' = I\,du\,du';$$

thus, by supposing

$$\begin{aligned}
F =&\text{S}n^{(i)^2}(\text{S}m^{(i)}p^{(i)})^2 - 2\text{S}m^{(i)}n^{(i)}\,\text{S}m^{(i)}p^{(i)}\text{S}n^{(i)}p^{(i)} + \text{S}m^{(i)^2}(\text{S}n^{(i)}p^{(i)})^2\\
G =&\text{S}n^{(i)^2}\,\text{S}m^{(i)}p^{(i)}\,\text{S}m^{(i)}q^{(i)} + \text{S}m^{(i)^2}\text{S}n^{(i)}p^{(i)}\text{S}n^{(i)}q^{(i)}\\
&- \text{S}m^{(i)}n^{(i)}(\text{S}n^{(i)}p^{(i)}\text{S}m^{(i)}q^{(i)} + \text{S}m^{(i)}p^{(i)}\text{S}n^{(i)}q^{(i)}),\\
H =&\text{S}n^{(i)^2}(\text{S}m^{(i)}q^{(i)})^2 - 2\text{S}m^{(i)}n^{(i)}\text{S}m^{(i)}q^{(i)}\text{S}n^{(i)}q^{(i)} + \text{S}m^{(i)^2}(\text{S}n^{(i)}q^{(i)})^2,
\end{aligned}$$

the function (3), multiplied by $dl\,dl'$ and next affected with the integral sign, becomes

$$(4)\qquad \iint \frac{K}{4K''\pi}\,\frac{I}{\sqrt{E}}\,\frac{du\,du'}{a^2}e^{-\frac{K(Fu^2+2Guu'+Hu''2)}{4K''aE}}.$$

We integrate first this function with respect to u' and in all the extent of its limits. The value of $\frac{u'}{a}$ is finite in these limits; but, as in the exponential it is multiplied by $\frac{G}{E}$ and $\frac{H}{E}$, and these quantities being of order s, because G and H are of order s^3, while E is of order s^2, this exponential becomes insensible to these limits, and we can extend the integral from $u' = -\infty$ to $u'' = \infty$. By making

$$t = \frac{\sqrt{\frac{KH}{4K''}}\left(u' + \frac{Gu}{H}\right)}{a\sqrt{E}},$$

and taking the integral relative to t from $t=-\infty$ to $t=\infty$, the function (4) is reduced to

$$\int \sqrt{\frac{K}{4K''\pi}}\,\frac{du}{a}\,\frac{I}{\sqrt{H}}e^{-\frac{KIu^2}{4K''a^2H}},$$

because

$$\frac{FH-G^2}{E}=I^2.$$

Now, if we imagine a curve of which u is the abscissa and of which the ordinate is

$$\sqrt{\frac{K}{4K''\pi}}\,\frac{I}{\sqrt{H}}e^{-\frac{KIu^2}{4K''a^2H}},$$

this curve, which we can extend to infinity on each side of the ordinate which corresponds to u null, will be the curve of the probabilities of the errors u of the correction of the first element. This put, every error, either positive or negative, must be considered as a disadvantage or a real loss in any game; now, by the known principles of the Calculus of probabilities, we evaluate this disadvantage by taking the sum of the products of each error by its probability; the mean value of the error to fear, to more or to less, on the first element, is therefore

$$\pm\sqrt{\frac{K}{4K''\pi}}\frac{I}{a\sqrt{H}}\int_0^\infty u\,du\,e^{-\frac{KIu^2}{4K''a^2H}},$$

the $+$ sign indicating the mean error to fear to more, and the $-$ sign indicating the error to fear to less. This error becomes thus

$$\pm\sqrt{\frac{K''}{K\pi}}\frac{a\sqrt{H}}{I}.$$

By changing H into F, we will have

$$\pm\sqrt{\frac{K''}{K\pi}}\frac{a\sqrt{F}}{I}$$

for the mean error to fear on the second element.

We determine presently the factors $m^{(i)}$ and $n^{(i)}$, so that this error is a minimum. By making $m^{(i)}$ to vary alone, we have

$$\begin{aligned} d\log\frac{\sqrt{H}}{I} =& dm^{(i)}\frac{q^{(i)}\mathrm{S}n^{(i)}p^{(i)}-p^{(i)}\mathrm{S}n^{(i)}q^{(i)}}{I} \\ &+dm^{(i)}\frac{\begin{bmatrix} q^{(i)}\mathrm{S}n^{(i)^2}\mathrm{S}m^{(i)}q^{(i)}-n^{(i)}\mathrm{S}m^{(i)}q^{(i)}\mathrm{S}n^{(i)}q^{(i)} \\ -q^{(i)}\mathrm{S}m^{(i)}n^{(i)}\mathrm{S}n^{(i)}q^{(i)}+m^{(i)}(\mathrm{S}n^{(i)}q^{(i)})^2\end{bmatrix}}{H}\end{aligned}$$

It is easy to see that this differential disappears if we suppose in the coefficients of $dm^{(i)}$

$$m^{(i)}=\mu p^{(i)},\qquad n^{(i)}=\mu q^{(i)},$$

μ being an arbitrary coefficient independent of i, and by means of which we can render m, $m^{(i)}$, ... some whole numbers, as the preceding analysis requires. The preceding supposition renders therefore null the differential of $\frac{\sqrt{H}}{I}$ taken with respect to $m^{(i)}$. We will see in the same manner that it renders null the differential of the same quantity, taken with respect to $n^{(i)}$; thus this supposition renders a minimum the mean error to fear on the correction of the first element, and we will see in the same manner that it renders a minimum the mean error to fear on the correction of the second element. Under this supposition, the corrections of the two elements are

$$z = \frac{\text{S}q^{(i)^2}\text{S}p^{(i)}\alpha^{(i)} - \text{S}p^{(i)}q^{(i)}\text{S}q^{(i)}\alpha^{(i)}}{\text{S}p^{(i)2}\text{S}q^{(i)2} - (\text{S}p^{(i)}q^{(i)})^2},$$
$$z' = \frac{\text{S}q^{(i)^2}\text{S}q^{(i)}\alpha^{(i)} - \text{S}p^{(i)}q^{(i)}\text{S}p^{(i)}\alpha^{(i)}}{\text{S}p^{(i)2}\text{S}q^{(i)2} - (\text{S}p^{(i)}q^{(i)})^2}.$$

These corrections are those which the method of least squares gives of the errors of the observations, or the condition of the minimum of the function

$$\text{S}(p^{(i)}z + q^{(i)}z' - \alpha^{(i)})^2,$$

where it follows that this method holds generally, whatever be the number of elements to determine; because it is clear that the preceding analysis can be extended to any number of elements. The mean error to fear on the first element becomes then

$$\pm a\frac{\sqrt{\frac{K''}{K\pi}}\sqrt{\text{S}q^{(i)^2}}}{\sqrt{\text{S}p^{(i)2}\text{S}q^{(i)2} - (\text{S}p^{(i)}q^{(i)})^2}}.$$

and on the second element it becomes

$$\pm a\frac{\sqrt{\frac{K''}{K\pi}}\sqrt{\text{S}p^{(i)^2}}}{\sqrt{\text{S}p^{(i)2}\text{S}q^{(i)2} - (\text{S}p^{(i)}q^{(i)})^2}}.$$

We see thus that the first element will be more or less well determined, that the second, according as $\text{S}q^{(i)^2}$, will be smaller or greater than $\text{S}p^{(i)^2}$.

If the first r equations of condition contain q not at all, and if the last $s - r$ contain p not at all, then $\text{S}p^{(i)}q^{(i)}$ is null and the first of the two preceding formulas becomes

$$\pm\frac{a\sqrt{\frac{K''}{K\pi}}}{\sqrt{\text{S}p^{(i)^2}}}.$$

The sign S is relating to all the values of i, from $i = 0$ to $i = r - 1$, this is the formula relative to a single element determined by a great number r of observations; it agrees with that which we have found in article VI.

In all these formulas, the factor $a\sqrt{\frac{K''}{K}}$ is unknown. We can take for a the deviation of the mean result, which would make to reject an observation. If we suppose $\phi\left(\frac{x}{a}\right)$ equal to a constant, we have

$$\frac{K''}{K} = \frac{1}{6};$$

this is the greatest value that we can suppose to the fraction $\frac{K''}{K}$, as we have seen in the article cited; but the following remark removes all uncertainty on the factor of which there is question. I have recognized, and I will prove in a Work[11] that I am going to publish soon on the probabilities, that the sum of the squares of the errors of a great number s of observations can be supposed very nearly equal to $2s\frac{a^2K''}{K}$; now we have this sum by substituting, into each equation of condition, the corrections of the elements, determined by the method of least squares of the errors of the observations; because, if we name $\epsilon^{(i)}$ that which remains after these substitutions into the $(i+1)^{\text{st}}$ equation of condition, this sum will be very nearly $S\epsilon^{(i)^2}$; by equating it therefore to $2sa^2\frac{K''}{K}$, we will have

$$a\sqrt{\frac{K''}{K}} = \sqrt{\frac{\mathrm{S}\epsilon^{(i)^2}}{2s}}.$$

For a single element, the mean error becomes therefore thus

$$(a) \qquad \sqrt{\frac{\mathrm{S}\epsilon^{(i)^2}}{2s\pi}}\,\frac{1}{\sqrt{\mathrm{S}p^{(i)^2}}}.$$

Thence results this general rule in order to have the mean error to fear, whatever be the number of elements. We represent generally the equations of condition by the following

$$\epsilon^{(i)} = p^{(i)}z + q^{(i)}z' + r^{(i)}z'' + t^{(i)}z''' + \cdots - \alpha^{(i)},$$

$z, z', z'', z''', \ldots$ being the corrections of these elements.

When there are two elements, we will have the mean error to fear on the first element, by changing in the function (a)

$$\mathrm{S}p^{(i)^2} \quad \text{into} \quad \mathrm{S}p^{(i)^2} - \frac{(\mathrm{S}p^{(i)}q^{(i)})^2}{\mathrm{S}q^{(i)^2}}.$$

We will have thus an expression which we will designate by (a').

When there will be three elements, we will have the error to fear on the first element, by changing in the expression (a')

$$\mathrm{S}p^{(i)^2} \quad \text{into} \quad \mathrm{S}p^{(i)^2} - \frac{(\mathrm{S}p^{(i)}r^{(i)})^2}{\mathrm{S}r^{(i)^2}}$$

$$\mathrm{S}q^{(i)^2} \quad \text{into} \quad \mathrm{S}q^{(i)^2} - \frac{(\mathrm{S}q^{(i)}r^{(i)})^2}{\mathrm{S}r^{(i)^2}}$$

and

$$\mathrm{S}p^{(i)}q^{(i)} \quad \text{into} \quad \mathrm{S}p^{(i)}q^{(i)} - \frac{\mathrm{S}p^{(i)}r^{(i)}\mathrm{S}q^{(i)}r^{(i)}}{\mathrm{S}r^{(i)^2}}.$$

We will form thus an expression which we will designate by (a'').

[11] *Translator's note*: This refers, of course, to the *Théorie Analytique des Probabilités* published in 1812.

When there are four elements, we will have the mean error to fear on the first element, by changing in the expression (a'')

$$\mathrm{S}p^{(i)^2} \quad \text{into} \quad \mathrm{S}p^{(i)^2} - \frac{(\mathrm{S}p^{(i)}t^{(i)})^2}{\mathrm{S}t^{(i)^2}}$$

$$\mathrm{S}p^{(i)}q^{(i)} \quad \text{into} \quad \mathrm{S}p^{(i)}q^{(i)} - \frac{\mathrm{S}p^{(i)}t^{(i)}\mathrm{S}q^{(i)}t^{(i)}}{\mathrm{S}t^{(i)^2}}, \cdots$$

By continuing thus, we will have the mean error to fear on the first element, whatever be the number of elements. By changing in the expression of this error that which is relative to the first element, into that which is relative to the second and reciprocally, we will have the mean error to fear on that element, and thus of the rest.