MÉMOIRE SUR LES APPROXIMATIONS DES FORMULES

QUI SONT FONCTIONS DE TRÈS GRANDS NOMBRES

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We are often led in analysis, and principally in that of chance, to some formulas of which usage becomes impossible when we substitute some large numbers into it. The numerical solution of the problems of which they are the analytic solution presents then great difficulties that we are yet successful in overcoming only in some particular cases, of which the two chief are related to the product of the natural numbers $1,\ 2,\ 3,\ 4,\ \dots$ and to the mean term of the binomial raised to a great power. If we suppose this power even and equal to $2s$, this term will be, as we know

$$\frac{2s(2s-1)(2s-2)(2s-3)\cdots(s+1)}{1.2.3.4\dots s}.$$

Although this expression is quite simple, however if s is very large, for example, equal to 10000, it becomes very difficult to reduce in numbers, because of the multiplication of these factors. Mr. Stirling has happily attained in transforming into some series the more so convergent as s is a greater number (*see* his good work *De summatione et interpolatione serierum*). This transformation, which we can regard as one of the most ingenious discoveries that one has made in the theory of series, is especially remarkable in that in one research, which seems to admit only some algebraic quantities, it introduces a transcendent quantity: namely the square root of the ratio of the semi-circumference to the radius. But the method of Mr. Stirling, founded on the interpolation of the series and on some theorems of Wallis, leaves wanting a direct method which extends to all functions composed of a great number of terms and of factors. I have given, in our Mémoires for the year 1778, p. 289,[1] a general method to reduce in a convergent series the integrals of the differential functions which contain some factors raised to some great powers; but, occupied by a different object, I myself am content now to draw from this method the beautiful theorems of Mr. Stirling, in reserving to myself the resumption and the more thorough study in another Memoir. Some new reflections have lead me to extend it generally to any functions of very great numbers and

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[1] "Mémoire sur la probabilités."

to reduce these functions to some series so much more convergent as these numbers are greater, so that this method is the better approximation as it becomes more necessary. I intend to develop it in this Memoir with all the detail due to the novelty of the subject and to its importance in the applications of Analysis.

The difficulty which the reduction of numbers presents in very composite analytic formulas comes from the multiplication of their terms and of their factors: we will make it therefore vanish, if we come to reduce these formulas to some convergent enough series in order that we have need to consider only the first terms, and if, moreover, each of these terms contains only a small number of factors which can be raised moreover to some great powers. It will be easy then to have these factors and their products, by the known artifices, in order to obtain, by means of Tables, the logarithms of very great numbers and the numbers of very great logarithms. The question is reduced therefore to transform the composite functions into convergent series. This appears impossible when we consider them under their natural form; but, as little that we are versed in the infinitesimal Analysis, we have often observed differential functions of a very simple form, and which contain some factors raised to some great powers, to produce, by their integration, some very composite functions, that which gives occasion to think that each composite function is reducible to some similar integrals which there will be no more further concern but to convert to convergent series. The problem that we propose to resolve, considered under this point of view, is divided thus into two others, of which one consists to integrate by approximation the differential functions which contain some very elevated factors, and of which the other has for object to restore to this kind of integrals the functions of which we seek some approximate values.

In article I of this memoir, I give the solution of the first problem, which, by itself, is very useful in this branch of the Analysis of hazards, where we propose to go back from the observed events to their causes and to recognize, by these events, the probability of future events (*see* the *Mémoires de l'Académie* for the year 1778). This solution leads me to different series which serve to supplement the ones to the others, the first needing to be employed for the points of the integral extended from the maximum of the differential function, and the second needing to serve for the points neighboring this maximum: these last series contain the transcendent quantities which, most often, are reduced to this one

$$\int dt\, e^{-t^2},$$

e being the number of which the hyperbolic logarithm is unity; and, as this integral, taken from $t = 0$ to $t = \infty$, is the half of the square root of the ratio of the semicircumference to the radius, there results that the approximate value of the integrals determined from the differential functions which contain some very elevated factors depends nearly always on this root, even in the case where these integrals are algebraic; thus this transcendent quantity that Mr. Stirling has introduced first in the approximate value of the mean term of the binomial is not particular to it, but it enters equally in the approximate values of a great number of other algebraic functions.

I consider in article II the problem which consists in restoring the functions in which one seeks the approximate values in the integration of differential functions multiplied by some factors raised to some great powers; in order to arrive to a general method, I represent by y_s, y'_s, y''_s, ... some functions of s, very composite and in which s is a great number. I suppose these functions given by some linear equations in the differences, either finite, or infinitely small, of which the coefficients are some rational functions of s; by making next, in these equations

$$y_s = \int x^s \phi \, dx, \quad y'_s = \int x^s \phi' dx, \quad \ldots,$$

and by preparing them in a convenient manner, each of them is divided into two parts, of which the one is affected by an integral sign $\int$ of which the other is outside of this sign: equating to zero of the parts under the sign gives as many linear equations in the infinitely small differences as there are variables ϕ, ϕ', ϕ'', ... We can, consequently, determine at their mean these variables in functions of x; as for the parts under the integral sign, by equating them to zero and by eliminating the arbitrary constants of the values of ϕ, ϕ', ϕ'', ..., we arrive to a final equation in x, of which the roots serve to determine the limits in which we must take the integrals $\int x^s \phi \, dx$, $\int x^s \phi' dx, \ldots$ A very important remark in this analysis, and which gives the means to extend to some functions of frequent usage, is that the series which we obtain for y_s, y'_s, ... holds generally by changing in it the sign of the constants which they contain, although, by this change, the final equation in x, which determines the limits of the integrals, ceases to have many real roots. The principal obstacle which we encounter in the application of this method comes from the nature of the differential equations in ϕ, ϕ', ϕ'', ..., which can not be integrable: we can often obviate this inconvenience by representing the functions y_s, y'_s, ... by some multiple integrals such as $\int x^s x'^s \phi \, dx dx'$, $\int x^s x'^s \phi' dx dx', \ldots$; we will arrive thus to determine ϕ, ϕ', ... by some equations of an order less elevated and susceptible to being integrated by known methods.

The preceding analysis, applied to linear equations in partial differentials, gives likewise their integrals in convergent series, so that it extends generally to very composite functions which can be represented by some differential equations linear in the ordinary or partial differences, finite or infinitely small, or by finite parts and by infinitely small parts, this which embraces all the functions which are met in the ordinary use of Analysis.

In article III, I apply the preceding method to diverse differential equations; I deduce from them the values, by highly convergent series, the product of the natural numbers $1, 2, 3, 4, \ldots$ of the mean term of the binomial, of that of the trinomial, etc., of very elevated differences, either finite or infinitely small of the functions or of any part whatever of these differences.

Finally, in article IV, I give the solution of many interesting problems in the Analysis of chances, which it would be impossible to resolve numerically by known means.

ARTICLE I.

On the integration by approximation of differential functions which contain factors raised to some great powers.

I.

If we designate by u, u', u'', ... and ϕ some arbitrary functions of x, and by s, s', s'', ... some large numbers, each differential function which contains some factors raised to some great powers will be comprehended in this form $u^s u'^{s'} u''^{s''} \cdots \phi \, dx$.

In order to have by convergent series its integral taken from $x = \theta$ to $x = \theta'$, we will make $u^s u'^{s'} \cdots \phi = y$, and, by designating by Y that which y becomes when we change x to θ in it, we will suppose $y = \mathrm{Y} e^{-t}$, e being the number of which the hyperbolic logarithm is unity; we will have thus

$$\log \frac{\mathrm{Y}}{y} = t.$$

If we consider x as a function of t given by this equation, we will have, by supposing dt constant,

$$x = \theta + t\frac{dx}{dt} + \frac{t^2}{1.2}\frac{d^2x}{dt^2} + \frac{t^3}{1.2.3}\frac{d^3x}{dt^3} + \cdots,$$

t needing to be supposed null, after the differentiations, in the values of $\frac{dx}{dt}$, $\frac{d^2x}{dt^2}$, ... Now we have generally

$$\frac{d^nx}{dt^n} = \frac{1}{dt}d\frac{1}{dt}d\frac{1}{dt}\cdots d\frac{dx}{dt},$$

the differential characteristic d relating to all that which follows it, and dt being able to vary in any manner whatever in the second member of this formula; moreover, if we differentiate the equation $\log\frac{\mathrm{Y}}{y} = t$, and if we designate $-\frac{y\,dx}{dy}$ by ν, we will have

$$dt = \frac{dx}{\nu};$$

therefore, we will have

$$\frac{d^nx}{dt^n} = \frac{\nu d[\nu\, d(\nu \ldots d\nu)]}{dx^{n-1}},$$

dx being supposed constant in the second member of this equation. By naming therefore U that which ν becomes when we change x to θ, the value of $\frac{d^nx}{dt^n}$, which corresponds to $x = \theta$, or, what returns to the same, to $t = 0$, will be equal to $\frac{\mathrm{U}\,d[\mathrm{U}\,d(\mathrm{U}\ldots d\mathrm{U})]}{d\theta^{n-1}}$; we will have thus

$$x = \theta + \mathrm{U}t + \frac{\mathrm{U}\,d\mathrm{U}}{1.2\,d\theta}t^2 + \frac{\mathrm{U}\,d(\mathrm{U}\,d\mathrm{U})}{1.2.3\,d\theta^2}t^3 + \cdots,$$

whence we deduce

$$dx = \mathrm{U}\,dt\left[1 + \frac{d\mathrm{U}}{d\theta}t + \frac{d(\mathrm{U}\,d\mathrm{U})}{1.2\,d\theta^2}t^2 + \cdots\right]$$

and, consequently,

$$\int y\,dx = \mathrm{UY}\int dt\,e^{-t}\left[1 + \frac{d\mathrm{U}}{d\theta}t + \frac{d(\mathrm{U}\,d\mathrm{U})}{1.2\,d\theta^2}t^2 + \cdots\right].$$

If we take the integral from $t = 0$ to $t = \infty$, we will have generally

$$\int t^n dt\,e^{-t^n} = 1.2.3\ldots n,$$

therefore

$$\int y\,dx = \mathrm{UY}\left\{1 + \frac{d\mathrm{U}}{d\theta} + \frac{d(\mathrm{U}\,d\mathrm{U})}{d\theta^2} + \frac{d[\mathrm{U}\,d(\mathrm{U}d\mathrm{U})]}{d\theta^3} + \cdots\right\},$$

the integral relative to x being taken from $x = \theta$ to the value of x which agrees to t infinity.

We name Y' and U' that which y and ν become when we change x to θ'; we will have likewise

$$\int y\,dx = \mathrm{U}'\mathrm{Y}'\left\{1 + \frac{d\mathrm{U}'}{d\theta'} + \frac{d(\mathrm{U}'\,d\mathrm{U}')}{d\theta'^2} + \frac{d[\mathrm{U}'\,d(\mathrm{U}'d\mathrm{U}')]}{d\theta'^3} + \cdots\right\},$$

the integral relative to x being taken from $x = \theta'$ to the value of x which corresponds to t infinity; by subtracting therefore these two expressions from one another, we will have

$$\text{(A)}\qquad \begin{cases} \displaystyle\int y\,dx = \mathrm{UY}\left\{1 + \frac{d\mathrm{U}}{d\theta} + \frac{d(\mathrm{U}\,d\mathrm{U})}{d\theta^2} + \frac{d[\mathrm{U}\,d(\mathrm{U}d\mathrm{U})]}{d\theta^3} + \cdots\right\} \\ \displaystyle\qquad\qquad - \mathrm{U}'\mathrm{Y}'\left\{1 + \frac{d\mathrm{U}'}{d\theta} + \frac{d(\mathrm{U}'\,d\mathrm{U}')}{d\theta^2} + \frac{d[\mathrm{U}'\,d(\mathrm{U}'d\mathrm{U}')]}{d\theta^3} + \cdots\right\}, \end{cases}$$

the integral relative to x being taken from $x = \theta$ to $x = \theta'$, so that the consideration of t disappears in this formula. If θ and θ' were originally contained in y, it would be necessary to vary only the quantities θ and θ' which introduce into U and U' the changes of x to θ and to θ' in the function ν.

Formula (A) will be very convergent if ν or $-\frac{y\,dx}{dy}$ is a very small quantity; now, y being, by assumption, equal to $u^s u'^{s'} u''^{s''} \cdots \phi$, we have

$$\nu = -\frac{1}{\frac{s\,du}{u\,dx} + s'\frac{du'}{u'dx} + \cdots + \frac{d\phi}{\phi\,dx}};$$

thus, in the case where s, s', s'', ... will be very great numbers, ν will be very small; and, if we make $\frac{1}{s} = \alpha$, α being a very small coefficient, the function ν will be of the order α and the successive terms of formula (A) will be respectively of the orders α, α^2, α^3,....

This formula will cease to be convergent if the assumption of $x = \theta$ rendered very small the denominator of the expression of ν. We suppose, for example, that $(x-a)^\mu$ is a factor of this denominator; it is clear that the successive terms of the series, which, in formula (A), multiplied UY, will be divided respectively by $(\theta-a)^\mu$, $(\theta-a)^{2\mu+1}$, $(\theta-a)^{3\mu+2}$, ... and will become very large if θ is little different from a. The convergence of this formula requires therefore that $(\theta-a)^\mu$ and $(\theta'-a)^\mu$ be greater than α; it cannot, consequently, be used in the interval where $(x-a)^\mu$ is equal or less than α; but, in this case, we can make use of the following method.

II.

If we name Y that which y becomes when we change x to a, it is clear that, $(x-a)^\mu$ being a factor of $-\frac{dy}{y\,dx}$ or, what returns to the same, of $\frac{d\log\frac{\mathrm{Y}}{y}}{dx}$, $(x-a)^{\mu+1}$ will be a factor of $\log\frac{\mathrm{Y}}{y}$. Let therefore

$$y = \mathrm{Y}e^{-t^{\mu+1}}$$

and

$$\nu = \frac{x-a}{(\log\mathrm{Y} - \log y)^{\frac{1}{\mu+1}}};$$

we will have

$$x - a = \nu t,$$

ν at no point becoming infinity by the assumption $x = a$. If we designate next by U, $\frac{d\mathrm{U}^2}{dx}$, $\frac{d^2\mathrm{U}^3}{dx^2}, \dots$ that which ν, $\frac{d\nu^2}{dx}$, $\frac{d^2\nu^3}{dx^2}$, ... become when we change x to a, after the differentiations, we will have

$$x = a + \mathrm{U}t + \frac{d\mathrm{U}^2}{1.2\,dx}t^2 + \frac{d^2\mathrm{U}^3}{1.2.3\,dx^2}t^3 + \cdots ,$$

whence it is easy to conclude

$$\text{(B)} \qquad \int y\,dx = \mathrm{Y}\int dt\,e^{-t^{\mu+1}}\left[\mathrm{U} + \frac{d\mathrm{U}^2}{dx}t + \frac{d^2\mathrm{U}^3}{1.2\,dx^2}t^2 + \frac{d^3\mathrm{U}^4}{1.2.3\,dx^3}t^3\cdots\right].$$

This formula can be used in each interval where x differs little from a; it can consequently serve as supplement to formula (A) of the preceding section; but, instead of being ordered as it with respect to the powers of α, it will be only relatively to the powers of $\alpha^{\frac{1}{\mu+1}}$, because it is clear that, in this last case, ν is only of order $\alpha^{\frac{1}{\mu+1}}$.

In order to determine easily the quantities U, $\frac{d\mathrm{U}^2}{dx}$, $\frac{d^2\mathrm{U}^3}{dx^2}, \dots$, we suppose

$$\log\mathrm{Y} - \log y = (x-a)^{\mu+1}\left[\mathrm{A} + \mathrm{B}(x-a) + \mathrm{C}(x-a)^2 + \cdots\right];$$

we will have, by changing x to a, after the differentiations

$$\mathrm{A} = -\frac{d^{\mu+1}\log y}{1.2.3\dots(\mu+1)\,dx^{\mu+1}},$$
$$\mathrm{B} = -\frac{d^{\mu+2}\log y}{1.2.3\dots(\mu+2)\,dx^{\mu+2}},$$
$$\cdots$$

We will have next, whatever be r,

$$\begin{aligned}\nu^r =& [\mathrm{A} + \mathrm{B}(x-a) + \mathrm{C}(x-a)^2 + \cdots]^{-\frac{r}{\mu+1}}\\ =& \mathrm{A}^{-\frac{r}{\mu+1}} - \frac{r}{\mu+1}\mathrm{A}^{-\frac{r+\mu+1}{\mu+1}}\mathrm{B}(x-a)\\ &+ \left[\frac{r(r+\mu+1)}{1.2\,(\mu+1)^2}\mathrm{A}^{-\frac{r+2\mu+2}{\mu+1}}\mathrm{B}^2 - \frac{r}{\mu+1}\mathrm{A}^{-\frac{r+\mu+1}{\mu+1}}\mathrm{C}\right](x-a)^2 + \cdots\end{aligned}$$

If we make successively in this formula $r = 1$, $r = 2$, $r = 3$, …, it will be easy to conclude the values of U, $\frac{d\text{U}^2}{dx}$, $\frac{d^2\text{U}^3}{dx^2}, \ldots$, and formula (B) will present no more difficulties other than those which result from the integration of the quantities of this form $\int t^n dt\, e^{-t^{\mu+1}}$; now we have

$$\begin{aligned}\int t^n dt\, e^{-t^{\mu+1}} =& \frac{-e^{t^{\mu+1}}}{\mu+1}\Bigg[t^{n-\mu} + \frac{n-\mu}{\mu+1}t^{n-2\mu-1} + \frac{(n-\mu)(n-2\mu-1)}{(\mu+1)^2}t^{n-3\mu-2} + \cdots \\ &+ \frac{(n-\mu)(n-2\mu-1)(n-3\mu-2)\cdots(n-r\mu+\mu-r+2)t^{n-r\mu-r+1}}{(\mu+1)^{r-1}}\Bigg] \\ &+ \frac{(n-\mu)(n-2\mu-1)\cdots(n-r\mu+\mu-r+1)}{(\mu+1)^r}\int t^{n-r\mu-r}dt\, e^{-t^{\mu+1}},\end{aligned}$$

r being equal to the quotient of the division of n by $\mu+1$ if the division is possible, or to the whole number immediately inferior if it is not. The determination of the integral $\int y\, dx$ depends therefore on the integrals of the form

$$\int dt\, e^{-t^{\mu+1}}, \quad \int t\, dt\, e^{-t^{\mu+1}}, \quad \ldots, \quad \int t^{\mu-1}dt\, e^{-t^{\mu+1}};$$

if it is not possible to obtain exactly these integrals by the known methods; but it will be easy in all the cases to have their approximate values.

III.

We will have principally need in the following of the value of $\int y\, dx$ for the whole interval contained between two consecutive values of x which render y null; we are going consequently to expose the simplifications of which this value is thus susceptible. y having been supposed in the preceding section equal to $\text{Y}e^{-t^{\mu+1}}$, it is clear that the two values of x which render y null render similarly null the quantity $e^{-t^{\mu+1}}$, that which supposes that $\mu+1$ is an even number, and that one of these values of x correspond to $t = -\infty$ and the other to $t = \infty$; Y is therefore then the maximum of y contained between these values. Let $\mu + 1 = 2i$; if we take the integral $\int t^{2n+1}dt\, e^{-t^{2i}}$from $t = -\infty$ to $t = \infty$, its value will be null, because it is clear that the elements of this integral which correspond to the negative values of t are equal and of contrary sign to those which correspond to the positive values of t. The integral $\int t^{2n}\, dt\, e^{-t^{2i}}$ will be equal to $2\int t^{2n}\, dt\, e^{-t^{2i}}$, this latter integral being taken from $t = 0$ to $t = \infty$, and, in this case, we have, by the preceding number,

$$\int t^{2n}\, dt\, e^{-t^{2i}} = \frac{(2n-2i+1)(2n-4i+1)\cdots(2n-ri+1)}{(2i)^r}\int t^{2n-2r}dt\, e^{-t^{2i}},$$

r being equal to the quotient of the division of n by i if the division is possible, or to the whole number immediately smaller if the division is not possible. Let therefore

$$\begin{aligned}
\mathrm{K} &= \int dt\, e^{-t^{2i}},\\
\mathrm{K}^{(1)} &= \int t^2\, dt\, e^{-t^{2i}},\\
\mathrm{K}^{(2)} &= \int t^4\, dt\, e^{-t^{2i}},\\
&\cdots\\
\mathrm{K}^{(i-1)} &= \int t^{2i-2}\, dt\, e^{-t^{2i}};
\end{aligned}$$

formula (B) of the preceding number will become

$$\text{(C)}\quad \left\{\begin{aligned}
\int y\,dx =& 2\mathrm{KY}\left(\mathrm{U} + \frac{1}{2i}\frac{d^{2i}\mathrm{U}^{2i+1}}{1.2.3\ldots 2i\,dx^{2i}} + \frac{2i+1}{4i^2}\frac{d^{4i}\mathrm{U}^{4i+1}}{1.2.3\ldots 4i\,dx^{4i}} + \cdots\right)\\
&+ 2\mathrm{K}^{(1)}\mathrm{Y}\left[\frac{d^2\mathrm{U}^3}{1.2\,dx^2} + \frac{3}{2i}\frac{d^{2i+2}\mathrm{U}^{2i+3}}{1.2.3\cdots(2i+2)\,dx^{2i+2}}\right.\\
&\qquad\left. + \frac{3(2i+3)}{4i^2}\frac{d^{4i+2}\mathrm{U}^{4i+3}}{1.2.3\cdots(4i+2)\,dx^{4i+2}} + \cdots\right]\\
&+ \cdots\\
&+ 2\mathrm{K}^{(i-1)}\mathrm{Y}\left[\frac{d^{2i-2}\mathrm{U}^{2i-1}}{1.2.3\cdots(2i-2)\,dx^{2i-2}} + \frac{2i-1}{2i}\frac{d^{4i-2}\mathrm{U}^{4i-1}}{1.2.3\cdots(4i-2)\,dx^{4i-2}}\right.\\
&\qquad\left. + \frac{(2i-1)(4i-1)}{4i^2}\frac{d^{6i-2}\mathrm{U}^{6i-1}}{1.2.3\cdots(6i-2)\,dx^{6i-2}} + \cdots\right].
\end{aligned}\right.$$

This formula is the sum of a number i of different series, decreasing as the powers of α, since U is of the order $\alpha^{\frac{1}{\alpha^{2i}}}$, and multiplied respectively by the transcendents $\mathrm{K}, \mathrm{K}^{(1)}, \mathrm{K}^{(2)}, \ldots$ which it is consequently important to know. Let us see that which the analysis takes us in this regard.

IV.

We will consider generally the integral

$$\int ds\,dx\,dx^{(1)}\,dx^{(2)}\cdots dx^{(r-2)}\,e^{-s\left(1+x^n+x^{(1)^n}+\cdots+x^{(r-2)^n}\right)},$$

the successive integrals being taken from $s,\ x,\ x^{(1)},\ x^{(2)},\ \ldots,$ equal to zero to the infinite values of these variables. By integrating first with respect to s, we will reduce the preceding integral to this

$$\int \frac{dx\,dx^{(1)}dx^{(2)} + \cdots + dx^{(r-2)}}{1 + x^n + x^{(1)^n} + \cdots + x^{(r-2)^n}}.$$

Let

$$\frac{x}{\left(1 + x^{(1)^n} + x^{(2)^n} + \cdots + x^{(r-2)^n}\right)^{\frac{1}{n}}} = z;$$

we will have

$$\int \frac{dx}{1 + x^n + x^{(1)^n} + \cdots + x^{(r-2)^n}}$$
$$= \frac{1}{\left(1 + x^{(1)^n} + x^{(2)^n} \cdots + x^{(r-2)^n}\right)^{\frac{n-1}{n}}} \int \frac{dz}{1+z^n},$$

the integral relative to z being taken from $z = 0$ to $z = \infty$. Let next

$$\frac{x^{(1)}}{\left(1 + x^{(2)^n} + \cdots + x^{(r-2)^n}\right)^{\frac{1}{n}}} = z^{(1)};$$

we will have

$$\int \frac{dx^{(1)}}{\left(1 + x^{(1)^n} + \cdots + x^{(r-2)^n}\right)^{\frac{n-1}{n}}}$$
$$= \frac{1}{\left(1 + x^{(2)^n} + \cdots + x^{(r-2)^n}\right)^{\frac{n-2}{n}}} \int \frac{dz^{(1)}}{\left(1 + z^{(1)^n}\right)^{\frac{n-1}{n}}},$$

the integral relative to $z^{(1)}$ being taken from $z^{(1)} = 0$ to $z^{(1)} = \infty$. By continuing to operate so, we will find

$$\int ds\, dx\, dx^{(1)} \cdots dx^{(r-2)}\, e^{-s\left(1+x^n+x^{(1)^n}+\cdots+x^{(r-2)^n}\right)}$$
$$= \int \frac{dz}{1+z^n} \int \frac{dz}{(1+z^n)^{\frac{n-1}{n}}} \int \frac{dz}{(1+z^n)^{\frac{n-2}{n}}} \cdots \int \frac{dz}{(1+z^n)^{\frac{n-r+2}{n}}},$$

the integrals relative to z being taken from $z = 0$ to $z = \infty$.

We integrate presently, in another way, the differential

$$ds\, dx\, dx^{(1)} \cdots e^{-s\left(1+x^n+x^{(1)^n}+\cdots\right)},$$

and, instead of beginning the integrations with s, we terminate them with this variable; for this, we will observe that we have

$$\int dx\, e^{-sx^n} = \frac{1}{s^{\frac{1}{n}}} \int s^{\frac{1}{n}}\, dx\, e^{-sx^n} = \frac{1}{s^{\frac{1}{n}}} \int dt\, e^{-t^n},$$

t being supposed equal to $s^{\frac{1}{n}} x$. The integral relative to x must be taken from $x = 0$ to $x = \infty$, the integral relative to t must be taken from $t = 0$ to $t = \infty$. Let therefore $\int dt\, e^{-t^n} = \mathrm{K}$, we will have

$$\int dx\, e^{-sx^n} = \frac{\mathrm{K}}{s^{\frac{1}{n}}};$$

we will have similarly

$$\int dx^{(1)}\, e^{-sx^{(1)^n}} = \frac{\mathrm{K}}{s^{\frac{1}{n}}};$$

and thus in sequence; therefore,

$$\begin{aligned}&\int ds\, dx\, dx^{(1)}\, \cdots dx^{(r-2)}\, e^{-s\left(1+x^n+x^{(1)^n}+\cdots+x^{(r-2)^n}\right)}\\ &\quad = \mathrm{K}^{r-1}\int \frac{ds\, e^{-s}}{s^{\frac{r-1}{n}}} = n\mathrm{K}^{r-1}\int t^{n-r}dt\, e^{-t^n},\end{aligned}$$

t being here equal to $s^{\frac{1}{n}}$, and the integral relative to t being taken, as the integral relative to s, from the null value of this variable of this variable to its infinite value. By comparing the two expressions of

$$\int ds\, dx\, dx^{(1)}\, \cdots\, e^{-s\left(1+x^n+x^{(1)^n}+\cdots\right)}$$

and by observing that

$$\int \frac{dz}{1+z^n} = \frac{\pi}{n\sin\frac{\pi}{n}},$$

π being the ratio of the semi-circumference to the radius, we will have

$$\begin{aligned}&n^2\mathrm{K}^{r-1}\int t^{n-r}dt\, e^{-t^n}\\ &\quad = \frac{\pi}{\sin\frac{\pi}{n}}\int \frac{dz}{(1+z^n)^{\frac{n-1}{n}}}\int \frac{dz}{(1+z^n)^{\frac{n-2}{n}}}\cdots\int \frac{dz}{(1+z^n)^{\frac{n-r+2}{n}}},\end{aligned}$$

all the integrals being taken from the null values of the variables to their infinite values.

If we make $1+z^n = \frac{1}{1-u^n}$, we will have

$$dz = \frac{du}{(1-u^n)^{\frac{n-1}{n}}};$$

the preceding formula will become thus

$$\text{(Z)}\qquad \left\{\begin{aligned}&n^2\mathrm{K}^{r-1}\int t^{n-r}dt\, e^{-t^n}\\ &\quad = \frac{\pi}{\sin\frac{\pi}{n}}\int \frac{du}{(1-u^n)^{\frac{2}{n}}}\int \frac{du}{(1-u^n)^{\frac{3}{n}}}\cdots\int \frac{du}{(1-u^n)^{\frac{r-1}{n}}},\end{aligned}\right.$$

the integrals relative to u being taken from $u=0$ to $u=1$, because the assumption of $z=0$ gives $u=0$ and because that of $z=\infty$ gives $u=1$. It is necessary in this formula to take as many of the factors affected by the integral sign as there are units in $r-2$.

Formula (Z) offers many interesting corollaries which we are going to develop; if we suppose $r=n$ in it, the integral $\int t^{n-r}dt\, e^{-t^n}$ will be changed to K, and we will have

$$\text{(V)}\qquad n^2\mathrm{K}^n = \frac{\pi}{\sin\frac{\pi}{n}}\int \frac{du}{(1-u^n)^{\frac{2}{n}}}\int \frac{du}{(1-u^n)^{\frac{3}{n}}}\cdots\int \frac{du}{(1-u^n)^{\frac{n-1}{n}}}.$$

Thus K or $\int dt\, e^{-t^n}$ will be given by this equation in functions of algebraic integrals, and formula (Z) will give the value of $\int t^{n-r} dt\, e^{-t^n}$ in similar functions, r being any positive whole number whatever and less than n; these values depend on $n-2$ algebraic integrals

$$\int \frac{du}{(1-u^n)^{\frac{2}{n}}} \int \frac{du}{(1-u^n)^{\frac{3}{n}}} \cdots \int \frac{du}{(1-u^n)^{\frac{n-1}{n}}};$$

but we can diminish to half the number of these integrals by the following method.

If, in formula (Z), we make $r = 2$, it will give

$$n^2 \int dt\, e^{-t^n} \int t^{n-2} dt\, e^{-t^n} = \frac{\pi}{\sin\frac{\pi}{n}}.$$

This equation is generally true, whatever be n, supposing it even fractional; therefore, if we change n into $\frac{n}{r-1}$, we will have

$$n^2 \int dt\, e^{-t^{\frac{n}{r-1}}} \int t^{\frac{n}{r-1}-2} dt\, e^{-t^{\frac{n}{r-1}}} = \frac{(r-1)^2\pi}{\sin\frac{(r-1)\pi}{n}}.$$

and, if in this new equation we change t into t^{r-1}, it will become

$$\text{(T)} \qquad n^2 \int t^{r-2} dt\, e^{-t^n} \int t^{n-r} dt\, e^{-t^n} = \frac{\pi}{\sin\frac{(r-1)\pi}{n}}.$$

If, in this equation, we suppose $r-2 = n-r$, that which gives $r = \frac{n}{2}+1$, we will have

$$n^2 \left(\int t^{\frac{n}{2}-1} dt\, e^{-t^n}\right)^2 = \pi,$$

and, if we change $t^{\frac{n}{2}}$ into t, we will have this remarkable result

$$\int dt\, e^{-t^2} = \frac{1}{2}\sqrt{\pi},$$

that is that the integral $\int dt\, e^{-t^2}$, taken from $t = 0$ to t infinity, is the half of the square root of the ratio of the semi-circumference to the radius.

We suppose now n even and equal to $2i$; if we make $r = i+1$ in formula (Z), it will become

$$4i^2\mathrm{K}^i \int t^{i-1} dt\, e^{-t^{2i}} = \frac{\pi}{\sin\frac{\pi}{2i}} \int \frac{du}{(1-u^{2i})^{\frac{2}{2i}}} \int \frac{du}{(1-u^{2i})^{\frac{3}{2i}}} \cdots \int \frac{du}{(1-u^{2i})^{\frac{1}{2}}}.$$

Now, by changing t^i into t, the integral $\int t^{i-1} dt\, e^{-t^{2i}}$ will become

$$\frac{1}{i} \int dt\, e^{-t^{2i}} = \frac{\sqrt{\pi}}{2i};$$

we will have therefore

$$\text{(R)} \qquad 2i\mathrm{K}^i = \frac{\sqrt{\pi}}{\sin\frac{\pi}{2i}} \int \frac{du}{(1-u^{2i})^{\frac{2}{2i}}} \cdots \int \frac{du}{(1-u^{2i})^{\frac{1}{2}}};$$

thus K will be given as function of the first $i-1$ algebraic integrals of formula (Z), and this same formula will give the values of all the transcendent integrals $\int t^{2i-r}dt\, e^{-t^{2i}}$, in functions of these same integrals, when r will be equal or less than $i+1$ or, what returns to the same, when the exponent $2i-r$ will be equal or greater than $i-1$. If this exponent is less, then $r-2$ will be greater than $i-1$, and formula (T) giving the value of the integral $\int t^{2i-r}dt\, e^{-t^{2i}}$, by means of this one here $\int t^{r-2}dt\, e^{-t^{2i}}$, this value will depend only on the first $i-1$ algebraic integrals of formula (Z); thus all the values of the integral $\int t^{2i-r}dt\, e^{-t^{2i}}$ will depend, whatever be r, only on these first i algebraic integrals, and, as the values corresponding to r greater than i are given by formula (Z) in functions of these integrals and of the following

$$\int \frac{du}{(1-u^{2i})^{\frac{i+1}{2i}}}, \quad \int \frac{du}{(1-u^{2i})^{\frac{i+2}{2i}}}, \quad \ldots, \quad \int \frac{du}{(1-u^{2i})^{\frac{2i-1}{2i}}},$$

there results that each of these last integrals will be given as function of the $i-1$ first algebraic integrals of formula (Z).

If n is odd and equal to $2i+1$, formula (Z) will give, by making successively $r=i+1$ and $r=i+2$,

$$\begin{aligned}
&(2i+1)^2\mathrm{K}^i \int t^i dt\, e^{-t^{2i+1}} \\
&\quad = \frac{\pi}{\sin\frac{\pi}{2i+1}} \int \frac{du}{(1-u^{2i+1})^{\frac{2}{2i+1}}} \int \frac{du}{(1-u^{2i+1})^{\frac{3}{2i+1}}} \cdots \int \frac{du}{(1-u^{2i+1})^{\frac{i}{2i+1}}}, \\
&(2i+1)^2\mathrm{K}^{i+1} \int t^{i-1} dt\, e^{-t^{2i+1}} \\
&\quad = \frac{\pi}{\sin\frac{\pi}{2i+1}} \int \frac{du}{(1-u^{2i+1})^{\frac{2}{2i+1}}} \cdots \int \frac{du}{(1-u^{2i+1})^{\frac{i+1}{2i+1}}};
\end{aligned}$$

by multiplying these two equations by one another and by observing that equation (T) gives, by making $r=i+1$,

$$(2i+1)^2 \int t^{i-1}dt\, e^{-t^{2i+1}} \int t^i dt\, e^{-t^{2i+1}} = \frac{\pi}{\sin\frac{i\pi}{2i+1}},$$

we will have

$$\begin{aligned}
&(2i+1)^2\mathrm{K}^{2i+1} \\
&\quad \frac{\pi \sin\frac{i\pi}{2i+1}}{\left(\sin\frac{\pi}{2i+1}\right)^2} \left[\int \frac{du}{(1-u^{2i+1})^{\frac{2}{2i+1}}} \cdots \int \frac{du}{(1-u^{2i+1})^{\frac{i}{2i+1}}}\right]^2 \int \frac{du}{(1-u^{2i+1})^{\frac{i+1}{2i+1}}}.
\end{aligned}$$

K will therefore be given as a function of the i first algebraic integrals of formula (Z), and this same formula will give the values of $\int t^{2i+1-r}dt\, e^{-t^{2i+1}}$, as a function of the same integrals, when r will be equal or less than $i+2$; formula (T) will give next the value of this transcendent integral when r will be greater than $i+2$, whence we can conclude that each of the integrals

$$\int \frac{du}{(1-u^{2i+1})^{\frac{i+2}{2i+1}}}, \quad \int \frac{du}{(1-u^{2i+1})^{\frac{i+3}{2i+1}}}, \quad \cdots \int \frac{du}{(1-u^{2i+1})^{\frac{2i}{2i+1}}}$$

will be given as a function of the first i algebraic integrals of formula (Z).

Thence it follows generally that all the values of $\int t^r dt\, e^{-t^n}$ will depend, whatever be r, only on $\frac{n}{2}-1$ algebraic integrals taken in formula (Z) if n is even, or on $\frac{n-1}{2}$ of these same integrals if n is odd.

V.

We take now formula (C) of No. III; if we make $i=1$, it will contain the sole transcendent K or $\int dt\, e^{-t^2}$, which, by the preceding section, is equal to $\frac{1}{2}\sqrt{\pi}$ or to 0.886227.

If we make $i=2$, this formula will contain the two transcendents K and $\mathrm{K}^{(1)}$, which are respectively equal to $\int dt\, e^{-t^2}$ and to $\int t^2 dt\, e^{-t^4}$; now formula (R) of the preceding section gives, by making $i=2$ and by observing that then $\sin\frac{\pi}{2i}=\frac{1}{\sqrt{2}}$,

$$4\left(\int dt\, e^{-t^4}\right)^2 = \sqrt{2\pi}\int \frac{du}{(1-u^4)^{\frac{1}{2}}}.$$

This last integral represents the length of the elastic curve which M. Stirling has found equal to

$$1.31102877714605987;$$

by designating therefore π' this value, we will have

$$\mathrm{K} = \int dt\, e^{-t^4} = \frac{1}{2}\sqrt{\pi'\sqrt{2\pi}};$$

formula (Z) will give next, by making $n=4$ and $r=2$,

$$16\int dt\, e^{-t^4}\int t^2 dt\, e^{-t^4} = \pi\sqrt{2},$$

therefore

$$\mathrm{K}^{(1)} = \int t^2 dt\, e^{-t^4} = \frac{\pi^{\frac{3}{4}}}{4\sqrt{2\pi'\sqrt{2\pi}}}.$$

We will not push further this examination of the values of $\mathrm{K}, \mathrm{K}^{(1)}, \ldots$ corresponding to the different values of i, because the case where i surpasses unity are very rare in the applications of Analysis.

VI.

The case in which $i=1$ being most ordinary, we are going to exhibit here the simplest formulas in order to determine in this case the approximate value of the integral $\int y\, dx$.

If we suppose $\nu = -\frac{y\,dx}{dy}$ and if we name Y and U that which y and ν become when x changes to θ, and Y′ and U′ that which these same quantities become when we change x to θ', we will have

$$(a)\qquad \begin{cases} \displaystyle\int y\,dx = \mathrm{YU}\left\{1 + \frac{d\mathrm{U}}{d\theta} + \frac{d(\mathrm{U}\,d\mathrm{U})}{d\theta^2} + \frac{d[\mathrm{U}\,d(\mathrm{U}\,d\mathrm{U})]}{d\theta^3} + \cdots\right\} \\ \displaystyle\qquad\qquad - \mathrm{Y'U'}\left\{1 + \frac{d\mathrm{U'}}{d\theta'} + \frac{d(\mathrm{U'}\,d\mathrm{U'})}{d\theta'^2} + \frac{d[\mathrm{U'}\,d(\mathrm{U'}\,d\mathrm{U'})]}{d\theta'^3} + \cdots\right\}, \end{cases}$$

the integral $\int y\,dx$ being taken from $x = \theta$ to $x = \theta'$. This formula will be very convergent all the time that $\frac{dy}{dx}$ will be very great with respect to y, that which takes place when, the factors of y being raised to some great powers, the integral $\int y\,dx$ is taken in the intervals distant from the maximum of y.

In order to have this same integral in the intervals near this maximum, we suppose that it corresponds to $x = a$, and name Y the maximum of y or that which it becomes when we change x into a; we suppose next, as this occurs most often, that the value a of x makes only the first difference of y vanish: in this case, we will make

$$t = \sqrt{\log \mathrm{Y} - \log y}, \quad \nu = \frac{x-a}{\sqrt{\log \mathrm{Y} - \log y}},$$

and, by designating by U, $\frac{d\mathrm{U}^2}{dx}$, $\frac{d^2\mathrm{U}^3}{dx^2}, \ldots$ that which ν, $\frac{du^2}{dx}$, $\frac{d^2u^3}{dx^2}$, … become when we change x into a, we will have

$$(b)\qquad \int y\,dx = \mathrm{Y}\int dt\,e^{-t^2}\left(\mathrm{U} + \frac{d\mathrm{U}^2}{dx}t + \frac{d^2\mathrm{U}^3}{1.2\,dx^2}t^2 + \frac{d^3\mathrm{U}^4}{1.2.3\,dx^3}t^3 + \cdots\right).$$

If in formula (a) we suppose $\log y$, and consequently $-\frac{y\,dx}{dy}$ very small of order α, this formula cannot serve in any interval where $(x-a)^2$ is less than α; in this case, we can make use of formula (b), which itself ceases to be convergent when νt or, that which reverts to the same, $x - a$ is not a very small quantity of order α^{λ}, λ being positive; but, in the interval where it is not, the series (a) can be employed, so that these two series serve to supplement each other; there are likewise some intervals where both can be used, because, since the convergence of series (a) requires that $x-a$ be of order $\alpha^{\frac{1}{2}-\lambda}$, λ being positive, and since that of the series (b) requires that $\frac{1}{2} - \lambda$ is positive, these two series can serve all at once for all the positive values of λ less than $\frac{1}{2}$. The first will be ordered with respect to the powers of $\alpha^{2\lambda}$, and the second will be with respect to the powers of $\alpha^{\frac{1}{2}-\lambda}$; it will be necessary to prefer the first or the second, according as 2λ will be greater or lesser than $\frac{1}{2} - \lambda$, that is according as we will have λ greater or lesser than $\frac{1}{2}$.

Formula (b) gives, by integrating from $t =$T to $t =$T′,

$$(c)\qquad \begin{cases} \displaystyle\int y\,dx = \mathrm{Y}\left(\mathrm{U} + \frac{1}{2}\frac{d^2\mathrm{U}^3}{1.2\,dx^2} + \frac{1.3}{2^2}\frac{d^4\mathrm{U}^5}{1.2.3.4\,dx^4} + \cdots\right)\int dt\,e^{-t^2} \\ \displaystyle\qquad + \frac{\mathrm{Y}}{2}e^{-\mathrm{T}^2}\left[\frac{d\mathrm{U}^2}{dx} + \mathrm{T}\,\frac{d^2\mathrm{U}^3}{1.2\,dx^2}t^2 + (\mathrm{T}^2+1)\frac{d^3\mathrm{U}^4}{1.2.3\,dx^3}t^3 + \cdots\right] \\ \displaystyle\qquad + \frac{\mathrm{Y}}{2}e^{-\mathrm{T'}^2}\left[\frac{d\mathrm{U}^2}{dx} + \mathrm{T'}\,\frac{d^2\mathrm{U}^3}{1.2\,dx^2}t^2 + (\mathrm{T'}^2+1)\frac{d^3\mathrm{U}^4}{1.2.3\,dx^3}t^3 + \cdots\right], \end{cases}$$

the integral $\int y\,dx$ being taken from the value of x which agrees with t =T to that which agrees with t =T′.

If we suppose T= $-\infty$ and T′ $=\infty$, we will have generally

$$\mathrm{T}^r e^{-\mathrm{T}^2}=0,\quad \mathrm{T}'^r e^{-\mathrm{T}'^2}=0;$$

we have besides in this case (no. IV)

$$\int dt\, e^{-t^2}=\sqrt{\pi}.$$

The preceding formula becomes thus

$$(d)\qquad \int y\,dx=\mathrm{Y}\sqrt{\pi}\left(\mathrm{U}+\frac{1}{2}\,\frac{d^2\mathrm{U}^3}{1.2\,dx^2}+\frac{1.3}{2^2}\,\frac{d^4\mathrm{U}^5}{1.2.3.4\,dx^4}+\cdots\right),$$

the integral $\int y\,dx$ being taken between the two consecutive values of x which render y null, and Y being the maximum of y contained between these values. The different terms of this formula will be determined easily by observing that, if we make

$$\mathrm{A}=-\frac{d^2\log y}{1.2\,dx^2},\quad \mathrm{B}=-\frac{d^3\log y}{1.2.3\,dx^3},\quad \mathrm{C}=-\frac{d^4\log y}{1.2.3.4\,dx^4},\quad ,\cdots,$$

x being changed into a, after the differentiations, we will have generally

$$\begin{aligned}\nu^r&=\mathrm{A}^{-\frac{r}{2}}-\frac{r}{2}\mathrm{A}^{-\frac{r}{2}-1}\mathrm{B}(x-a)\\&\qquad+\left[\frac{r(r+2)}{8}\mathrm{A}^{-\frac{r}{2}-2}\mathrm{B}^2-\frac{r}{2}\mathrm{A}^{-\frac{r}{2}-1}\mathrm{C}\right](x-a)^2+\cdots\end{aligned}$$

We have

$$d^2\log y=\frac{d^2y}{y}-\frac{dy^2}{y^2};$$

the assumption of $x=a$ makes dy disappear: we will have therefore

$$\frac{d^2\log y}{dx^2}=-2\mathrm{A}=\frac{d^2\mathrm{Y}}{\mathrm{Y}dx^2},$$

Y and $\frac{d^2\mathrm{Y}}{dx^2}$ being that which y and $\frac{d^2y}{dx^2}$ become when we make $x=a$; therefore, if in formula (d) we consider only the first term of the series, we will have very nearly

$$\int y\,dx=\frac{\mathrm{Y}^{\frac{3}{2}}\sqrt{2\pi}}{\sqrt{-\frac{d^2\mathrm{Y}}{dx^2}}}\quad\text{or}\quad\left(\int y\,dx\right)^2=\frac{2\pi\mathrm{Y}^3}{-\frac{d^2\mathrm{Y}}{dx^2}},$$

the integral $\int y\,dx$ being taken between the two consecutive values of x which make y disappear, Y and $\frac{d^2\mathrm{Y}}{dx^2}$ corresponding to the intermediate value of x which makes dy disappear. This expression of $\int y\,dx$ will be as much nearer as the factors of y will be raised to higher powers.

Formula (c) contains the indefinite integral $\int dt\, e^{-t^2}$, which it is not possible to obtain in finite terms; but we can, in all cases, determine it in a manner quite near by known methods. If t is not very large, we can make use of the following series

$$\int dt\, e^{-t^2} = \mathrm{T} - \frac{1}{3}\mathrm{T}^3 + \frac{1}{1.2}\frac{\mathrm{T}^5}{5} - \frac{1}{1.2.3}\frac{\mathrm{T}^7}{7} + \frac{1}{1.2.3.4}\frac{\mathrm{T}^9}{9} - \cdots,$$

the integral being taken from $t = 0$ to $t =$T.

If t is large, we can be served by this series

$$\int dt\, e^{-t^2} = \frac{e^{-\mathrm{T}^2}}{2\mathrm{T}}\left(1 - \frac{1}{2\mathrm{T}^2} + \frac{1.3}{2^2\mathrm{T}^4} - \frac{1.3.5}{2^3\mathrm{T}^6} + \cdots\right),$$

the integral $\int dt\, e^{-t^2}$ being taken from $t =$T to $t = \infty$, so that, in order to have the value of this integral from $t = 0$ to $t =$T, it is necessary to subtract the preceding value from $\frac{1}{2}\sqrt{\pi}$. This series is alternatively greater and lesser than the $\int dt\, e^{-t^2}$, in a manner that the value of this integral, taken from $t =$T to $t = \infty$, is always contained between the sum of any finite number of its terms and this same sum increased by the following term. This kind of series, which we can name *series of limits*, has the advantage of showing with precision the limits of the errors of the approximations. In a great number of cases, formulas (a), (b), (c) and (d) lead to some series of this nature.

VII.

We can easily extend the preceding analysis to double, triple, ...integrals; for that, we will consider the double integral $\int y\, dx\, dx'$, y being a function of x and of x' which contains some factors raised to some great powers. We suppose that the integral relative to x must be taken from a function X of x to another function X$'$ of the same variable; by making $x' =$X$+u$X$'$, the integral $\int y\ dx\ dx'$ will be changed into $\int y\, \mathrm{X}' dx\, du$, the integral relative to u needing to be taken from $u = 0$ to $u = 1$. We can thus therefore reduce the integral $\int y\ dx\ dx'$ to some limits constant and independent of the variables which it contains; we will suppose consequently that it has this form and that the integral relative to x is taken from $x = \theta$ to $x = \varpi$, while the integral relative to x' is taken from $x' = \theta'$ to $x' = \varpi'$. This put, by naming Y that which y becomes when we change x and x' into θ and θ', we will make

$$y = \mathrm{Y}e^{-t-t'};$$

by supposing next $x = \theta + u$ and $x' = \theta' + u'$, we will reduce the function $\log \frac{\mathrm{Y}}{y}$ to a series ordered with respect to the powers of u and of u', and we will have an equation of this form

$$\mathrm{M}u + \mathrm{M}'u' = t + t',$$

in which M is the part of the expansion of $\log \frac{\mathrm{Y}}{y}$ which contains all the terms multiplied by u, and M$'$ is the other part which contains the terms multiplied by u' and which are independent of u. We will divide the preceding equation into the two following

$$\mathrm{M}u = t, \quad \mathrm{M}'u' = t',$$

whence we will deduce this, by the reversion of the series,

$$u = \mathrm{N}t, \quad u' = \mathrm{N}'t',$$

N being a series ordered with respect to the powers of t and of t', and N$'$ being uniquely ordered with respect to the powers of t' and independently of t. These two series will be very convergent if y contains some very elevated factors. Now we have

$$dx\,dx' = du\,du',$$

and it is easy to be assured that this last product is equal to $\frac{\partial u}{\partial t}\frac{\partial u'}{\partial t'}dt\,dt'$, that is $\frac{\partial(\mathrm{N}t)}{\partial t}\frac{\partial(\mathrm{N}'t')}{\partial t'}dt\,dt'$, therefore

$$\int y\;dx\;dx' = \mathrm{Y}\int \frac{\partial(\mathrm{N}t)}{\partial t}\frac{\partial(\mathrm{N}'t')}{\partial t'}dt\,dt'e^{-t-t'}.$$

It will be easy to integrate the different terms of the second member of this equation, since the question is only of integrating the terms of this form $\int t^n dt\;e^{-t}$ or $\int t'^n dt'\;e^{-t'}$.

If we take the integral relative to t' from $t' = 0$ to $t' = \infty$, and if we name Q the result of the integration, we will have

$$\int y\;dx' = \mathrm{YQ},$$

the integral being taken from $x' = \theta'$ to the value of x' which agrees with t' infinity; if we change next, in Y and Q, θ' into ϖ', and if we name Y$'$ and Q$'$ that which these quantities then become, we will have

$$\int y\;dx' = \mathrm{Y}'\mathrm{Q}',$$

the integral being taken from $x' = \varpi'$ to the value of x' which agrees with t' infinite; we will have therefore

$$\int y\;dx = \mathrm{YQ} - \mathrm{Y}'\mathrm{Q}',$$

the integral relative to x' being taken from $x' = \theta'$ to $x' = \varpi'$.

By naming R and R$'$ the integrals $\int \mathrm{Q}dt$ and $\int \mathrm{Q}'dt$, taken from $t = 0$ to $t = \infty$, we will have

$$\int y\;dx\,dx' = \mathrm{YR} - \mathrm{Y}'\mathrm{R}',$$

the integral relative to x' being taken from $x' = \theta'$ to $x' = \varpi'$, and the integral relative to x being taken from $x = \theta$ to the value of x which agrees with t infinity. If in $\mathrm{Y}, \mathrm{R}, \mathrm{Y}', \mathrm{R}'$ we change θ into ϖ, and if we name $\mathrm{Y}_1, \mathrm{R}_1, \mathrm{Y}_1', \mathrm{R}_1'$ that which these quantities then become, we will have

$$\int y\;dx\,dx' = \mathrm{Y}_1\mathrm{R}_1 - \mathrm{Y}_1'\mathrm{R}_1',$$

the integral relative to x' being taken between the limits θ' and ϖ', and the integral relative to x being taken from $x = \varpi$ to the value x which agrees with $t = \infty$; therefore

$$\int y\,dx\,dx' = \mathrm{YR} - \mathrm{Y'R'} - \mathrm{Y_1R_1} + \mathrm{Y'_1R'_1},$$

the integral relative to x being taken between the limits θ and ϖ, and the integral relative to x' being taken between the limits θ' and ϖ'. This formula corresponds to formula (A) of No. I, which is relative only to a single variable. It has the same inconvenience, that of not being able to be extended it to the intervals near the maximum of y; it is necessary for these intervals to use a method analogous to that of No. II. Thus, by supposing that, in the interval contained between θ and ϖ, y becomes a maximum and that the condition of the maximum makes only the first difference of y vanish, as previously, $y =$Y$e^{-t-t'}$, we will make

$$y = \mathrm{Y}e^{-t^2-t'};$$

and if, in the interval contained between θ' and ϖ', y becomes a maximum, we will make

$$y = \mathrm{Y}e^{-t^2-t'^2}.$$

As we will have principally need in the following of the integral $\int y\ dx\,dx'$, taken between the limits of x and x' which renders y null, we are going to discuss this case in a general manner.

We will consider the integral $\int y\ dx\,dx'\ dx'' \ldots$, y being a function of r variables x, x', x'', ... which contain some factors raised to some great powers. If we name a, a', a'', ... the values of x, x', x'', ... which correspond to the maximum of y, and if we designate by Y this maximum, we will make

$$y = \mathrm{Y}e^{-t^2-t'^2-t''^2-\cdots};$$

by supposing next

$$x = a + \theta, \quad x' = a' + \theta', \quad x'' = a'' + \theta'', \quad \ldots,$$

we will substitute these values into the function $\log \frac{\mathrm{Y}}{y}$, and, by expanding it into a series ordered with respect to the powers of θ, θ', θ'', ... we will have an equation of this form

$$\mathrm{M}\theta^2 + \mathrm{M}'\theta'^2 + \mathrm{M}''\theta''^2 + \cdots = t^2 + t'^2 + t''^2 + \cdots ,$$

M being the part of the expansion of $\log \frac{\mathrm{Y}}{y}$ multiplied by θ^2; M$'$ being the part of this expansion multiplied by θ'^2 and independent of θ; M$''$ being the part multiplied by θ''^2 and independent of θ and of θ', and thus the rest. We will divide this equation into the following

$$\mathrm{M}\theta^2 = t^2, \quad \mathrm{M}'\theta'^2 = t'^2, \quad \mathrm{M}''\theta''^2 = t''^2, \quad \cdots ,$$

whence we will deduce that, by the reversion of the series,

$$\theta = \mathrm{N}t, \quad \theta' = \mathrm{N}'t', \quad \theta'' = \mathrm{N}''t'', \quad \ldots,$$

N being a series ordered with respect to the powers of $t,\ t', t'', \ldots$;N$'$ being a series ordered with respect to the powers of $t', t'', \ldots$; N$''$ being a series ordered with respect to the powers of $t'', \ldots$. These series will be so much more convergent as the factors of y will be raised to higher powers.

Now we have

$$dx\,dx'\,dx''\ldots = d\theta\ d\theta'\,d\theta''\ldots,$$

and it is easy to be assured that this last product is equal to

$$\frac{d(\mathrm{N}t)}{dt}\frac{d(\mathrm{N}'t')}{dt'}\frac{d(\mathrm{N}''t'')}{dt''}\cdots dt\,dt'\,dt''\ldots;$$

therefore

$$\int y\ dx\,dx'\ dx''\ldots = \mathrm{Y}\int\frac{d(\mathrm{N}t)}{dt}\frac{d(\mathrm{N}'t')}{dt'}\frac{d(\mathrm{N}''t'')}{dt''}\cdots dt\,dt'\,dt''\ldots e^{-t^2-t'^2-t''^2-\cdots},$$

the integrals relative to $dt\,dt'\,dt''\ldots$being taken from these variable equal to $-\infty$ to these variables equal to $+\infty$. It will be easy to have the integrals of the different terms of the second member of this equation by observing that we have generally

$$\int t^n t'^{n'} t''^{n''}\cdots dt\,dt'\,dt''\ldots e^{-t^2-t'^2-t''^2-\cdots} = 0,$$

when any one of the numbers $n,\ n',\ n'',\ldots$ is odd, and

$$\int t^{2i} t'^{2i'} t''^{2i^{n''}}\cdots dt\,dt'\,dt''\ldots e^{-t^2-t'^2-t''^2-\cdots}$$
$$= \frac{1.3.5\ldots(2i-1).1.3.5\ldots(2i'-1).1.3.5\ldots(2i''-1)\ldots\pi^{\frac{r}{2}}}{2^{i+i'+i''+\cdots}}$$

If the powers to which the factors of y are raised are very large, we will have very nearly

$$\mathrm{M} = -\frac{\frac{\partial^2\mathrm{Y}}{\partial x^2}}{\mathrm{Y}},\quad \mathrm{M}' = -\frac{\frac{\partial^2\mathrm{Y}}{\partial x'^2}}{\mathrm{Y}},\quad \mathrm{M}'' = -\frac{\frac{\partial^2\mathrm{Y}}{\partial x''^2}}{\mathrm{Y}},\quad \ldots,$$

$\frac{\partial^2\mathrm{Y}}{\partial x^2},\ \frac{\partial^2\mathrm{Y}}{\partial x'^2},\ \frac{\partial^2\mathrm{Y}}{\partial x''^2},\ldots$ being that which $\frac{\partial^2 y}{\partial x^2},\ \frac{\partial^2 y}{\partial x'^2},\ \frac{\partial^2 y}{\partial x''^2},\ldots$ become when we change $x,\ x',\ x'',\ \ldots$ into $a,\ a',\ a'',\ \ldots$. We will have thus very nearly

$$\theta = \frac{\mathrm{Y}^{\frac{1}{2}}}{\sqrt{-\frac{\partial^2\mathrm{Y}}{\partial x^2}}},\quad \theta' = \frac{\mathrm{Y}^{\frac{1}{2}}}{\sqrt{-\frac{\partial^2\mathrm{Y}}{\partial x'^2}}},\quad \theta'' = \frac{\mathrm{Y}^{\frac{1}{2}}}{\sqrt{-\frac{\partial^2\mathrm{Y}}{\partial x''^2}}},\quad \ldots,$$

whence we deduce this general theorem:

The integral $\int y\ dx\ dx'\ dx''\ldots$, taken between the consecutive values of $x,\ x',\ x'',\ \ldots$ which render y null, is very nearly equal to

$$\frac{(-2\pi)^{\frac{r}{2}}Y^{\frac{r+2}{2}}}{\sqrt{\frac{\partial^2 Y}{\partial x^2}\frac{\partial^2 Y}{\partial x'^2}\frac{\partial^2 Y}{\partial x''^2}\cdots}}$$

if the factors of y are raised to great powers.

ARTICLE II.

On integration by approximation with linear equations to the finite and infinitesimal differences.

We will consider the equation linear in finite differences

$$S = \mathrm{A}y_s + \mathrm{B}\triangle y_s + \mathrm{C}\triangle^2 y_s + \cdots, \tag{1}$$

S being a function of s; $\mathrm{A}, \mathrm{B}, \mathrm{C}$ being some rational and entire functions of the same variable, and the characteristic $\triangle$ being that of the finite differences, so that $\triangle y_s = y_{s+1} - y_s$. Let

$$\begin{aligned}
\mathrm{A} &= a + a^{(1)}s + a^{(2)}s^2 + a^{(3)}s^3 + \cdots,\\
\mathrm{B} &= b + b^{(1)}s + b^{(2)}s^2 + b^{(3)}s^3 + \cdots,\\
\mathrm{C} &= c + c^{(1)}s + c^{(2)}s^2 + c^{(3)}s^3 + \cdots,\\
&\cdots,
\end{aligned}$$

and we represent the value of y_s by the integral $\int e^{-sx}\phi\,dx$, ϕ being a function of x, independent of s, and the integral being taken between some limits independently of this variable; we will have

$$\begin{aligned}
\triangle y_s &= \int e^{-sx}(e^{-x}-1)\phi\,dx,\\
\triangle^2 y_s &= \int e^{-sx}(e^{-x}-1)^2\phi\,dx,\\
&\cdots
\end{aligned}$$

Moreover, if we designate e^{-sx} by δy, we will have

$$se^{-sx} = -\frac{d\,\delta y}{dx}, \qquad s^2e^{-sx} = -\frac{d^2\,\delta y}{dx^2}, \qquad s^3e^{-sx} = -\frac{d^3\,\delta y}{dx^3}, \qquad \ldots;$$

equation (1) will become thus

$$\begin{aligned}
\mathrm{S} = \int \phi\,dx\{\delta y[a + b(e^{-x}-1) + c(e^{-x}-1)^2 + \cdots]&\\
-\frac{d\,\delta y}{dx}[a^{(1)} + b^{(1)}(e^{-x}-1) + c^{(1)}(e^{-x}-1)^2 + \cdots]&\\
+\frac{d^2\,\delta y}{dx^2}[a^{(2)} + b^{(2)}(e^{-x}-1) + c^{(2)}(e^{-x}-1)^2 + \cdots]&\\
+\cdots\}.&
\end{aligned}$$

If we represent y_s by the integral $\int x^s\phi\,dx$, we will have, by designating x^s by δy,

$$sx^s = x\frac{d\,\delta y}{dx}, \qquad s(s-1)x^s = x^2\frac{d^2\,\delta y}{dx^2}, \qquad \ldots;$$

we would have next

$$\triangle y_s = \int \delta y\,(x-1)\,\phi\,dx, \qquad \triangle^2 y_s = \int \delta y\,(x-1)^2\,\phi\,dx, \qquad \ldots$$

Hence, if in this case we put the values of A, B, C, ... under this form

$$\begin{aligned}
&\mathrm{A} = a + a^{(1)}s + a^{(2)}s(s-1) + a^{(3)}s(s-1)(s-2) + \cdots,\\
&\mathrm{B} = b + b^{(1)}s + b^{(2)}s(s-1) + b^{(3)}s(s-1)(s-2) + \cdots,\\
&\mathrm{C} = c + c^{(1)}s + c^{(2)}s(s-1) + c^{(3)}s(s-1)(s-2) + \cdots,\\
&\quad\cdots,
\end{aligned}$$

equation (1) will become

$$\begin{aligned}
\mathrm{S} = \int \phi\, dx \{ & \delta y [a + b(x-1) + c(x-1)^2 + \cdots] \\
& + x\frac{d\,\delta y}{dx}[a^{(1)} + b^{(1)}(x-1) + c^{(1)}(x-1)^2 + \cdots] \\
& + x^2\frac{d^2\,\delta y}{dx^2}[a^{(2)} + b^{(2)}(x-1) + c^{(2)}(x-1)^2 + \cdots] \\
& + \cdots\}.
\end{aligned}$$

By representing generally y_s by $\int \delta y\ \phi\, dx$, the two preceding forms that equation (1) takes under the suppositions of $\delta y = e^{-sx}$ and of $\delta y = x^s$ will be contained in the following

$$\mathrm{S} = \int \phi\, dx \left(\mathrm{M}\,\delta y + \mathrm{N}\frac{d\,\delta y}{dx} + \mathrm{P}\frac{d^2\,\delta y}{dx^2} + \mathrm{Q}\frac{d^3\,\delta y}{dx^3} + \cdots \right),$$

M, N, P, Q, ... being functions of x independent of the variable s, which enters into the second member of this equation only as far as δy and its differences are functions of it.

Now, in order to satisfy it, we will integrate by parts its different terms; now we have

$$\begin{aligned}
\int \mathrm{N}\phi\, dx \frac{d\,\delta y}{dx} &= \delta y \mathrm{N}\phi - \int \delta y\ \frac{d(\mathrm{N}\phi)}{dx} dx,\\
\int \mathrm{P}\phi\, dx \frac{d^2\,\delta y}{dx^2} &= \frac{d\,\delta y}{dx}\mathrm{P}\phi - \delta y \frac{d(\mathrm{P}\phi)}{dx} - \int \delta y\ \frac{d^2(\mathrm{P}\phi)}{dx^2} dx,\\
&\cdots
\end{aligned}$$

The preceding equation becomes thus

$$\begin{aligned}
\mathrm{S} = \int \delta y\, dx & \left[\mathrm{M}\phi - \frac{d(\mathrm{N}\phi)}{dx} + \frac{d^2(\mathrm{P}\phi)}{dx^2} + \frac{d^3(\mathrm{Q}\phi)}{dx^3} + \cdots \right] \\
& + \mathrm{C} + \delta y \left[\mathrm{N}\phi - \frac{d(\mathrm{P}\phi)}{dx} + \frac{d^2(\mathrm{Q}\phi)}{dx^2} + \cdots \right] \\
& + \frac{d\,\delta y}{dx} \left[\mathrm{P}\phi - \frac{d(\mathrm{Q}\phi)}{dx} + \cdots \right] \\
& + \frac{d^2\,\delta y}{dx^2} [\mathrm{Q}\phi - \cdots] \\
& + \cdots
\end{aligned}$$

C being an arbitrary constant.

Since the function ϕ must be independent of s and, consequently, of δy, we must separately equate to zero the part of this equation affected by the $\int$ sign, that which produces the following two equations:

$$(2) \qquad 0 = \mathrm{M}\phi - \frac{d(\mathrm{N}\phi)}{dx} + \frac{d^2(\mathrm{P}\phi)}{dx^2} + \frac{d^3(\mathrm{Q}\phi)}{dx^3} + \cdots ,$$

$$(3) \qquad \begin{cases} \mathrm{S} = \mathrm{C} + \delta y \left[\mathrm{N}\phi - \dfrac{d(\mathrm{P}\phi)}{dx} + \dfrac{d^2(\mathrm{Q}\phi)}{dx^2} + \cdots\right] \\ \qquad + \dfrac{d\,\delta y}{dx}\left[\mathrm{P}\phi - \dfrac{d(\mathrm{Q}\phi)}{dx} + \cdots\right] \\ \qquad + \dfrac{d^2\,\delta y}{dx^2}[\mathrm{Q}\phi - \cdots] \\ \qquad + \cdots \end{cases}$$

The first equation serves to determine the function ϕ, and the second determines the limits in which the integral $\int \delta y \phi\, dx$ must be contained.

We can observe here that equation (2) is the equation of condition which must hold in order that the differentiable function

$$\phi\, dx \left(\mathrm{M}\,\delta y + \mathrm{N}\frac{d\,\delta y}{dx} + \mathrm{P}\frac{d^2\,\delta y}{dx^2} + \cdots\right)$$

is an exact difference, whatever be δy, and, in this case, the integral of this function is equal to the second member of equation (3); ϕ is therefore the factor in x alone which must multiply the equation

$$0 = \mathrm{M}\,\delta y + \mathrm{N}\frac{d\,\delta y}{dx} + \mathrm{P}\frac{d^2\,\delta y}{dx^2} + \cdots$$

in order to render it integrable. If ϕ were known, we could lower this equation by a degree, and reciprocally; if this equation were lowered by a degree, the coefficient of δy, in its differential, divided by Mdx, would give a value of ϕ; this equation and equation (2) are consequently linked between them, in a manner that the integral of one of the two gives the integral of the other.

IX.

We will consider particularly equation (3), and we make first S$= 0$; if we suppose that δy, $\frac{d\,\delta y}{dx}$, $\frac{d^2\,\delta y}{dx^2}$,... become null, by means of one same value of x, which we will designate by h, and which is independent of s, it is clear that by supposing C$= 0$ this value will satisfy equation (3), and that thus it will be one of the limits between which we must take the integral $\int \delta y \phi\, dx$. The preceding supposition holds clearly in the two cases of $\delta y = x^s$ and of $\delta y = e^{-sx}$; because, in the first case, the equation $x = 0$, and, in the second case, the equation $x = \infty$, render null the quantities δy,

$\frac{d\,\delta y}{dx}$, $\frac{d^2\,\delta y}{dx^2}$,.... In order to have some other limits of the integral $\int \delta y \phi\, dx$, we will observe that, these limits needing to be independent of s, by the preceding section, it is necessary, in equation (3), to equate separately to zero the coefficients of δy, $\frac{d\,\delta y}{dx}$, $\frac{d^2\,\delta y}{dx^2}$,..., that which gives the following equations:

$$
\begin{aligned}
&0 = \mathrm{N}\phi - \frac{d(\mathrm{P}\phi)}{dx} + \frac{d^2(\mathrm{Q}\phi)}{dx^2} - \cdots \\
&0 = \mathrm{P}\phi - \frac{d(\mathrm{Q}\phi)}{dx} + \cdots \\
&0 = \mathrm{Q}\phi - \cdots \\
&\ldots
\end{aligned}
$$

These equations will be in number i if i is the order of the differential equation (2); we can therefore eliminate, by their means, all the arbitrary constants of the value of ϕ, less one, and we will have a final equation in x, of which the roots will be as many as limits of the integral $\int \delta y \phi\, dx$; we will seek, by means of this equation, a number of different values of x, equal to the degree of the differential equation (1). Let q, $q^{(1)}$, $q^{(2)}$, ... be these values, they will give as many different values of ϕ, since the arbitrary constants of ϕ, less one, are determined by functions of these values. We can thus represent the values of ϕ, corresponding to the limits q, $q^{(1)}$, $q^{(2)}$, ..., by $\mathrm{B}\lambda, \mathrm{B}^{(1)}\lambda^{(1)}, \mathrm{B}^{(2)}\lambda^{(2)}, \ldots$; B,$\mathrm{B}^{(1)}$, $\mathrm{B}^{(2)}$,... being some arbitrary constants, and we will have for the complete value of y_s

$$y_s = \mathrm{B}\int \delta y \lambda\, dx + \mathrm{B}^{(1)}\int \delta y \lambda^{(1)} dx + \mathrm{B}^{(2)}\int \delta y \lambda^{(2)}\, dx + \cdots,$$

the integral of the first term being taken from $x = h$ to $x = q$, that of the second term being taken from $x = h$ to $x = q^{(1)}$, that of the third term being taken from $x = h$ to $x = q^{(2)}$,..., and thus the rest. We will determine the arbitrary constants B,$\mathrm{B}^{(1)}$, $\mathrm{B}^{(2)}$,... by means of as many particular values of y_s.

X.

We suppose now that in equation (3) S is not null; if we take the integral $\int \delta y \phi\, dx$ from $x = h$ to x equal to any quantity p, it is clear that we will have C$=0$ and that S will be that which the function

$$\delta y \left[\mathrm{N}\phi - \frac{d(\mathrm{P}\phi)}{dx} + \cdots\right] + \frac{d\,\delta y}{dx}(\mathrm{P}\phi - \cdots) + \cdots$$

becomes when we change x into p; thus, for the success of the preceding method, it is necessary that S be the form of this function. We suppose, for example, $\delta y = x^s$, and

$$\mathrm{S} = p^s[l + l^{(1)}s + l^{(2)}s(s-1) + l^{(3)}s(s-1)(s-2) + \cdots];$$

by comparing this value of S to the preceding, we will have

$$
\begin{aligned}
l &= \mathrm{N}\phi - \frac{d(\mathrm{P}\phi)}{dx} + \cdots, \\
l^{(1)}p &= \mathrm{P}\phi - \cdots, \\
&\cdots,
\end{aligned}
$$

x needing to be changed into p in the second members of these equations of which the number is equal to the degree of the differential equation (2): we can therefore, by their means, determine all the arbitrary constants of the value of ϕ; and, if we designate by ψ that which ϕ becomes when we have thus determined its arbitrary constants, we will have

$$y_s = \int x^s \psi\, dx.$$

Thence, and because equation (1) is linear, it is easy to conclude that, if S is equal to

$$\begin{aligned}
&p^s[l + l^{(1)}s + l^{(2)}s(s-1) + l^{(3)}s(s-1)(s-2) + \cdots] \\
&+ p_1^s[l_1 + l_1^{(1)}s + l_1^{(2)}s(s-1) + l_1^{(3)}s(s-1)(s-2) + \cdots] \\
&+ p_2^s[l_2 + l_2^{(1)}s + l_2^{(2)}s(s-1) + l_2^{(3)}s(s-1)(s-2) + \cdots] \\
&+ \cdots
\end{aligned}$$

by naming $\psi_1,\ \psi_2,\ \ldots$ that which ψ becomes when we change successively $p, l,\ l^{(1)},\ \ldots$ into $p_1,\ l_1,\ l_1^{(1)}, \ldots, p_2, l_2,\ l_2^{(1)},\ \ldots,$ we will have

$$y_s = \int x^s\, \psi\, dx + \int x^s\, \psi_1\, dx + \int x^s\, \psi_2\, dx + \cdots\ ,$$

the first integral being taken from $x = 0$ to $x = p$, the second integral being taken from $x = 0$ to $x = p_1$, etc. This value of y_s, contains no arbitrary constant; but, by joining it to that which we just found in the preceding section, for the case of S$= 0$, we will have for the complete expression of y_s

$$(4) \qquad \left\{ \begin{aligned} y_s = \mathrm{B} \int x^s\, \lambda\, dx + \mathrm{B}^{(1)} \int x^s\, \lambda^{(1)}\, dx + \mathrm{B}^{(2)} \int x^s\, \lambda^{(2)}\, dx + \cdots \\ + \int x^s\, \psi\, dx + \int x^s\, \psi_1\, dx + \int x^s\, \psi_2\, dx + \cdots \end{aligned} \right.$$

It will be easy, by the methods of No. VI, to have in convergent series the different terms of this expression when s will be a large number.

XI.

In order to determine the function y_s of s, that we arrive thus to reduce to a convergent series, we return to equation (1) of No. VIII and we suppose that it is differential of order n; if we designate by $u_s,\ {}^1u_s,\ {}^2u_s, \ldots$ the n particular values which satisfy it, when we make S$= 0$, so that its complete integral is then

$$y_s = \mathrm{H}u_s + {}^1\mathrm{H}\,{}^1u_s + {}^2\mathrm{H}\,{}^2u_s + \cdots + {}^{n-1}\mathrm{H}\,{}^{n-1}u_s;$$

if we form next the following quantities

$$u_s^1 = u_s \triangle \frac{{}^1u_{s-1}}{u_{s-1}},$$

$$^1u_s^1 = u_s \triangle \frac{{}^2u_{s-1}}{u_{s-1}},$$

$$^2u_s^1 = u_s \triangle \frac{{}^3u_{s-1}}{u_{s-1}},$$

$$\cdots,$$

$$u_s^2 = u_s^1 \triangle \frac{{}^1u_{s-1}^1}{u_{s-1}^1},$$

$$^1u_s^2 = u_s^1 \triangle \frac{{}^2u_{s-1}^1}{u_{s-1}^1},$$

$$^2u_s^2 = u_s^1 \triangle \frac{{}^3u_{s-1}^1}{u_{s-1}^1},$$

$$\cdots,$$

$$u_s^3 = u_s^2 \triangle \frac{{}^1u_{s-1}^2}{u_{s-1}^2},$$

$$\cdots;$$

by continuing thus to that which we arrive to form u_s^{n-1}, let

$$u_s^{n-1} = \frac{1}{{}^{n-1}z_{x-n}},$$

and we name $\frac{1}{{}^{n-2}u_{x-n}}$, $\frac{1}{{}^{n-3}u_{x-n}}$, $\ldots$ that which u_s^{n-1} becomes when we change ${}^{n-1}u_s$ successively into ${}^{n-2}u_s$, ${}^{n-3}u_s$, $\ldots$ and reciprocally; finally we designate by L the coefficient of $\triangle^n y_s$ in equation (1). The complete integral of this equation will be, as I have shown moreover (Vol. VII of the *Mémoires des Savants étranges*, p. 56[2])

$$y_s = u_s\left(\mathrm{H} + \sum \frac{\mathrm{S}}{\mathrm{L}} z_x\right) + {}^1u_s\left(\mathrm{H} + \sum \frac{\mathrm{S}}{\mathrm{L}}\,{}^1z_x\right) + \cdots + {}^{n-1}u_s\left(\mathrm{H} + \sum \frac{\mathrm{S}}{\mathrm{L}}\,{}^{n-1}z_x\right),$$

the characteristic Σ being that of finite integrals; we can therefore always reduce to convergent series all the functions of this nature, provided that S has the form which we have assigned to it in the preceding section.

XII.

We will consider generally the case where we have any number of equations linear in the finite differences, among a like number of variables y_s, y_s', y_s'', ..., and of which

[2]"Recherches sur l'intégration des équations différentelles aux différences finies, et sur leur usage dans la théorie des hasards." *Oeuvres de Laplace*, T. VIII, p. 87.

the coefficients are some rational and entire functions of s. If we suppose

$$y_s = \int x^s \phi \, dx,$$
$$y'_s = \int x^s \phi' \, dx,$$
$$y''_s = \int x^s \phi'' \, dx,$$
$$\cdots,$$

these different integrals being all extended into the same limits independent of s, we will have

$$\triangle y_s = \int x^s (x-1) \phi \, dx,$$
$$\triangle^2 y_s = \int x^s (x-1)^2 \phi \, dx,$$
$$\cdots,$$
$$\triangle y'_s = \int x^s (x-1) \phi' \, dx,$$
$$\triangle^2 y'_s = \int x^s (x-1)^2 \phi' \, dx,$$
$$\cdots,$$

We can therefore put the equations of which there is concern under the following forms

$$\mathrm{S} = \int x^s z \, dx,$$
$$\mathrm{S}' = \int x^s z' \, dx,$$
$$\mathrm{S}'' = \int x^s z'' \, dx,$$
$$\cdots,$$

$\mathrm{S}, \mathrm{S}', \mathrm{S}'', \ldots$ being functions of s, and z, z', z'', $\ldots$ being some rational and entire functions of the same variable, and of x, ϕ, ϕ', ϕ'', …, in which ϕ, ϕ', ϕ'', …are under a linear form.

We will consider first the equation

$$\mathrm{S} = \int x^s z \, dx;$$

we have

$$z = \mathrm{Z} + s \triangle \mathrm{Z} + \frac{s(s-1)}{1.2} \triangle^2 \mathrm{Z} + \frac{s(s-1)(s-2)}{1.2.3} \triangle^3 \mathrm{Z} + \cdots,$$

Z, $\triangle$Z, $\triangle^2$Z, ... being that which z, $\triangle z$, $\triangle^2 z$, ... become when we suppose $s = 0$ in it. Hence, we will have

$$\mathrm{S} = \int x^s dx \left[\mathrm{Z} + s\triangle\mathrm{Z} + \frac{s(s-1)}{1.2}\triangle^2\mathrm{Z} + \cdots\right].$$

Now, if we make $x^s = \delta y$, we will have

$$sx^s = x\frac{d\,\delta y}{dx}, \qquad s(s-1)x^s = x^2\frac{d^2\,\delta y}{dx^2}, \qquad \ldots$$

The preceding equation will become thus

$$\mathrm{S} = \int dx \left(\mathrm{Z}\,\delta y + x\triangle\mathrm{Z}\frac{d\,\delta y}{dx} + \frac{x^2\triangle^2\mathrm{Z}}{1.2}\frac{d^2\,\delta y}{dx^2} + \cdots\right),$$

whence we deduce, by integrating by parts as in No. VIII, the following two equations:

$$(a) \qquad 0 = \mathrm{Z} - \frac{d(x\,\triangle\mathrm{Z})}{dx} + \frac{d^2(x^2\,\triangle^2\mathrm{Z})}{1.2\,dx^2} - \cdots,$$

$$(b) \qquad \begin{cases} \mathrm{S} = \mathrm{C} + \delta y \left[x\triangle\mathrm{Z} - \dfrac{d(x^2\,\triangle^2\mathrm{Z})}{1.2\,dx} + \dfrac{d^2(x^3\triangle^3\mathrm{Z})}{1.2.3\,dx} + \cdots\right] \\ \qquad + \dfrac{d\,\delta y}{dx}\left[\dfrac{x^2\,\triangle^2\mathrm{Z}}{1.2} - \dfrac{d(x^3\triangle^3\mathrm{Z})}{1.2.3\,dx} + \cdots\right] \\ \qquad + \dfrac{d^2\,\delta y}{dx^2}\left(\dfrac{x^3\triangle^3\mathrm{Z}}{1.2.3} - \cdots\right) \\ \qquad + \cdots \end{cases}$$

The equation

$$\mathrm{S}' = \int x^s\, z'\, dx,$$

treated in the same manner, will give the following two:

$$(a') \qquad 0 = \mathrm{Z}' - \frac{d(x\,\triangle\mathrm{Z}')}{dx} + \frac{d^2(x^2\,\triangle^2\mathrm{Z}')}{1.2\,dx^2} - \cdots,$$

$$(b') \qquad \begin{cases} \mathrm{S}' = \mathrm{C}' + \delta y \left[x\triangle\mathrm{Z}' - \dfrac{d(x^2\,\triangle^2\mathrm{Z}')}{1.2\,dx} + \cdots\right] \\ \qquad + \dfrac{d\,\delta y}{dx}\left(\dfrac{x^2\,\triangle^2\mathrm{Z}'}{1.2} - \cdots\right) \\ \qquad + \cdots \end{cases}$$

The equations

$$\mathrm{S}'' = \int x^s\, z''\, dx, \qquad \mathrm{S}''' = \int x^s\, z'''\, dx, \qquad \ldots$$

will produce some similar equations which we will designate by (a''), (b''), (a'''), (b'''), ...

Equations (a), (a'), (a''), ...will determine the variables ϕ, ϕ', ϕ'', ... in x, and the equations (b), (b'), (b''), ...will determine the limits in which we must take the integrals $\int x^s\phi\,dx, \int x^s\phi'\,dx, \ldots$. For this, we will suppose first $\mathrm{S}, \mathrm{S}', \mathrm{S}''$, ... nulls; by making next $\mathrm{C}, \mathrm{C}', \mathrm{C}''$, ... nulls in equations (b), (b'), (b''), ...and by equating separately to zero the coefficients of δy, $\frac{d\,\delta y}{dx}$, ... in these equations, we will have the following:

$$
\begin{aligned}
&0 = x\,\triangle\mathrm{Z} - \frac{d(x^2\triangle^2\mathrm{Z})}{1.2\,dx} + \cdots,\\
&0 = \frac{x^2\triangle^2\mathrm{Z}}{1.2} - \cdots,\\
&\quad\cdots,\\
&0 = x\,\triangle\mathrm{Z}' - \frac{d(x^2\triangle^2\mathrm{Z}')}{1.2\,dx} + \cdots,\\
&0 = \frac{x^2\triangle^2\mathrm{Z}'}{1.2} - \cdots,\\
&\quad\cdots,
\end{aligned}
$$

We will eliminate by means of these equations all the arbitrary constants, less one, of the values of ϕ, ϕ', ϕ'', ... and we will arrive to one final equation in x of which the roots are the limits of the integrals $\int x^s\phi\,dx, \int x^s\phi'\,dx, \ldots$; we will determine as many of these limits as it will be necessary in order that the values of y_s, y'_s, ... are complete.

We suppose now that S is not null and that it is equal to

$$p^s[l + l^{(1)}s + l^{(2)}s(s-1) + \cdots];$$

by making C$= 0$ in equation (b) and by putting x^s in the place of δy, we will have

$$
\begin{aligned}
&p^s[l + l^{(1)}s + l^{(2)}s(s-1) + \cdots]\\
&\quad = x^s\left[x\,\triangle\mathrm{Z} - \frac{d(x^2\triangle^2\mathrm{Z})}{1.2\,dx} + \cdots\right] + sx^s\left(\frac{x^2\triangle^2\mathrm{Z}}{1.2} - \cdots\right) + \cdots,
\end{aligned}
$$

whence we deduce first $x = p$, so that the integrals $\int x^s\phi\,dx, \int x^s\phi'\,dx, \ldots$ must be taken from $x = 0$ to $x = p$. The comparison of the coefficients of s, $s(s-1)$, ... will give as many equations among the arbitrary constants of the values of ϕ, ϕ', ϕ'', ...; the equality to zero of these same coefficients in equations (b'), (b''), ... will give some new equations among these arbitraries, which we can consequently determine by means of all these equations. We will have thus the particular values of y_s, which satisfy in the case where, $\mathrm{S}', \mathrm{S}''$, ...being nulls, S has the form which we just supposed to it, or, more generally, is equal to any number of functions of the same form. Similarly, if we suppose that S, S'', ...being nulls, S' is the sum of any number of similar functions, we will determine the particular values of y_s, y'_s, y''_s, ... which they satisfy in this case, and thus the rest. By joining next all these values to those which we have determined in the case where $\mathrm{S}, \mathrm{S}', \mathrm{S}''$, ... are zero, we will have the complete expressions of these variables corresponding to the case where $\mathrm{S}, \mathrm{S}', \mathrm{S}''$ have the preceding form.

XIII.

It is easy to extend the method of the preceding section to the equations linear in infinitely small differences, or in finite parts, and in infinitely small parts and in which the coefficients of the principal variables are some rational and entire functions of s; because, if we designate, as previously, by $y_{s,}\ y'_s,\ y''_s, \ldots$ these principal values, we will make

$$y_s = \int x^s \phi \, dx, \quad y'_s = \int x^s \phi' \, dx, \quad y''_s = \int x^s \phi'' \, dx, \quad \ldots,$$

that which gives

$$\begin{array}{lll}
\dfrac{dy_s}{ds} = \displaystyle\int x^s \phi \, dx \log x, & \dfrac{d^2 y_s}{ds^2} = \displaystyle\int x^s \phi \, dx \, (\log x)^2, & \ldots, \\
\triangle y_s = \displaystyle\int x^s (x-1) \phi \, dx, & \triangle^2 y_s = \displaystyle\int x^s (x-1)^2 \phi \, dx, & \ldots, \\
\ldots, & \ldots, & \ldots, \\
\dfrac{dy'_s}{ds} = \displaystyle\int x^s \phi' \, dx \log x, & \ldots, & \ldots, \\
\ldots, & \ldots, & \ldots,
\end{array}$$

The proposed equations will take thus the following forms

$$\mathrm{S} = \int x^s z \, dx, \quad \mathrm{S}' = \int x^s z' \, dx, \quad \mathrm{S}'' = \int x^s z'' \, dx, \quad \ldots,$$

z, z', z'', … being some rational functions of s, in which ϕ, ϕ', ϕ'', … are under a linear form. By treating them therefore as in the preceding section, we will determine the values of ϕ, ϕ', ϕ'', … and the limits of the integrals $\int x^s \phi \ dx$, $\int x^s \phi' \, dx, \ldots$. Thus the method exhibited in this section extends to all the linear differential equations of which the coefficients are rationals.

By making $y_s = \int e^{-sx} \phi \, dx$, $y'_s = \int e^{-sx} \phi' \, dx, \ldots$, we would arrive to some similar results. In many circumstances, these forms of y_s, y'_s, … will be more suitable than the previous.

XIV.

The principal difficulty which the application of the preceding method presents consists in the integration of the linear differential equations which determine ϕ, ϕ', ϕ'', … in x. The degree of these equations depends not at all on that of the proposed equations in y_s, y'_s, y''_s, …: it depends uniquely on the highest powers of s in their coefficients. Thus, relatively to the finite differential equation of the first order,

$$0 = \mathrm{A} y_s + \mathrm{B} \triangle y_s,$$

in which A and B are some rational and entire functions of s, if we suppose $y_s = \int x^s \phi \, dx$,and if we determine by No. VIII the value of ϕ in x, we will arrive to a differential equation of an order equal to the highest exponent of s in A and B.

We can, in this particular case, prevent this inconvenience by decomposing the proposed equation into finite differences. For it to happen, we will put it under this form

$$y_{s+1} = \frac{q(s+a)(s+a')(s+a'')\cdots}{(s+b)(s+b')(s+b'')\cdots} y_s.$$

If we suppose next

$$\begin{aligned}
z_{s+1} &= q(s+a)z_s, \qquad & z'_{s+1} &= q(s+a')z'_s, \qquad & z''_{s+1} &= q(s+a'')z''_s, \qquad & \ldots,\\
t_{s+1} &= q(s+b)t_s, \qquad & t'_{s+1} &= q(s+b')t'_s, \qquad & t''_{s+1} &= q(s+a'')t''_s, \qquad & \ldots,
\end{aligned}$$

we will have

$$y_s = \frac{z_s z'_s z''_s \cdots}{t_s t'_s t''_s \cdots}.$$

It will be easy to have z_s, z'_s, z''_s,...,t_s, t'_s, t''_s, ... in convergent series, and we will have need for that only to integrate the equations linear in the infinitely small differences of the first order. Every time that we can decompose thus a proposed equation into other linear equations, in which the variable s will not pass the first degree, we will have always in convergent series the value of its integral, if s is a large number.

In many cases where we are led to a differential equation in ϕ, of an order superior to the first, we can make use of multiple integrals by representing y_s by the double integral $\int x^s x'^s \phi\, dx\, dx'$, in which ϕ is a function of x and of x', or by the triple integral $\int x^s x'^s x''^s \phi\, dx\, dx'\, dx''$, ϕ being a function of x, x', x'', and thus in sequence. We will arrive often to determine ϕ directly or by an equation of the first order; we will see some examples in the following article.

XV.

The case in which the equation which determines the value of ϕ is differential of the first order being the only one which is generally solvable, we are going to develop it here by applying directly the method of approximation of article I.

We suppose that we have a linear equation of any order in finite or infinitely small differences, or in finite parts and in infinitely small parts, in the coefficients of which the variable s does not pass the first degree; this equation will have the following form

$$0 = \mathrm{V} + s\mathrm{T},$$

V and T being some linear functions in the principal variable y_s and of its differences. If we make $y_s = \int \delta y\, \phi\, dx$, δy being equal to x^s or to e^{-sx}, it will become

$$0 = \int \phi\, dx \left(\mathrm{M} + \mathrm{N}\frac{d\,\delta y}{dx} \right),$$

M and N being some functions of x; we will have therefore, by the method of No. VIII, the two equations

$$\begin{aligned}
0 &= \mathrm{M}\phi - \frac{d(\mathrm{N}\phi)}{dx},\\
0 &= \mathrm{C} + \delta y\, \mathrm{N}\phi.
\end{aligned}$$

The first gives, by integrating it,

$$\phi = \frac{\mathrm{H}}{\mathrm{N}} e^{\int \frac{\mathrm{M}}{\mathrm{N}} dx},$$

H being an arbitrary constant. We suppose, in the second equation, C$= 0$; if we designate by a the value of x given by the equation

$$0 = d(\mathrm{N}\,\phi\,\delta y),$$

and by Q that which the function $\mathrm{N}\,\phi\,\delta y$ becomes when we change x into a in it, we will make

$$\mathrm{N}\,\phi\,\delta y = \mathrm{Q}e^{-t^2};$$

we will have thus

$$t = \sqrt{\log \mathrm{Q} - \log(\mathrm{N}\phi) - \log \delta y}$$

$\log \delta y$ being of order s, if we make $\frac{1}{s} = \alpha$, α being a very small coefficient, the quantity under the radical will take this form $\frac{(x-a)^2}{\alpha}$X, X being a function of $x - a$; we will have therefore, by the reversion of the series, the value of x in t by a series of this form

$$x = a + \alpha^{\frac{1}{2}} ht + \alpha h^{(1)} t^2 + \alpha^{\frac{3}{2}} h^{(2)} t^3 + \cdots$$

Now, y_s being equal to $\int \delta y\,\phi\,dx$, if we substitute into this integral in the place of $\delta y\,\phi$ its value $\frac{\mathrm{Q}e^{-t^2}}{\mathrm{N}}$, it will become $\mathrm{Q}\int \frac{dx}{\mathrm{N}} e^{-t^2}$, and if in $\frac{dx}{\mathrm{N}}$ we put in the place of x its preceding value in t, we will have y_s by a series of this form

$$y_s = \alpha^{\frac{1}{2}} \mathrm{Q} \int dt\, e^{-t^2} \left(l + \alpha^{\frac{1}{2}} l^{(1)} t + \alpha l^{(2)} t^2 + \alpha^{\frac{3}{2}} l^{(3)} t^3 + \cdots \right).$$

The limits of the integral relative to t must be determined by this condition, that at these limits the quantity $\mathrm{N}\,\phi\,\delta y$ or its equivalent $\mathrm{Q}e^{-t^2}$ is null; whence it follows that these limits are $t = -\infty$ and $t = \infty$. We will have therefore, by article I,

$$y_s = \alpha^{\frac{1}{2}} \mathrm{Q}\sqrt{\pi} \left(l + \frac{1}{2}\alpha l^{(2)} + \frac{1.3}{2^2}\alpha^2 l^{(4)} + \frac{1.3.5}{2^3}\alpha^3 l^{(6)} + \cdots \right).$$

This expression has the advantage of being independent of the determination of the limits in x which render null the function $\mathrm{N}\,\phi\,\delta y$, so that it would subsist always in the same case where this function equated to zero will not have many real roots. This remark is important in this analysis and gives the means to extend it to a great number of cases in which it seems first to be refused.

The preceding value of y_s contains only one arbitrary constant H, and consequently, if the proposed equation is differentiable of order n, it will be only one particular value of it. In order to have the complete integral, it will be necessary to seek n different values of x in the equation

$$0 = d(\mathrm{N}\,\phi\,\delta y).$$

Let a, a', a'', ... be these n values; we will change successively, in the preceding expression of y_s, a to a', a'',... and H to H$'$, H$''$, ...; we will have thus n particular values of y_s, which will contain each an arbitrary constant; their sum will be the complete expression of this variable.

XVI.

We can obtain directly by the preceding method the value of y_s in the differential equation $0 =$V$+s$T, by means of definite integrals; in order to show by a very general example, we will consider the differential equation

$$0 = (a+bs)y_s + (a'+b's)\frac{dy_s}{ds} + (a''+b''s)\frac{d^2y_s}{ds^2} + (a'''+b'''s)\frac{d^3y_s}{ds^3} + \cdots;$$

if we suppose

$$y_s = \int \delta y\,\phi\,dx,$$

δy being equal to e^{-sx}, we will have

$$\begin{aligned}0 = \int \phi\,dx[&\delta y(a - a'x + a''x^2 - a'''x^3 + \cdots)\\ &- \frac{d\,\delta y}{dx}(b - b'x + b''x^2 - \cdots)],\end{aligned}$$

whence we deduce the two equations

$$\begin{aligned}&0 = \phi(a - a'x + a''x^2 - a'''x^3 + \cdots) + \frac{d[\phi(b - b'x + b''x^2 - \cdots)]}{dx},\\ &0 = e^{-sx}\phi(b - b'x + b''x^2 - \cdots).\end{aligned}$$

We decompose the function $b - b'x + b''x^2 - \cdots$ into its factors, and we suppose that it is equal to

$$b(1-qx)(1-q'x)(1-q''x)\cdots,$$

the first equation will give for ϕ an expression of this form

$$\phi = \mathrm{H}e^{lx}(1-qx)^r(1-q'x)^{r'}(1-q''x)^{r''}\cdots,$$

H being an arbitrary constant; hence

$$y_s = \mathrm{H}\int e^{-(s-l)x}dx(1-qx)^r(1-q'x)^{r'}(1-q''x)^{r''}\cdots;$$

and the equation which will determine the limits of the integral will be

$$0 = e^{-(s-l)x}(1-qx)^{r+1}(1-q'x)^{r'+1}(1-q''x)^{r''+1}\cdots$$

These limits will be consequently $x = \frac{1}{q}$ and $x = \infty$, or $x = \frac{1}{q'}$ and $x = \infty$, etc., so that the complete expression of y_s will be

$$\begin{aligned}y_s =&\mathrm{H}\int e^{-(s-l)x}dx(1-qx)^r(1-q'x)^{r'}(1-q''x)^{r''}\cdots\\ &+\mathrm{H}'\int e^{-(s-l)x}dx(1-qx)^r(1-q'x)^{r'}(1-q''x)^{r''}\cdots\\ &+\mathrm{H}''\int e^{-(s-l)x}dx(1-qx)^r(1-q'x)^{r'}(1-q''x)^{r''}\cdots\\ &+\cdots\end{aligned}$$

the first integral being taken from $x = \frac{1}{q}$ to $x = \infty$; the second integral being taken from $x = \frac{1}{q'}$ to $x = \infty$; the third being taken from $x = \frac{1}{q''}$ to $x = \infty$, and thus in sequence, $\mathrm{H}, \mathrm{H}', \mathrm{H}''$, ... being some arbitrary constants.

It can happen that the numbers $s - l$, $r + 1$, $r' + 1, \ldots$ are negatives and, in this case, the equation

$$0 = e^{-(s-l)x}(1 - qx)^{r+1}(1 - q'x)^{r'+1} \cdots$$

is not satisfied by making $x = \infty$, $x - \frac{1}{q}$, $x = \frac{1}{q'}, \ldots$; but we can observe that the results obtained under the assumption where these numbers are positive take place equally when these numbers are negative. Thus, by designating by S the integral, either finite, or reduced to a series, by the method of article I, of the differential function

$$e^{-(s-l)x}dx(1 - qx)^r(1 - q'x)^{r'} \cdots,$$

integrated from $x = \frac{1}{q}$ to $x = \infty$ in the case where $s - l$ and r are positives, if we change, in S, r into $-r$, and if we designate by S′ that which S becomes, the function HS′ will be a particular value of y_s in the case where the number r, instead of being positive, is negative and equal to $-r$; because it is clear that the equation $y_s =$HS, satisfying the proposed equation, r being positive and anything, the equation $y_s =$HS′ must similarly satisfy it, r being negative and anything. Thus, we will not hesitate at all in the series to extend generally to all possible cases the results obtained in the case where the equation which determines the limits of the integrals is satisfied.

It is easy to extend the preceding method to the equation in finite differences

$$0 = (a + bs)y_s + (a' + b's)\triangle y_s + (a'' + b''s)\triangle^2 y_s + \cdots$$

or to the equation in the differences in finite parts and in infinitely small parts,

$$\begin{aligned} 0 = (a + bs)y_s + (a' + b's)\triangle y_s + (a''' + b'''s)\triangle^2 y_s + \cdots \\ +(a'' + b''s)\frac{dy_s}{ds} + (a^{iv} + b^{iv}s)\triangle\frac{dy_s}{ds} + \cdots \\ + \cdots \end{aligned}$$

We can always obtain, by the preceding method, the integral of these equations by definite integrals, and its approximate value by some series which will be highly convergent when s will be a large number.

XVII.

The same method can be further extended to the equations linear in the partial differentials, either finite, or infinitely small. For this, we will consider first the equation linear in the partial differences of which the coefficients are constants; by designating by $y_{s,s'}$ the principal variable, s, s' being the two variables of which it is a function, we can represent this equation by this one 0 =V, V being a linear function of $y_{s,s'}$ and of its partial differences, either finite, or infinitely small. We suppose now

$$y_{s,s'} = \int x^s u^{s'} \phi\, dx;$$

by substituting this value into the preceding equation, it will become

$$0 = \int \mathrm{M} x^s u^{s'} \phi \, dx,$$

M being a function of x and of u, with neither s nor s'; by equating it to zero, we will have the value of u in x, and this value, substituted into the integral $\int x^s u^{s'} \phi dx$, will give the general expression of $y_{s,s'}$, ϕ being an arbitrary function of x, and the limits of the integral being indeterminate. If the proposed equation $0 =$V is of order n, it will be necessary, by means of the equation M$=0$, to determine n values of u in x, and the sum of the n integrals $\int x^s u^{s'} \phi dx$ which will result from it will be the complete expression of $y_{s,s'}$.

We will consider presently the equation in the partial differences

$$0 = \mathrm{V} + s\mathrm{T} + s'\mathrm{R},$$

in which $\mathrm{V}, \mathrm{T}, \mathrm{R}$ are some linear functions any whatsoever of $y_{s,s'}$ and of its finite and infinitely small partial differences. If we suppose

$$y_{s,s'} = \int x^s x'^{s'} \phi dx,$$

x' being a function of x which the concern is to determine, we will have an equation in this form

$$0 = \int x^s x'^{s'} \phi dx (\mathrm{M} + \mathrm{N}s + \mathrm{P}s'),$$

$\mathrm{M}, \mathrm{N}, \mathrm{P}$ being some functions of x and x', with neither s nor s'; now we have

$$\frac{d(x^s x'^{s'})}{dx} = x^s x'^{s'} \left(\frac{s}{x} + \frac{s' dx'}{x' dx} \right).$$

Therefore, if we determine x' by this equation

$$\frac{dx'}{x'} = \frac{\mathrm{P} \, dx}{\mathrm{N} x},$$

we will have

$$x^s x'^{s'} (\mathrm{N}s + \mathrm{P}s') = \mathrm{N}x \frac{d(x^s x'^{s'})}{dx};$$

consequently, if we designate $x^s x'^{s'}$ by δy, we will have

$$0 = \int \phi \, dx \left(\mathrm{M} \, \delta y + \mathrm{N}x \frac{d \, \delta y}{dx} \right).$$

This equation gives the following two

$$0 = \mathrm{M}\phi - \frac{d(\mathrm{N}x\phi)}{dx},$$

$$0 = \mathrm{N}x\phi \, \delta y;$$

the first determines the function ϕ in x, and the second determines the limits of the integral $\int \delta y\, \phi\, dx$. This value of $y_{s,s'}$ contains no arbitrary function at all, it is only a particular integral of the proposed equation in the partial differences; in order to render it complete, we will observe that the integral of the equation $\frac{dx'}{x'} = \frac{\mathrm{P}dx}{\mathrm{N}x}$, which determine x' in x, is

$$x' = u\mathrm{Q},$$

Q being a function of x and u being an arbitrary constant. By designating therefore by ψ an arbitrary function of u, we will have

$$y_{s,s'} = \iint u^{s'} \mathrm{Q}^{s'} x^s \phi\psi\, dx\, du,$$

the integral relative to x being taken between the limits determined by the equation $0 = \mathrm{N}x\phi\delta y$, and the integral relative to u being taken between some limits any whatsoever. This value of $y_{s,s'}$ will be, because of the arbitrary ψ, the complete integral of the proposed equation if this equation is of first order; but, if it is of a superior order, it will be necessary, by means of the equation $0 = \mathrm{N}x\phi\delta y$, to determine as many values of x in u as there are units in that order; and the sum of the expressions of $y_{s,s'}$ to which we will arrive will be the complete value of $y_{s,s'}$.

XVIII.

By considering with attention the form of the series in which the preceding method leads in order to determine $y_{s,s'}$, we see that it can always be reduced to the following

$$\mathrm{H}p^s s^{is+r} \left(1 + \frac{q}{s^{r'}} + \frac{q'}{s^{r''}} + \cdots\right),$$

H being an arbitrary constant and the numbers $r', r'', \ldots$ being positives and forming an increasing series. If the proposed equation in y_s is in infinitely small differences, then $i = 0$, since, without this, the differences of y_s would introduce the logarithmic quantities $\log s,\ (\log s)^2, \ldots$, which, by assumption, are encountered not at all in the coefficients of this equation; we will have therefore then

$$y_s = \mathrm{H}p^s s^r \left(1 + \frac{q}{s^{r'}} + \frac{q'}{s^{r''}} + \cdots\right),$$

and it will be easy, by known methods, to determine the exponents r, r', r'', ... and the constants p, q, p', q', ...

If the proposed equation in y_s is in finite differences, i can not be null, and the determination of the quantities r, r', r'', ...; p, q, q', ... can then present some difficulties which we are going to resolve.

For this, we will observe that

$$\begin{aligned}\log(s+n)^{is+in+r} &= (is+in+r)\left[\log s + \log\left(1+\frac{n}{s}\right)\right]\\ &= (is+in+r)\left(\log s + \frac{n}{s} - \frac{n^2}{2s^2} + \frac{n^3}{3s^3} - \cdots\right)\end{aligned}$$

that which gives

$$(s+n)^{is+in+r} = s^{is+in+r} e^{in+\frac{in^2+2rn}{2s}+\cdots}.$$

We can put the second member of this equation under this form

$$s^{is+in+r} e^{in} \left(1 + \frac{a_n}{s} + \frac{b_n}{s^2} + \cdots\right),$$

$a_n,\ b_n, \ldots$ being some functions of n; we will have therefore

$$y_{s+n} = \mathrm{H} p^{s+n} s^{is+in+r} e^{in} \left(1 + \frac{a_n}{s} + \frac{b_n}{s^2} + \cdots\right) \left[1 + \frac{q}{s^{r'}}\left(1 - \frac{s}{r'n} + \cdots\right)\right.$$
$$\left. + \frac{q'}{s^{r''}}\left(1 - \frac{s}{r''n} + \cdots\right) + \cdots\right],$$

whence it is easy to conclude the values of $y_{s+1},\ y_{s+2},\ y_{s+3},\ \ldots$, by making successively in this expression $n = 1,\ n = 2,\ n = 3, \ldots$. Now, if we substitute these values into the proposed equation in finite differences, we will determine easily by the known methods the exponents $i,\ r,\ r', \ldots$ and the constants $p,\ q,\ q', \ldots$.

This new method has the advantage of being independent of all integration and of being extended to the case where the coefficients of the proposed equation in y_s would be irrational; but the arbitrary constants $\mathrm{H}, \mathrm{H}', \ldots$ which we introduce can be determined then only by means of given values of y_s, when s is already a large number, instead that, following the method exhibited in the preceding sections, these constants can be determined by means of the first values of y_s, that which gives the means to know that which this function becomes when s is very great or even infinite, by supposing that it has begun in a determined manner; it is in this that the principal advantage of this method consists.

ARTICLE III.

Application of the preceding method to the approximation of diverse functions of very great numbers.

XIX.

We propose to integrate by approximation the equation in finite differences

$$0 = (s+1)y_s - y_{s+1}.$$

If we suppose

$$y_s = \int x^s \phi dx,$$

we will have, by designating x^s by δy,

$$0 = \int \phi\, dx \left[(1-x)\delta y + x\frac{d\,\delta y}{dx}\right],$$

whence we deduce, by the preceding article, the following two equations:

$$0 = \phi(1-x) - \frac{d(x\phi)}{dx},$$
$$0 = \phi x^{s+1}.$$

The first equation gives, by integrating it,

$$\phi = \mathrm{A}e^{-x},$$

and the second gives, in order to determine the limits of the integral $\int x^s \phi \, dx$,

$$0 = x^{s+1}e^{-x};$$

these limits are, consequently, $x = 0$ and $x = \infty$. Thus we have

$$y_s = \mathrm{A}\int x^s e^{-x} dx,$$

the integral being taken from $x = 0$ to $x = \infty$.

In order to have this integral in a series, we make, following the method of article I,

$$x^s e^{-x} = s^s e^{-s} e^{-t^2},$$

s being the value of x which corresponds to the maximum of the function $x^s e^{-x}$; if we suppose next $x = s + \theta$, we will have

$$\left(1 + \frac{\theta}{s}\right)^s e^{-\theta} = e^{-t^2},$$

hence

$$t^2 = -s\log\left(1 + \frac{\theta}{s}\right) + \theta = \frac{\theta^2}{2s} - \frac{\theta^3}{3s^2} + \frac{\theta^4}{4s^3} - \cdots,$$

that which gives, by the reversion of the series,

$$\theta = t\sqrt{2s} + \frac{2t^2}{3} + \frac{t^3}{9\sqrt{2s}} + \cdots$$

and, consequently,

$$dx = d\theta = dt\left(\sqrt{2s} + \frac{4t}{3} + \frac{t^2}{3\sqrt{2s}} + \cdots\right);$$

the function $\int x^s \, dx \, e^{-x}$ will become therefore

$$s^s e^{-s}\int dt e^{-t^2}\left(\sqrt{2s} + \frac{4t}{3} + \frac{t^2}{3\sqrt{2s}} + \cdots\right),$$

the integral being taken from $t = -\infty$ to $t = \infty$. By integrating by the method of article I, we will have

$$\int x^s \, dx \, e^{-x} = s^{s+\frac{1}{2}} e^{-s}\sqrt{2\pi}\left(1 + \frac{1}{12s} + \cdots\right),$$

hence

$$y_s = \mathrm{A}s^{s+\frac{1}{2}}e^{-s}\sqrt{2\pi}\left(1+\frac{1}{12s}+\cdots\right).$$

We will determine the arbitrary constant A by means of a particular value of y_s; by supposing, for example, that, s being equal to μ, we have $y_s =$Y, we will have

$$\mathrm{Y} = \mathrm{A}\int x^\mu\,dx\,e^{-x},$$

that which gives

$$\mathrm{A} = \frac{\mathrm{Y}}{\int x^\mu\,dx\,e^{-x}}$$

and, consequently,

$$(q) \qquad y_s = \mathrm{Y}\frac{s^{s+\frac{1}{2}}e^{-s}\sqrt{2\pi}\left(1+\frac{1}{12s}+\cdots\right)}{\int x^\mu\,dx\,e^{-x}};$$

if μ is a large number, we will have

$$\int x^\mu\,dx\,e^{-x} = \mu^{\mu+\frac{1}{2}}e^{-\mu}\sqrt{2\pi}\left(1+\frac{1}{12s}+\cdots\right),$$

that which gives

$$(q') \qquad y_s = \mathrm{Y}\frac{s^{s+\frac{1}{2}}}{\mu^{\mu+\frac{1}{2}}}e^{\mu-s}\left(1+\frac{\mu-s}{12\mu s}+\cdots\right);$$

thus, in this case, the ratio of the semi-circumference to the radius disappears, and there remains only the sole transcendental quantity e.

Let us see now of what nature is the function y_s; for this it is necessary to integrate the equation in finite differences

$$0 = (s+1)y_s - y_{s+1};$$

now we will find easily that its integral is

$$y_s = \mathrm{Y}(\mu+1)(\mu+2)(\mu+3)\cdots s.$$

We will have therefore, by comparing this expression with that of formula (q),

$$(q'') \qquad (\mu+1)(\mu+2)(\mu+3)\cdots s = \frac{s^{s+\frac{1}{2}}e^{-s}\sqrt{2\pi}\left(1+\frac{1}{12s}+\cdots\right)}{\int x^\mu\,dx\,e^{-x}}.$$

If we suppose $\mu = 0$, we will have

$$\int x^\mu\,dx\,e^{-x} = 1,$$

hence

$$1.2.3\ldots s = s^{s+\frac{1}{2}}e^{-s}\sqrt{2\pi}\left(1+\frac{1}{12s}+\cdots\right).$$

If we make $\mu = \frac{m}{n}$, m being less than n, we will have

$$s = s' + \frac{m}{n},$$

s' being a whole number; thus

$$s^{s+\frac{1}{2}} = \left(s'+\frac{m}{n}\right)^{s'+\frac{m}{n}+\frac{1}{2}};$$

now it is easy to assure ourselves by the preceding number that, if s' is a large number, we have

$$\left(s'+\frac{m}{n}\right)^{s'+\frac{m}{n}+\frac{1}{2}} = s'^{s'+\frac{m}{n}+\frac{1}{2}}e^{\frac{m}{n}}\left(1+\frac{nm+m^2}{2n^2s'}+\cdots\right).$$

We have moreover, by making $x = t^n$,

$$\int x^{\frac{m}{n}}dx\,e^{-x} = n\int t^{m+n-1}dt\,e^{-t^n} = m\int t^{m-1}dt\,e^{-t^n},$$

the integral relative to t being taken from $t=0$ to $t=\infty$; the formula (q'') will give therefore

$$\begin{aligned} &m(m+n)(m+2n)(m+3n)\cdots(m+s'n) \\ &\qquad = n^{s'}\frac{s'^{s'+\frac{m}{n}+\frac{1}{2}}e^{-s'}\sqrt{2\pi}\left(1+\frac{n^2+6mn+6m^2}{12n^2s'}+\cdots\right)}{\int t^{m-1}dt\,e^{-t^n}} \end{aligned}$$

so that the approximate value of the product of all the terms of the arithmetic progression m, $m+n, m+2n, \ldots, m+s'n$ depend on the three transcendentals e, π and $\int t^{m-1}dt\,e^{-t^n}$.

XX.

The expressions of y_s, given by formulas (q) and (q'), hold further, according to the remark of No. XVI, in the case where s and μ are negatives, although, in this case, the equation $0 = x^{s+1}e^{-x}$, which determines the limits of the integral $\int x^s\phi\,dx$, does not have many real roots; we can be assured of it moreover by supposing the function $x^{s+1}e^{-x}$, which must become null at the two extremities of this integral, equal to $\mathrm{Q}e^{-t^2}$, according to the method of No. XV, because then we would arrive at some expressions of y_s easily reducible to formulas (q) and (q'), and we have observed in the section cited that, by following this method, the consideration of the roots of the equation $0 = x^{s+1}e^{-x}$ becomes useless.

Now, if in formula (q) we change s into $-s$ and μ into $-\mu$, we will have

$$y_{-s} = \mathrm{Y}\frac{\sqrt{-1}e^s\sqrt{2\pi}\left(1-\frac{1}{12s}+\cdots\right)}{(-1)^s s^{s-\frac{1}{2}}\int\frac{dx\,e^{-x}}{x^\mu}},$$

Y being the value of y_s which corresponds to $s = -\mu$; all the difficulty is reduced therefore to integrating the differential function $\frac{e^{-x}dx}{x^\mu}$. In order to arrive to it, it is necessary to follow a method similar to that of which we have made use to reduce into series the integral $\int \frac{dx\, e^{-x}}{x^s}$. We will make therefore

$$x = -\mu + \varpi\sqrt{-1},$$

$-\mu$ being the value of x given by the condition $0 = d\frac{e^{-x}}{x^\mu}$ of the maximum or of the minimum of $\frac{e^{-x}}{x^\mu}$; we will have thus

$$\int \frac{e^{-x}dx}{x^\mu} = \frac{e^\mu\sqrt{-1}}{(-1)^\mu}\int \frac{d\varpi e^{-\varpi\sqrt{-1}}}{(\mu - \varpi\sqrt{-1})^\mu}.$$

The integral relative to x must extend between the two limits which return null the quantity $\frac{e^{-x}}{x^\mu}$, it is clear that the integral relative to ϖ must extend from $\varpi = -\infty$ to $\varpi = \infty$: by joining therefore the two quantities $\frac{e^{-\varpi\sqrt{-1}}}{(\mu-\varpi\sqrt{-1})^\mu}$ and $\frac{e^{\varpi\sqrt{-1}}}{(\mu+\varpi\sqrt{-1})^\mu}$ which correspond to the same values of ϖ affected with contrary signs, we will have

$$\int \frac{e^{-x}dx}{x^\mu} = \frac{e^\mu\sqrt{-1}}{(-1)^\mu}\int d\varpi \times$$
$$\frac{\{\cos\varpi[(\mu+\varpi\sqrt{-1})^\mu + (\mu-\varpi\sqrt{-1})^\mu] + \sqrt{-1}\sin\varpi[(\mu-\varpi\sqrt{-1})^\mu - (\mu+\varpi\sqrt{-1})^\mu]\}}{(\mu^2+\varpi^2)^\mu},$$

the integral relative to ϖ being taken from $\varpi = 0$ to $\varpi = \infty$. If we expand the quantities under the $\int$ sign, the imaginaries disappear, and there will remain only a real function which we will designate by Qdx; we will have thus

$$\int \frac{e^{-x}dx}{x^\mu} = \frac{e^\mu\sqrt{-1}}{(-1)^\mu}\int \mathrm{Q}d\varpi,$$

hence

$$y_{-s} = \frac{\mathrm{Y}e^{s-\mu}\sqrt{2\pi}\left(1 - \frac{1}{12s} + \cdots\right)}{(-1)^{s-\mu}s^{s-\frac{1}{2}}\int \mathrm{Q}d\varpi}.$$

Let us see presently which function of s is y_{-s}. For this, we represent the proposed equation

$$0 = (s+1)y_s - y_{s+1};$$

by changing s into $-s$, it becomes

$$0 = (1-s)y_{-s} - y_{1-s}.$$

Let $y_{-s} = u_s$; we will have

$$0 = (s-1)u_s + u_{s-1},$$

an equation of which the integral

$$u_s = \frac{(-1)^{s-\mu}\mathrm{Y}}{\mu(\mu+1)(\mu+2)(\mu+3)\cdots(s-1)},$$

Y being equal to $y_{-\mu}$. We will have therefore

$$y_{-s} = \frac{(-1)^{s-\mu}\mathrm{Y}}{\mu(\mu+1)(\mu+2)(\mu+3)\cdots(s-1)}.$$

If we compare this expression of y_{-s} to the preceding, we will have, by observing that $(-1)^{2s-2\mu} = 1$,

$$\frac{1}{\mu(\mu+1)(\mu+2)(\mu+3)\cdots(s-1)} = \frac{e^{s-\mu}\mu\sqrt{2\pi}\left(1-\frac{1}{12s}+\cdots\right)}{s^{s-\frac{1}{2}}\int \mathrm{Q}d\varpi};$$

by dividing the two members of this equation by s and by turning them upside down, we will have

$$(\mu+1)(\mu+2)\cdots s = \frac{s^{s+\frac{1}{2}}e^{\mu-s}}{\mu\sqrt{2\pi}}\left(1-\frac{1}{12s}+\cdots\right)\int \mathrm{Q}d\varpi.$$

By comparing this equation to formula (q') of the preceding section, we will have this rather remarkable result

$$\int \mathrm{Q}d\varpi = \frac{2\pi e^{-\mu}\mu}{\int x^{\mu}e^{-x}dx};$$

we suppose, for example, $\mu = 1$, we will have

$$\int \mathrm{Q}d\varpi = 2\int d\varpi\frac{\cos\varpi\ +\varpi\sin\varpi}{1+\varpi^2} = 2\int\frac{\varpi\, d\varpi\sin(3+\varpi^2)}{(1+\varpi^2)^2},$$

these integrals being taken from $\varpi = 0$ to $\varpi = \infty$; hence

$$\int\frac{d\varpi\sin\varpi(3+\varpi^2)}{(1+\varpi^2)^2} = \frac{\pi}{e}.$$

We can observe further that, $\int\frac{e^{-x}dx}{x^{\mu}}$ being equal to $\frac{e^{-\mu\sqrt{-1}}}{(-1)^{\mu}}\int \mathrm{Q}d\varpi$, we have

$$\int\frac{e^{-x}dx}{x^{\mu}} = \frac{2\pi\mu(-1)^{\mu-\frac{1}{2}}}{\int x^{\mu}dx\, e^{-x}} = \frac{2\pi(-1)^{\mu-\frac{1}{2}}}{\int x^{\mu-1}dx\, e^{-x}},$$

the integral of the first member of this equation being taken between the two imaginary values of x which render null the quantity $\frac{e^{-x}}{x^{\mu}}$, and the integral of the second member being taken between the two real values of x which render null the quantity $x^{\mu}e^{-x}$, that is from $x = 0$ to $x = \infty$.

We could easily arrive to the preceding results, by considering the equation in finite differences

$$0 = y_s - sy_{s+1};$$

but I have wished to show, by a very simple example, that the same expressions, found in the case of s positive, subsist still when s is negative.

XXI.

We will consider the equation in the finite differences

$$p^s = sy_s - (m-s)y_{s+1};$$

by supposing there

$$y_s = \int x^s \phi\, dx \qquad \text{and} \qquad x^s = \delta y,$$

it will become

$$p^s = \int \phi\, dx \left[-mx\,\delta y + x(1+x)\frac{d\,\delta y}{dx} \right];$$

whence we deduce the two equations

$$0 = mx\phi + \frac{d[x(1+x)\phi]}{dx},$$
$$p^s = x^{s+1}(1+x)\phi.$$

The first gives, by integrating it,

$$\phi = \frac{\mathrm{A}}{x(1+x)^{m+1}},$$

that which changes the second into this one

$$\frac{\mathrm{A}x^s}{(1+x)^m} = p^s.$$

We suppose first $p = 0$, we will have $x = 0$ and $x = \infty$ for the limits of the integral $\int x^s \phi\, dx$, s being supposed less than m; thus, in this case, the integral $\int x^s \phi\, dx$ must be extended from $x = 0$ to $x = \infty$, and we will have, with this condition,

$$y_s = \mathrm{A} \int \frac{x^{s-1} dx}{(1+x)^{m+1}},$$

A being an arbitrary constant.

If p is not null, the two limits of x will be $x = 0$ and $x = p$; we will have next

$$\mathrm{A} = (1+p)^m,$$

hence

$$y_s = (1+p)^m \int \frac{x^{s-1} dx}{(1+x)^{m+1}},$$

the integral being taken from $x = 0$ to $x = p$. By reuniting this value to that which we just found in the case $p = 0$, we will have, for the complete expression of y_s,

$$y_s = \mathrm{A} \int \frac{x^{s-1} dx}{(1+x)^{m+1}} + (1+p)^m \int \frac{x^{s-1} dx}{(1+x)^{m+1}},$$

the integral of the first term being taken from $x = 0$ to $x = \infty$, and that of the second being taken from $x = 0$ to $x = p$. We can give yet to the expression of y_s this form

$$y_s = \mathrm{A}' \int \frac{x^{s-1}dx}{(1+x)^{m+1}} - (1+p)^m \int \frac{x^{s-1}dx}{(1+x)^{m+1}},$$

the integral of the first term being taken from $x = 0$ to $x = \infty$, and that of the second being taken from $x = p$ to $x = \infty$; A' is a arbitrary constant equal to A+1.

Now, the integral of the proposed equation

$$p^s = sy_s - (m-s)y_{s+1};$$

is

$$y_s = \frac{1.2.3\cdots(s-1)}{m(m-1)(m-2)\cdots(m-s+1)}\left[\mathrm{Q} - \sum \frac{m(m-1)\cdots(m-s+1)p^s}{1.2.3\cdots s}\right].$$

Q being an arbitrary and $\sum$ being the characteristic of the finite integrals, so that $\sum \frac{m(m-1)\cdots(m-s+1)p^s}{1.2.3\cdots s}$ is equal to

$$1 + mp + \frac{m(m-1)}{1.2}p^2 + \cdots + \frac{m(m-1)(m-2)\cdots(m-s+2)}{1.2.3\cdots(s-1)}p^{s-1},$$

that is to the sum of the s first terms of the expansion of the binomial $(1+p)^m$. If we compare this expression of y_s with that which we just found by definite integrals, we will have

$$\begin{aligned}\mathrm{A}' \int \frac{x^{s-1}dx}{(1+x)^{m+1}} &- (1+p)^m \int \frac{x^{s-1}dx}{(1+x)^{m+1}}\\ &= \frac{1.2.3\cdots(s-1)}{m(m-1)(m-2)\cdots(m-s+1)}\left[\mathrm{Q} - \sum \frac{m(m-1)\cdots(m-s+1)p^s}{1.2.3\cdots s}\right].\end{aligned}$$

If we make $s = 1$ in this equation, we will have $\mathrm{A}' =$Q; thus A' being arbitrary, this equation is divided into the following two

$$\frac{1.2.3\cdots(s-1)}{m(m-1)(m-2)\cdots(m-s+1)} = \int \frac{x^{s-1}dx}{(1+x)^{m+1}},$$

$$\frac{1.2.3\cdots(s-1)}{m(m-1)\cdots(m-s+1)} \sum \frac{m(m-1)\cdots(m-s+1)p^s}{1.2.3\cdots s} = (1+p)^m \int \frac{x^{s-1}dx}{(1+x)^{m+1}},$$

whence we deduce

$$\sum \frac{m(m-1)\cdots(m-s+1)}{1.2.3\cdots s}p^s = (1+p)^m \frac{\int \frac{x^{s-1}dx}{(1+x)^{m+1}}}{\int \frac{x^{s-1}dx}{(1+x)^{m+1}}},$$

the integral of the numerator being taken from $x = p$ to $x = \infty$, and that of the denominator being taken from $x = 0$ to $x = \infty$. It will be easy to reduce to series these two integrals by the method of article I, we will have thus the sum of the first s terms of the binomial $(1+p)^m$, by a series as much more convergent as s and m will be greater numbers.

XXII.

We propose again to integrate, by approximation, the equation in finite differences

$$0 = (2+4s)y_s - (s+1)y_{s+1}.$$

If we make in it

$$y_s = \int x^s \phi\, dx,$$

and if we suppose $x^s = \delta y$, we will have

$$0 = \int \phi\, dx \left[(2-x)\delta y + (4x - x^2)\frac{d\,\delta y}{dx}\right],$$

whence we deduce the two equations

$$\begin{aligned}
0 &= (2-x)\phi - \frac{d[x\phi(4-x)]}{dx},\\
0 &= x^{s+1}\phi(4-x).
\end{aligned}$$

The first equation gives, by integrating it,

$$\phi = \frac{\mathrm{A}}{\sqrt{4x - x^2}};$$

the second becomes thus

$$0 = x^{s+\frac{1}{2}}\sqrt{4-x}.$$

The limits of the integral $\int x^s \phi\, dx$ or $\mathrm{A}\int \frac{x^{s-\frac{1}{2}}\,dx}{\sqrt{4-x}}$ will be, consequently, $x = 0$ and $x = 4$. Let $\sqrt{4-x} = 2u$, we will have

$$y_s = \mathrm{A}2^{2s+1}\int (1-u^2)^{s-\frac{1}{2}}\, du,$$

this last integral being taken from $u = 0$ to $u = 1$.

In order to determine it by approximation, we will make

$$\frac{1}{s-\frac{1}{2}} = \alpha \quad \text{and} \quad 1 - u^2 = e^{-\alpha t^2},$$

that which gives

$$u = \sqrt{1 - e^{-\alpha t^2}}$$

and

$$\int (1-u^2)^{s-\frac{1}{2}}\, du = \int du\, e^{-t^2}.$$

We suppose

$$\sqrt{1-e^{-\alpha t^2}} = \alpha^{\frac{1}{2}} t(1+\alpha q^{(1)}t^2 + \alpha^2 q^{(2)}t^4 + \alpha^3 q^{(3)}t^6 + \alpha^4 q^{(4)}t^8 + \cdots);$$

by taking the logarithmic differences of the two members of this equation, we will have

$$\begin{aligned}&\frac{1+3\alpha q^{(1)}t^2 + 5\alpha^2 q^{(2)}t^4 + 7\alpha^3 q^{(3)}t^6 + \cdots}{t+\alpha q^{(1)}t^3 + \alpha^2 q^{(2)}t^5 + \alpha^3 q^{(3)}t^7 + \cdots}\\ &= \frac{\alpha t e^{-\alpha t^2}}{1-e^{-\alpha t^2}} = \frac{1-\alpha t^2 + \frac{1}{1.2}\alpha^2 t^4 - \frac{1}{1.2.3}\alpha^3 t^6 + \cdots}{t - \frac{\alpha t^3}{1.2} + \frac{\alpha^2 t^5}{1.2.3} - \frac{\alpha^3 t^7}{1.2.3.4} + \cdots};\end{aligned}$$

that which gives the general equation

$$\begin{aligned}0 = 2iq^{(i)} &- \frac{2i-3}{1.2} q^{(i-1)} + \frac{2i-6}{1.2.3} q^{(i-2)}\\ &- \frac{2i-9}{1.2.3.4} q^{(i-3)} + \frac{2i-12}{1.2.3.4.5} q^{(i-4)} - \cdots,\end{aligned}$$

$q^{(0)}$ being equal to unity. If we make successively, in this equation, $i=1$, $i=2$, $i=3,\ldots$, we will form as many equations, by means of which it will be easy to determine the coefficients $q^{(1)}$, $q^{(2)}$, $q^{(3)},\ldots$ Thus put, we will have

$$\int du(1-u^2)^{s-\frac{1}{2}} = \alpha^{\frac{1}{2}} \int dt\, e^{-t^2}(1+3\alpha q^{(1)}t^2 + 5\alpha^2 q^{(2)}t^4 + 7\alpha^3 q^{(3)}t^6 + \cdots).$$

The integral relative to u must be taken from $u=0$ to $u=1$; thus $-\alpha t^2$ being equal to $\log(1-u^2)$, the integral relative to t must be taken from $t=0$ to $t=\infty$; now we have, in this case,

$$\int t_{2r} dt\, e^{-t^2} = \frac{1.3.5\ldots(2r-1)}{2^r} \int dt e^{-t^2} = \frac{1.3.5\ldots(2r-1)}{2^{r+1}}\sqrt{\pi};$$

therefore

$$\begin{aligned}&\int du(1-u^2)^{s-\frac{1}{2}}\\ &= \frac{1}{2}\sqrt{\alpha\pi}\left(1+\frac{1.3}{2}\alpha q^{(1)} + \frac{1.3.5}{2^2}\alpha^2 q^{(2)} + \frac{1.3.5.7}{2^3}\alpha^3 q^{(3)} + \cdots\right)\end{aligned}$$

and, consequently,

$$y_s = \mathrm{A}2^{2s}\sqrt{\alpha\pi}\left(1+\frac{1.3}{2}\alpha q^{(1)} + \frac{1.3.5}{2^2}\alpha^2 q^{(2)} + \cdots\right).$$

Now, if s is a positive whole number, the integral of the proposed equation

$$0 = (2+4s)y_s - (s+1)y_{s+1}$$

is

$$y_s = \frac{y_1}{2}\frac{(s+1)(s+2)\cdots 2s}{1.2.3\ldots s};$$

but the equation

$$y_s = \mathrm{A}2^{2s+1} \int du(1-u^2)^{s-\frac{1}{2}}$$

gives

$$y_1 = 8\mathrm{A} \int du(1-u^2)^{\frac{1}{2}} = 2\mathrm{A}\pi,$$

whence we deduce

$$\mathrm{A} = \frac{y_1}{2\pi};$$

by comparing therefore the two preceding values of y_s, we will have

$$\frac{2^{2s}}{\sqrt{(s-\frac{1}{2})\pi}} \left(1 + \frac{1.3}{2}\alpha q^{(1)} + \frac{1.3.5}{2^2}\alpha^2 q^{(2)} + \frac{1.3.5.7}{2^3}\alpha^3 q^{(3)} + \cdots\right)$$
$$= \frac{(s+1)(s+2)(s+3)\cdots 2s}{1.2.3\ldots s}.$$

This last quantity is the middle term of the binomial $(1+1)^{2s}$; the preceding formula will give therefore this term by a highly convergent series, when s will be a large number. It follows thence that the ratio of the middle term of the binomial $(1+1)^{2s}$ to the sum of all its terms is equal to

$$\frac{1}{\sqrt{(s-\frac{1}{2})\pi}} \left(1 + \frac{1.3}{2}\alpha q^{(1)} + \cdots\right),$$

and consequently, when s is quite large, this ratio is very nearly equal to $\frac{1}{\sqrt{s\pi}}$.

XXIII.

We can arrive more simply to the preceding results by the following manner: for this, we name y_s the middle term of the binomial $(1+1)^{2s}$; it is clear that this term is equal to the term independent of $e^{\varpi\sqrt{-1}}$ in the expansion of the binomial $(e^{\varpi\sqrt{-1}} + e^{-\varpi\sqrt{-1}})^{2s}$; now, if we multiply this binomial by $d\varpi$, and if we take next the integral of it from $\varpi = 0$ to $\varpi = 180^\circ$, it is clear that this integral will be equal to πy_s; we will have therefore, by substituting $2\cos\varpi$ in the place of $e^{\varpi\sqrt{-1}} + e^{-\varpi\sqrt{-1}}$,

$$y_s = \frac{2^{2s}}{\pi} \int d\varpi \cos^{2s}\varpi.$$

This integral, taken from $\varpi = 0$ to $\varpi = 180^\circ$, is evidently the double of this same integral, taken from $\varpi = 0$ to $\varpi = 90^\circ$, that which gives

$$y_s = \frac{2^{2s+1}}{\pi} \int d\varpi \cos^{2s}\varpi,$$

this last integral being taken from $\varpi = 0$ to $\varpi = 90^\circ$; if we suppose $\sin\varpi = u$, we will have

$$y_s = \frac{2^{2s+1}}{\pi} \int du(1-u^2)^{s-\frac{1}{2}},$$

the integral being taken from $u = 0$ to $u = 1$, that which is conformed to that which we have found in the preceding section.

This method has the advantage of being extended to the determination of the middle term of the trinomial $(1 + 1 + 1)^s$, of the one of the quadrinomial $(1 + 1 + 1 + 1)^{2s}$, and thus in sequence. We will consider the trinomial $(1 + 1 + 1)^s$, and we name y_s its middle term; y_s will be equal to the term independent of $e^{\varpi\sqrt{-1}}$ in the expansion of the trinomial

$$(e^{\varpi\sqrt{-1}} + 1 + e^{-\varpi\sqrt{-1}})^s;$$

we will have consequently

$$y_s = \frac{1}{\pi}\int d\varpi(2\cos\varpi + 1)^s,$$

the integral being taken from $\varpi = 0$ to $\varpi = \pi$. The condition of the maximum of the function $(2\cos\varpi + 1)^s$ gives $\sin\varpi = 0$, so that the two limits $\varpi = 0$ and $\varpi = \pi$ correspond to the two maxima of this function; we will divide therefore the preceding integral into two others

$$\int d\varpi(2\cos\varpi + 1)^s \quad \text{and} \quad (-1)^s\int d\varpi(2\cos\varpi - 1)^s,$$

the first of these two integrals being taken from $\varpi = 0$ to the value of ϖ which renders null the quantity $2\cos\varpi + 1$, and the second integral being taken from $\varpi = 0$ to the value of ϖ which renders null the quantity $2\cos\varpi - 1$.

In order to obtain the first integral in a convergent series, we will make

$$(2\cos\varpi + 1)^s = 3^s e^{-t^2},$$

and, by supposing $\alpha = \frac{1}{s}$, we will have

$$3 - \varpi^2 + \frac{\varpi^4}{12} - \cdots = 3 - 3\alpha t^2 + \frac{3\alpha^2 t^4}{2} - \cdots,$$

whence we deduce, by the reversion of the series

$$\varpi = \alpha^{\frac{1}{2}} t\sqrt{3}\left(1 - \frac{\alpha t^2}{8} + \cdots\right),$$

hence

$$\int d\varpi(2\cos\varpi + 1)^s = \alpha^{\frac{1}{2}} 3^{s+\frac{1}{2}}\int dt\, e^{-t^2}\left(1 - \frac{3}{8}\alpha t^2 + \cdots\right).$$

The integral relative to t needing to be taken from $t = 0$ to $t = \infty$, we will have

$$\int d\varpi(2\cos\varpi + 1)^s = \frac{\alpha^{\frac{1}{2}} 3^{s+\frac{1}{2}}\sqrt{\pi}}{2}\left(1 - \frac{3\alpha}{8} + \cdots\right);$$

we will find in the same manner

$$\int d\varpi(2\cos\varpi - 1)^s = \frac{\alpha^{\frac{1}{2}}\sqrt{\pi}}{2}\left(1 - \frac{5\alpha}{16} + \cdots\right).$$

We will have therefore

$$y_s = \frac{3^{s+\frac{1}{2}}}{2\sqrt{s\pi}}\left(1 - \frac{3\alpha}{16} + \cdots\right) + \frac{(-1)^s}{2\sqrt{s\pi}}\left(1 - \frac{5\alpha}{16} + \cdots\right);$$

s being a very large number, this quantity is reduced very nearly to $\frac{3^{s+\frac{1}{2}}}{2\sqrt{s\pi}}$; the ratio of the middle term of the trinomial $(1+1+1)^s$ to the sum of all the terms is therefore then very nearly equal to $\frac{\sqrt{3}}{2\sqrt{s\pi}}$.

We can determine in the same manner the middle term of the polynomial $1+1+1+1+1+\cdots$, raised to a very great power; we will content ourselves to present here the first term of its value in series, to which it is reduced when the exponent of the power is infinite.

If the polynomial is composed of a number of terms even and equal to $2n$, it will have of middle term only so much as the power to which it is raised will be even; let $2s$ be this power and y_s the middle term of the polynomial raised to this power, we will have very nearly, by supposing n greater than unity

$$y_s = \frac{(2n)^{2s}\sqrt{3}}{\sqrt{(2n+1)(n+1)2s\pi}};$$

the ratio of this term to the sum of all the terms will be consequently very nearly equal to

$$\frac{\sqrt{3}}{\sqrt{(2n+1)(n+1)2s\pi}}.$$

If the polynomial is composed of a number of terms odd and equal to $2n+1$, by naming s the power to which it is raised, and y_s its middle term, we will have very nearly

$$y_s = \frac{(2n+1)^s\sqrt{3}}{\sqrt{n(n+1)2s\pi}};$$

thus the ratio of this term to the sum of all the terms of the polynomial is, in this case, very nearly equal to $\frac{\sqrt{3}}{\sqrt{n(n+1)2s\pi}}$.

XXIV.

We propose now to determine by approximation the quite extended terms of the expansion of any function of u. By representing this expanded function by the following series

$$y_0 + y_1 u + y_2 u^2 + y_3 u^3 + \cdots + y_s u^s + y_{s+1} u^{s+1} + \cdots,$$

we will seek the law which exists among the coefficients y_s, y_{s-1}, $y_{s-2}, \ldots$, and, if this law can be expressed by an equation linear in finite or infinitely small differences, of which the coefficients are some rational and entire functions of s, we will have, by article II, the value of y_s in a highly convergent series when s will be a large number.

We suppose, for example, that the proposed function is

$$(a + bu + cu^2 + hu^3 + \cdots)^\mu;$$

by taking the logarithmic differences of the two members of the equation

$$(a + bu + cu^2 + hu^3 + \cdots)^\mu = y_0 + y_1 u + y_2 u^2 + \cdots + y_s u^s + \cdots ,$$

we will have

$$\frac{\mu(b + 2cu + 3hu^2 + \cdots)}{a + bu + cu^2 + hu^3 + \cdots} = \frac{y_1 + 2y_2 u + \cdots + sy_s u^{s-1} + \cdots}{y_0 + y_1 u + y_2 u^2 + \cdots + y_s u^s + \cdots}.$$

If we deliver this equations from fractions and if we equate to zero the coefficients of the similar powers of u, we will have the general equation

$$0 = asy_s + b(s - 1 - \mu)y_{s-1} + c(s - 2 - 2\mu)y_{s-2} + \cdots ;$$

if we suppose

$$y_s = \int x^{s-1}\phi\, dx$$

and if we designate x^{s-1} by δy, we will have

$$0 = \int \phi\, dx \left[a - \frac{\mu b}{x} - (2\mu + 1)\frac{c}{x^2} - \cdots + \frac{d\,\delta y}{dx}\left(ax + b + \frac{c}{x} + \cdots\right)\right],$$

whence we deduce the two equations

$$0 = \phi\, dx \left[a - \frac{\mu b}{x} - (2\mu + 1)\frac{c}{x^2} - \cdots\right] - d\left[\phi\left(ax + b + \frac{c}{x} + \cdots\right)\right],$$

$$0 = x^s\phi\left(a + \frac{b}{x} + \frac{c}{x^2} + \cdots\right).$$

The first gives, by integrating it

$$\phi = \mathrm{A}\left(a + \frac{b}{x} + \frac{c}{x^2} + \frac{h}{x^3} + \cdots\right)^\mu,$$

so that we will have ϕ by changing, in the proposed function, u into $\frac{1}{x}$, and by multiplying it by an arbitrary constant A, that which is generally true, whatever be this function.

The second equation will become

$$0 = x^s\left(a + \frac{b}{x} + \frac{c}{x^2} + \frac{h}{x^3} + \cdots\right)^{\mu+1};$$

whence it follows that the limits of the integral $\int x^{s-1}\phi\,dx$ is $x = 0$, and x equal to any one of the roots of the equation

$$0 = a + \frac{b}{x} + \frac{c}{x^2} + \cdots$$

The number of these roots being equal to the degree of the differential equation

$$0 = asy_s + b(s-1-\mu)y_{s-1} + \cdots\ ,$$

we have as many particular values of y_s as there are units in this degree, and their sum will be the complete expression of this variable.

This method can serve further to determine the infinitely small highly elevated differences of the function $(a+bz+cz^2+hz^3+\cdots)^\mu$, taken relatively to z; because, if we name s the degree of this difference, we will have

$$\frac{d^s(a+bz+cz^2+hz^3+\cdots)^\mu}{dz^s} = \frac{d^s[a+b(z+u)+c(z+u)^2+h(z+u)^3+\cdots]^\mu}{du^s},$$

provided that we suppose $u = 0$ after the differentiations in the second member of this equation. Now, if we designate by y_s the coefficient of u^s in the expansion of $[a+b(z+u)+c(z+u)^2+\cdots]^\mu$, the second member of the preceding equation will be evidently equal to $1.2.3\ldots sy_s$; we will have therefore

$$\frac{d^s(a+bz+cz^2+hz^3+\cdots)^\mu}{dz^s} = 1.2.3\ldots sy_s.$$

s being a very large number, we will have, by No. XIX, the product $1.2.3\ldots s$ in a highly convergent series; we have besides, by that which precedes

$$y_s = \mathrm{A}\int x^{s-1}dx\left[a+b\left(z+\frac{1}{x}\right)+c\left(z+\frac{1}{x}\right)^2+h\left(z+\frac{1}{x}\right)^3+\cdots\right]^\mu,$$

by taking as many similar terms as there are units in the degree of the function $a+bz+cz^2+\cdots$ and by integrating them from $x = 0$ to x successively equal to the different roots of the equation

$$0 = a+b\left(z+\frac{1}{x}\right)+c\left(z+\frac{1}{x}\right)^2++\cdots$$

We will have easily these integrals in convergent series by the method of article I.

We will determine, by this method, the $(s+1)^{\text{st}}$ difference of the angle of which z is the sine; if we name θ that angle, we will have $\frac{d\theta}{dz} = \frac{1}{\sqrt{1-z^2}}$, hence

$$\frac{d^{s+1}\theta}{dz^{s+1}} = \frac{d^s(1-z^2)^{-\frac{1}{2}}}{dz^s};$$

by expanding this difference, we have

$$\frac{d^{s+1}\theta}{dz^{s+1}} = \frac{1.2.3\ldots s}{(1-z^2)^{s+\frac{1}{2}}}\left[z^s + \frac{1}{2}\frac{s(s-1)}{1.2}z^{s-2} + \frac{1.3}{2.4}\frac{s(s-1)(s-2)(s-3)}{1.2.3.4}z^{s-4} \right.$$
$$+\frac{1.3.5}{2.4.6}\frac{s(s-1)(s-2)(s-3)(s-4)(s-5)}{1.2.3.4.5.6}z^{s-6}$$
$$\left. + \cdots \right].$$

The law of this expression is easy to grasp; but the calculation of it would be impractical if s were a large number such as ten thousand. In order to have, in this case, its value by a highly convergent series, we name y_s the coefficient of u^s in the expansion of the function $[1-(z-u)^2]^{-\frac{1}{2}}$; we will have

$$\frac{d^s(1-z^2)^{-\frac{1}{2}}}{dz^s} = 1.2.3\ldots sy_s;$$

we have besides, by the preceding section,

$$y_s = \mathrm{A}\int x^{s-1}dx\left[1-\left(z+\frac{1}{x}\right)^2\right]^{-\frac{1}{2}} + \mathrm{A}'\int x^{s-1}dx\left[1-\left(z+\frac{1}{x}\right)^2\right]^{-\frac{1}{2}},$$

the first integral being taken from $x=0$ to one of the values of x which renders null the function $\left[1-\left(z+\frac{1}{x}\right)^2\right]^{-\frac{1}{2}}$, and the second integral being taken from $x=0$ to the other value of x which renders this same function null. These two values are

$$z = -\frac{1}{1+z} \qquad \text{and} \qquad x = \frac{1}{1-z};$$

by supposing therefore $x = \frac{z+\cos\varpi}{1-z^2}$, we will transform the preceding expression of y_s into this one

$$y_s = \frac{\mathrm{B}}{(1-z^2)^s}\int d\varpi(z+\cos\varpi)^s + \frac{\mathrm{B}'}{(1-z^2)^s}\int d\varpi(z+\cos\varpi)^s,$$

the first integral being taken from $\varpi=0$ to the value of ϖ, of which the cosine is $-z$, and the second integral being taken from that value to $\varpi=\pi$. In order to determine the two arbitraries B and B$'$, we will observe that

$$y_0 = \frac{1}{\sqrt{(1-z^2)}} = \mathrm{B}\int d\varpi + \mathrm{B}'\int d\varpi,$$
$$y_1 = \frac{z}{(1-z^2)^{\frac{3}{2}}} = \frac{\mathrm{B}}{1-z^2}\int d\varpi(z+\cos\varpi) + \frac{\mathrm{B}'}{1-z^2}\int d\varpi(z+\cos\varpi);$$

whence it is easy to conclude

$$\mathrm{B} = \mathrm{B}' = \frac{1}{\pi\sqrt{1-z^2}},$$

hence

$$y_s = \frac{1}{\pi(1-z^2)^{s+\frac{1}{2}}}\left[\int d\varpi(z+\cos\varpi)^s + (-1)^s\int d\varpi(\cos\varpi - z)^s\right],$$

the first integral being taken from $\varpi = 0$ to $z + \cos\varpi = 0$, and the second integral being taken from $\varpi = 0$ to $z - \cos\varpi = 0$. Let

$$\frac{1}{s} = \alpha \quad \text{and} \quad (z+\cos\varpi)^s = (1+z)^s e^{-t^2};$$

we will have

$$\varpi = \alpha^{\frac{1}{2}} t\sqrt{2(1+z)}\left[1 - \frac{\alpha(2-z)}{12}t^2 + \cdots\right],$$

whence it is easy to conclude

$$\int d\varpi(z+\cos\varpi)^s = \frac{\alpha^{\frac{1}{2}}(1+z)^{s+\frac{1}{2}}\sqrt{2\pi}}{2}\left[1 - \frac{\alpha(2-z)}{8} + \cdots\right].$$

By changing z into $-z$, we will have

$$\int d\varpi(z-\cos\varpi)^s = \frac{\alpha^{\frac{1}{2}}(1-z)^{s+\frac{1}{2}}\sqrt{2\pi}}{2}\left[1 - \frac{\alpha(2+z)}{8} + \cdots\right],$$

hence

$$\begin{aligned} y_s =& \frac{1}{(1-z)^{s+\frac{1}{2}}\sqrt{2s\pi}}\left[1 - \frac{\alpha(2-z)}{8} + \cdots\right] \\ &+ \frac{(-1)^s}{(1+z)^{s+\frac{1}{2}}\sqrt{2s\pi}}\left[1 - \frac{\alpha(2+z)}{8} + \cdots\right]. \end{aligned}$$

By multiplying this value by the product $1.2.3\ldots s$, which, by No. XIX, is equal to

$$s^{s+\frac{1}{2}}e^{-s}\left(1 + \frac{\alpha}{12} + \cdots\right),$$

we will have the value in series of $\frac{d^{s+1}\theta}{dz^{s+1}}$, and we will find that, s being very great, this value is reduced to very nearly $\frac{s^s e^{-s}}{(1-z)^{s+\frac{1}{2}}}$. It is remarkable that the expression which we have given above of this difference, and which becomes very composite when s is a great number, is reduced thus to an approximate value so simple.

XXV.

Here now is a general method to have in convergent series the differences and the integrals quite elevated, either finite, or infinitely small of a function y_s. We will begin by reducing this function to some terms of one or of the other of these two forms $\mathrm{A}\int x^s\phi\,dx$, $\mathrm{A}\int e^{-sx}\phi\,dx$; we will observe next that the infinitely small n^{th} difference of $\mathrm{A}\int x^s\phi\,dx$ is $\mathrm{A}\int x^s ds^n\,\phi\,dx\,(\log x)^n$, and that its n^{th} finite difference is $\mathrm{A}\int x^s\phi\,dx(x-1)^n$. We will have therefore

$$\frac{d^n y_s}{ds^n} = \mathrm{A}\int x^s\,\phi\,dx\,(\log x)^n + \cdots,$$

$$\triangle^n y_s = \mathrm{A}\int x^s\phi\,dx(x-1)^n + \cdots,$$

the $+$ sign being relative to the other terms of the form $\mathrm{A}\int x^s \phi\, dx$ which can enter in the expression of y_s. If we make use of the form $\mathrm{A}\int e^{-sx}\phi\, dx$, we will have

$$\frac{d^n y_s}{ds^n} = (-1)^n \mathrm{A} \int x^n \, \phi \, dx \, e^{-sx} + \cdots ,$$
$$\triangle^n y_s = \mathrm{A} \int \phi \, dx e^{-sx} (e^{-x} - 1)^n + \cdots .$$

In order to have the n^{th} integrals, either finite, or infinitely small of y_s, it suffices to make n negative in these expressions; we can observe that they are generally true whatever be n, by supposing it even fractional, so that they offer a very simple means to interpolate the differences and the integrals of these functions.

As we are principally lead in the analysis of chances to some expressions which are only highly elevated finite differences of functions or any part of these differences, we are going to apply the preceding method and determine their value in convergent series.

XXVI.

We will consider first the function $\frac{1}{s^i}$; by designating it by y_s, it will be determined by the equation in the infinitely small differences

$$0 = s\frac{dy_s}{ds} + iy_s.$$

If we suppose in this equation

$$y_s = \int e^{-sx} \phi \, dx \quad \text{and} \quad e^{-sx} = \delta y,$$

it will become

$$0 = \int \phi \, dx \left(i \, \delta y + x \frac{d\, \delta y}{dx} \right),$$

whence we deduce the two equations

$$0 = i\phi - \frac{d(x\phi)}{dx}, \qquad 0 = x\phi \, \delta y.$$

The first gives, by integrating it,

$$\phi = \mathrm{A} x^{i-1},$$

and the second, for the limits of the integral $\int e^{-sx} \phi \, dx$,

$$x = 0 \qquad \text{and} \qquad x = \infty;$$

we will have therefore then

$$\frac{1}{s^i} = \mathrm{A} \int x^{i-1} dx \, e^{-sx}.$$

In order to determine the arbitrary constant A, we will observe that, s being 1, the first member of this equation is reduced to unity, that which gives

$$\mathrm{A} = \frac{1}{\int x^{i-1} dx\, e^{-x}},$$

hence

$$\frac{1}{s^i} = \frac{\int x^{i-1} dx\, e^{-sx}}{\int x^{i-1} dx\, e^{-x}};$$

we will have therefore

$$(\mu) \qquad \triangle^n \frac{1}{s^i} = \frac{\int x^{i-1} dx\, e^{-sx}(e^{-x}-1)^n}{\int x^{i-1} dx\, e^{-x}},$$

the integrals of the numerator and of the denominator being taken from $x = 0$ to $x = \infty$. The consideration of this formula is going to furnish us some interesting remarks on this analysis.

In order to expand it in series, we suppose

$$x^{i-1} e^{-sx}(e^{-x}-1)^n = a^{i-1} e^{-sa}(e^{-a}-1)^n e^{-t^2},$$

a being the value of x which corresponds to the maximum of the first member of this equation. If we make $x = a + \theta$, we will have, by taking the logarithms of each member and by expanding the logarithm of the first into a series ordered with respect to the powers of θ,

$$h\theta^2 + h'\theta^3 + h''\,\theta^4 + \cdots = t^2,$$

the quantities a, h, h', h'', ... being given by the following equations:

$$0 = \frac{i-1}{a} - s - \frac{ne^{-a}}{e^{-a}-1},$$

$$h = \frac{i-1}{2a^2} - \frac{n}{2}\frac{e^{-a}}{e^{-a}-1} + \frac{n}{2}\left(\frac{e^{-a}}{e^{-a}-1}\right)^2,$$

$$h' = \frac{i-1}{3a^2} + \frac{n}{6}\frac{e^{-a}}{e^{-a}-1} - \frac{n}{2}\left(\frac{e^{-a}}{e^{-a}-1}\right)^2 + \frac{n}{3}\left(\frac{e^{-a}}{e^{-a}-1}\right)^3,$$

$$h'' = \frac{i-1}{4a^2} - \frac{n}{24}\frac{e^{-a}}{e^{-a}-1} + \frac{7n}{24}\left(\frac{e^{-a}}{e^{-a}-1}\right)^2 - \frac{n}{2}\left(\frac{e^{-a}}{e^{-a}-1}\right)^3 + \frac{n}{4}\left(\frac{e^{-a}}{e^{-a}-1}\right)^4,$$

$$\ldots$$

We will have therefore, by the reversion of the series,

$$\theta = \frac{t}{\sqrt{h}}\left(1 - \frac{h't}{2h\sqrt{h}} + \frac{5h'^2 - 4hh''}{8h^3}t^2 + \cdots\right),$$

and this series will be so much more convergent as one of the numbers n or i will be greater. By substituting this value of θ into the function $\int d\theta\, e^{-t^2}$ and by taking the integral from $t = -\infty$ to $t = \infty$, we will have

$$\int x^{i-1} dx\, e^{-sx}(e^{-x}-1)^n = a^{i-1} e^{-sa}(e^{-a}-1)^n \frac{\sqrt{\pi}}{\sqrt{h}}\left(1 + \frac{15h'^2 - 12hh''}{16h^3} + \cdots\right);$$

we have moreover

$$\int x^{i-1}dx\, e^{-x} = \frac{1}{i}\int x^i dx\, e^{-x},$$

and by No. XIX

$$\int x^i dx\, e^{-x} = i^{i+\frac{1}{2}}e^{-i}\sqrt{2\pi}\left(1+\frac{1}{12i}+\cdots\right).$$

By dividing therefore the one by the other the two values of

$$\int x^{i-1}dx\, e^{-sx}(e^{-x}-1)^n \quad \text{and of} \quad \int x^{i-1}dx\, e^{-x},$$

we will have

$$\triangle^n \frac{1}{s^i} = \frac{\left(\frac{a}{i}\right)^{i-1} e^{i-sa}(e^{-a}-1)^n}{\sqrt{2hi}}\left(1+\frac{15h'^2-12hh''}{16h^3}+\cdots-\frac{1}{12i}+\cdots\right).$$

XXVII.

In order to have the n^{th} finite difference of the positive power s^i, it suffices (No. XVI) to change in this equation i into $-i$, and we will have

$$(\mu') \quad \left\{\begin{aligned} \triangle^n s^i =& (s+n)^i - n(s+n-1)^i \\ &+\frac{n(n-1)}{1.2}(s+n-2)^i - \frac{n(n-1)(n-2)}{1.2.3}(s+n-3)^i+\cdots \\ =& \frac{\left(\frac{i}{a}\right)^{i+1} e^{sa-i}(e^a-1)^n}{\sqrt{\frac{i(i+1)}{a^2}-ni\frac{e^a}{(e^a-1)^2}}}\left(1+\frac{15l'^2-12ll''}{16l^3}+\cdots+\frac{1}{12i}+\cdots\right) \end{aligned}\right.$$

a, l, l', l'', … being given by the following equations:

$$\begin{aligned} 0 &= \frac{i+1}{a} - s - \frac{ne^a}{e^a-1}, \\ l &= -\frac{i+1}{2a^2} - \frac{n}{2}\frac{e^a}{e^a-1} + \frac{n}{2}\left(\frac{e^a}{e^a-1}\right)^2, \\ l' &= -\frac{i+1}{3a^2} + \frac{n}{6}\frac{e^a}{e^a-1} - \frac{n}{2}\left(\frac{e^a}{e^a-1}\right)^2 + \frac{n}{3}\left(\frac{e^a}{e^a-1}\right)^3, \\ l'' &= \frac{i+1}{4a^2} - \frac{n}{24}\frac{e^a}{e^a-1} + \frac{7n}{24}\left(\frac{e^a}{e^a-1}\right)^2 - \frac{n}{2}\left(\frac{e^a}{e^a-1}\right)^3 + \frac{n}{4}\left(\frac{e^a}{e^a-1}\right)^4, \\ &\cdots \end{aligned}$$

We will arrive to the same result by resolving directly, by the method of No. XV, the equation in the finite and infinitely small differences

$$0 = \triangle^n\left(iy_s - s\frac{dy_s}{ds}\right)$$

or this one

$$0=(s+n)\triangle\frac{dy'_s}{ds}+n\frac{dy'_s}{ds}-i\triangle y'_s,$$

in which $y'_s=\triangle^{n-1}y_s$.

We suppose $i+1$ rather great, relatively to $n+s$, in order that $e^{\frac{i+1}{n+s}}$ be of the same order as i; the equation

$$0=\frac{i+1}{a}-s-\frac{ne^a}{e^a-1}$$

will give very nearly

$$a=\frac{i+1}{n+s}\left(1-\frac{n}{n+s}e^{\frac{-i}{n+s}}\right),$$

and if, for brevity, we make $e^{\frac{-i}{n+s}}=q$, we will find, by considering only the first term of the expression of $\triangle^n s^i$ and by making all the convenient reductions, this quite simple expression

$$\triangle^n s^i=(n+s)^i e^{-nq},$$

so that, if i is infinite relatively to $n+s$, that which gives $q=0$, we will have

$$\triangle^n s^i=(s+n)^i;$$

it is easy moreover to be assured of it *a priori* by considering that the quantity $(s+n)^i-n(s+n-1)^i+\cdots$ is reduced then to its first term.

XXVIII.

The series (μ') ceases to be convergent when a is a very small number of the order $\frac{1}{n}$, because then it is clear that, the quantities l, $l', l'', \ldots$ forming an increasing progression, each term of the series is of the same order as the one which precedes it. In order to determine into what case a is very small, we take the equation

$$0=\frac{i+1}{a}-s-\frac{ne^a}{e^a-1};$$

we can transform it into the following

$$0=\frac{i+1}{a}-s-\frac{n}{a}\left(1+\frac{a}{2}+\cdots\right),$$

whence we deduce to very nearly, under the supposition of a not very large,

$$a=\frac{i+1-n}{s+\frac{n}{2}}.$$

Thus a will be very small whenever the difference $i-n$ will be not very large relatively to $s+\frac{n}{2}$; in this case, we will determine $\triangle^n s^i$ by the following method.

We take the equation

$$\triangle^n s^i=\frac{\int\frac{dx}{x^{i+1}}e^{-sx}(e^{-x}-1)^n}{\int\frac{dx}{x^{i+1}}e^{-x}},$$

into which formula (μ) is changed from No. XXVI when we make i negative and equal to $-i$; we can put the factor $(e^{-x}-1)^n$ under this form

$$\begin{aligned}e^{-\frac{nx}{2}}\left(e^{-\frac{x}{2}}-e^{+\frac{x}{2}}\right)^n =&(-1)^n e^{-\frac{nx}{2}}x^n\left(1+\frac{1}{1.2.3}\frac{x^2}{2^2}+\frac{1}{1.2.3.4.5}\frac{x^4}{2^4}+\cdots\right)^n\\ =&(-1)^n e^{-\frac{nx}{2}}x^n\left[1+\frac{nx^2}{24}+\frac{n(5n-2)}{15.16.24}x^4+\cdots\right];\end{aligned}$$

we will have therefore

$$\int\frac{dx}{x^{i+1}}e^{-sx}(e^x-1)^n=(-1)^n\int\frac{dx}{x^{i+1-n}}e^{-\left(s+\frac{n}{2}\right)x}\left(1+\frac{nx^2}{24}+\cdots\right).$$

If we make $\left(s+\frac{n}{2}\right)x=x'$, we will have generally

$$\int\frac{dx}{x^r}e^{-\left(s+\frac{n}{2}\right)x}=\left(s+\frac{n}{2}\right)^{r-1}\int\frac{dx^ie^{-x'}}{x'^r},$$

and by No. XX we have

$$\int\frac{dx^ie^{-x'}}{x'^r}=\frac{2\pi(-1)^{r-1}}{\int x'^{r-1}dx'e^{-x'}}=\frac{2\pi(-1)^{r-\frac{1}{2}}}{(r-1)(r-2)(r-3)\cdots};$$

hence, we will have

$$(\mu'')\quad\left\{\begin{aligned}\triangle^n s^i =&(i-n+1)(i-n+2)\cdots i\left(s+\frac{n}{2}\right)^{i-n}\\ &\times\left[1+(i-n)(i-n-1)\frac{n}{24\left(s+\frac{n}{2}\right)^2}\right.\\ &\left.+(i-n)(i-n-1)(i-n-2)(i-n-3)\frac{n(5n-2)}{15.16.24\left(s+\frac{n}{2}\right)^4}\right].\end{aligned}\right.$$

This series is highly convergent if $i-n$ is not very large relatively to $s+\frac{n}{2}$; it can moreover be used in the case where i is fractional; as for the product $(i-n+1)(i-n+2)\cdots i$, it will be easy to obtain it in series by No. XIX.

In the case where $i=n$, the preceding formula gives

$$\triangle^n s^i=1.2.3\ldots i,$$

that which conforms to that which we know besides.

XXIX.

Formulas (μ') and (μ'') of the two preceding sections suppose n equal or less than i; indeed, if we consider the expression

$$\triangle^n s^i=\frac{\int\frac{dx}{x^{i+1}}e^{-sx}(e^{-x}-1)^n}{\int\frac{dx}{x^{i+1}}e^{-x}},$$

of which the expansion has produced these formulas, we see that the limits of the integrals of the numerator and of the denominator being determined by equating to zero the quantities under the $\int$ signs, these limits will be all imaginaries when $i+1$ will be greater than n, instead that, in the case when $i+1$ will be less than n, the limits of the integrals of the numerator will be real, while those of the integral of the denominator will be imaginaries; it is necessary therefore then to restore these last limits to the real state. In order to arrive there, we will observe that we have generally

$$\int x^{i-1}dx\, e^{-x} = \frac{\int x^{i+r}dx\, e^{-x}}{i(i+1)(i+2)\cdots(i+r)};$$

if we make in this expression i negative and equal to $-r-\frac{m}{n}$, m being less than n, we will have

$$\int \frac{dx\, e^{-x}}{x^{i+1}} = \frac{(-1)^{r+1}\int x^{-\frac{m}{n}}\,dx\, e^{-x}}{\frac{m}{n}\left(1+\frac{m}{n}\right)\left(2+\frac{m}{n}\right)\cdots i}.$$

Now we have, by No. XIX,

$$\left(1+\frac{m}{n}\right)\left(2+\frac{m}{n}\right)\cdots i = \frac{\int x^{i}dx\, e^{-x}}{\int x^{\frac{m}{n}}\,dx\, e^{-x}},$$

hence

$$\int \frac{dx\, e^{-x}}{x^{i+1}} = \frac{(-1)^{r+1}\int x^{-\frac{m}{n}}\,dx\, e^{-x}\int x^{\frac{m}{n}}\,dx\, e^{-x}}{m\int x^{i}dx\, e^{-x}};$$

this is the expression of $\int\frac{dx\, e^{-x}}{x^{i+1}}$ of which we must make use in the case which we examine here.

If we make $x=t^n$, we will have

$$\begin{aligned}\frac{m}{n}\int x^{-\frac{m}{n}}\,dx\, e^{-x}\int x^{\frac{m}{n}}\,dx\, e^{-x} &= \frac{n^3}{m}\int t^{n-m-1}dt\, e^{-t^n}\int t^{n+m-1}dt\, e^{-t^n}\\ &= n^2\int t^{n-m-1}dt\, e^{-t^n}\int t^{m-1}dt\, e^{-t^n},\end{aligned}$$

and equation (T) of No. IV will give, by changing r into $m+1$,

$$n^2\int t^{m-1}dt\, e^{-t^n}\int t^{n-m-1}dt\, e^{-t^n} = \frac{\pi}{\sin\frac{m\pi}{n}};$$

we will have therefore

$$\int \frac{dx\, e^{-x}}{x^{i+1}} = \frac{(-1)^{r+1}\pi}{\sin\frac{m\pi}{n}\int x^{i}dx\, e^{-x}},$$

whence we deduce, by substituting this value into the preceding expression of $\triangle^n s^i$,

$$(\mu''') \qquad \triangle^n s^i = (-1)^{r+1}\frac{\sin\frac{m\pi}{n}}{\pi}\int x^{i}dx\, e^{-x}\int\frac{dx}{x^{i+1}}e^{-sx}(e^{-x}-1)^n,$$

the two integrals being taken from $x=0$ to $x=\infty$. If i is a very large number, we will have the first in a convergent series by No. XIX, and the method of No. XXVI will give the second in a similarly convergent series when the difference $n - i$ will be great; in the case where it will be not very large relatively to $s + \frac{n}{2}$, the method of No. XXVIII will give for the expression of $\triangle^n s^i$ a convergent series analogous to the series (μ''). We can observe that, if i is a whole number, we will have $m = 0$; formula (μ''') will give therefore then $\triangle^n s^i = 0$, that which accords with that which we know besides.

We suppose $i = \frac{m}{n} = 0$, we will have, r being equal to zero,

$$r = 0, \qquad \sin \frac{m}{n}\pi = \frac{m}{n}\pi = i\pi$$

and

$$\triangle^n s^i = \triangle^n \frac{s^i - 1}{i} = \triangle^n \log s;$$

formula (μ''') will give therefore

$$\triangle^n \log s = -\int \frac{e^{-sx}dx}{x}(e^{-x} - 1)^n,$$

whence it is easy to conclude, by No. XXVII,

$$\begin{aligned}\triangle^n \log s &= \log(s + n) - n\log(s + n - 1) + \frac{n(n-1)}{1.2}\log(s + n - 2) - \cdots \\ &= \frac{e^{sa}(e^a - 1)^n\sqrt{2\pi}}{\sqrt{\frac{na^2e^a}{(e^a-1)^2} - 1}}(1 + \cdots),\end{aligned}$$

a being given by the equation

$$0 = \frac{1}{a} - s - \frac{ne^a}{e^a - 1}.$$

XXX.

We can extend the method of the preceding sections to the determination of the n^{th} difference of any power of a rational function of s; it suffices for this to reduce this function to the form $\int x^s\phi\,dx$; now, by designating it by y_s, we will have between y_s and its difference dy_s and equation of this form

$$\frac{dy_s}{ds} = \mathrm{M}y_s,$$

M being a rational function of s. By applying therefore to this equation the methods of article II, we will have ϕ by a differential equation, of a degree equal to the highest power of s in M; this last equation we be generally integrable only in the case where the exponent of s in M does not surpass unity; but we will have in every case the n^{th} finite difference of y_s, by means of multiple integrals, in the following manner.

We will consider the function $\frac{1}{(s+p)^i(s+p')^{i'}\cdots}$, to which we can restore all the powers of the rational functions of s and their products, the exponents i, i', ... can be supposed negatives. If, in the integral $\int x^{i-1}dx\,e^{-(s+p)x}$, taken from $x = 0$ to $x = \infty$, we suppose $(s + p)x = x'$, it will become

$$\frac{1}{(s+p)^i}\int x'^{i-1}dx'\,e^{-x'},$$

the integral relative to x' being taken similarly from $x'=0$ to $x'=\infty$; by comparing these two integrals, we will have

$$\frac{1}{(s+p)^i}=\frac{\int x^{i-1}dx\,e^{-(s+p)x}}{\int x^{i-1}dx\,e^{-x}},$$

the integrals of the numerator and of the denominator being taken from $x=0$ to $x=\infty$.

It follows thence that

$$\frac{1}{(s+p)^i(s+p')^{i'}\cdots}=\frac{\int x^{i-1}x'^{i'-1}\cdots dx\,dx'\cdots\ e^{-px-p'x'-\cdots-s(x+x'+\cdots)}}{\int x^{i-1}x'^{i'-1}\cdots dx\,dx'\cdots\ e^{-x-x'-\cdots}},$$

the integrals relative to x, x', ... being taken from the null values of these variables to their infinite values; we will have therefore

$$\begin{aligned}&\triangle^n\frac{1}{(s+p)^i(s+p')^{i'}\cdots}\\ &\quad=\frac{\int x^{i-1}x'^{i'-1}\cdots dx\,dx'\cdots\ e^{-px-p'x'-\cdots-s(x+x'+\cdots)}(e^{-x-x'-\cdots}-1)^n}{\int x^{i-1}x'^{i'-1}\cdots dx\,dx'\cdots\ e^{-x-x'-\cdots}}.\end{aligned}$$

We will reduce easily into convergent series the numerator and the denominator of this expression by the method of No. VII; and, if we change in this series the signs of i, i', ..., we will have the approximate values of $\triangle^n(s+p)^i(s+p')^{i'}\cdots$, on which we must make some remarks analogous to those which we have made in the preceding sections on the approximate value of $\triangle^n s^i$.

If we suppose n, i, i', ... some very large numbers, we will find easily, by No. VII, that we have very nearly

$$\begin{aligned}&\triangle^n(s+p)^i(s+p')^{i'}\cdots\\ &\quad=\frac{\left(\frac{i}{a}\right)^{i+1}\left(\frac{i'}{a'}\right)^{i'+1}\cdots e^{(s+p)a+(s+p')a'+\cdots-i-i'-\cdots}(e^{a+a'+\cdots}-1)^n}{\sqrt{\left[\frac{i(i+1)}{a^2}-\frac{nie^{a+a'+\cdots}}{(e^{a+a'+\cdots}-1)^2}\right]\left[\frac{i'(i'+1)}{a'^2}-\frac{nie^{a+a'+\cdots}}{(e^{a+a'+\cdots}-1)^2}\right]\cdots}},\end{aligned}$$

a, a', ... being determined by the equations

$$\begin{aligned}0&=\frac{i+1}{a}-s-p-\frac{ne^{a+a'+\cdots}}{e^{a+a'+\cdots}-1},\\ 0&=\frac{i'+1}{a'}-s-p'-\frac{ne^{a+a'+\cdots}}{e^{a+a'+\cdots}-1},\\ &\cdots\end{aligned}$$

XXXI.

The n^{th} finite difference of $\frac{1}{(s+p)^i(s+p')^{i'}\cdots}$ is equal to the product of $(-1)^n$ by

$$\begin{aligned}&\frac{1}{(s+p)^i(s+p')^{i'}\cdots}-\frac{n}{(s+p+1)^i(s+p'+1)^{i'}\cdots}\\ &\quad+\frac{n(n-1)}{(s+p+2)^i(s+p'+2)^{i'}\cdots}-\cdots;\end{aligned}$$

we often have need, in the analysis of chances, to consider only the sum of any number of the first terms of this function; let's see how we can obtain it in convergent series.

We name S the sum of the r first terms of the preceding function; it is easy to be assured by the preceding section that, if we name Q the sum of the r first terms of the binomial $(1-e^{-x-x'-\cdots})^n$, we will have

$$\mathrm{S}=\frac{\int x^{i-1}x'^{i'-1}\cdots dx\,dx'\cdots\; e^{-px-p'x'-\cdots-s(x+x'+\cdots)}\mathrm{Q}}{\int x^{i-1}x'^{i'-1}\cdots dx\,dx'\cdots\; e^{-x-x'-\cdots}}$$

We have, by No. XXI,

$$\mathrm{Q}=\frac{(1-e^{-x-x'-\cdots})^n\int\frac{u^{r-1}du}{(1+u)^{n+1}}}{\int\frac{u^{r-1}du}{(1+u)^{n+1}}}$$

the integral of the numerator being taken from $u=-e^{-x-x'-\cdots}$ to $u=\infty$, and that of the denominator being taken from $u=0$ to $u=\infty$, so that we can put this expression of Q under the following form

$$\mathrm{Q}=(-1)^{r-1}\frac{(1-e^{-x-x'-\cdots})^n e^{-rx-rx'-\cdots}\int\frac{(1-u)^{r-1}du}{[1-e^{-x-x'-\cdots}(1+u)]^{n+1}}}{\int\frac{u^{r-1}du}{(1+u)^{n+1}}},$$

the integrals of the numerator and of the denominator being taken from $u=0$ to $u=\infty$; we will have therefore

$$\mathrm{S}=(-1)^{r-1}\frac{\int x^{i-1}x'^{i'-1}\cdots du\,dx\,dx'\cdots\; e^{-px-p'x'-\cdots-s(x+x'+\cdots)}(1-e^{-x-x'-\cdots})^n\frac{(1-u)^{r-1}}{[1-e^{-x-x'-\cdots}(1+u)]^{n+1}}}{\int x^{i-1}x'^{i'-1}\cdots du\,dx\,dx'\cdots\; e^{-x-x'-\cdots}\frac{u^{r-1}}{(1+u)^{n+1}}}$$

all the integrals being taken from the null values of the variables to their infinite values. There is no longer a question now but to reduce, by the method of No. VII, the numerator and the denominator of this expression into convergent series. The applications that we make, in the following article, of these researches, to diverse problems on chances, will shed a new day on this analysis.