BOOK I
CALCULUS OF GENERATING FUNCTIONS

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FIRST PART
GENERAL CONSIDERATIONS ON THE ELEMENTS OF MAGNITUDES

The notation of exponents, imagined by Descartes, has led Wallis and Newton to the consideration of fractional exponents, positive and negative, and to the interpolation of series. Leibnitz has rendered these exponents variables, that which has given birth to the exponential calculus and has completed the system of elements of finite functions. These functions are formed of exponential, algebraic and logarithmic quantities; quantities essentially distinct from one another. Integrations are not often reducible to finite functions. Leibnitz having adapted to his differential characteristic of the exponents, in order to express the repeated differentiations, he has been led by the analogy of the powers and of the differences, an analogy that Lagrange has followed by way of induction, in all his developments. The theory of generating functions extends this analogy to some unspecified characteristics and indicates it evidently. All theory of series and the integration of the equations in the differences result with an extreme facility from this theory. N° 1.

Chapter I. — CONCERNING GENERATING FUNCTIONS IN ONE VARIABLE

u being any function of a variable t and y_x being the coefficient of t^x in the development of this function, u is the *generating function* of y_x. If we multiply u by any function s of $\frac{1}{t}$, we will have a new generating function which will be that of a function of y_x, y_{x+1}, etc. By designating by ∇y_x this last function, us^i will be the generating function of $\nabla^i y_x$, so that the exponent of s, in the generating function, becomes the one of the characteristic ∇ in the engendered function. N° 2.

On the interpolation of the sequences in one variable, and on the integration of the linear differential equations.

Interpolation is reduced to determining the coefficient y_{x+1} of t^x in the development of $\frac{u}{t^i}$. We are able to give to $\frac{1}{t}$ an infinity of different forms: by elevating it

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. July 29, 2013 to the power i under these forms and passing again next from the generating functions to the coefficients, we have, under an infinity of corresponding forms, the expression of y_{x+i}. Application of this method to the series of which the successive differences of the terms decrease. N° 3.

Formulas in order to interpolate between an odd or even number of equidistant quantities. N° 4.

General formula of interpolation of series of which the last ratio of the terms is that of a series of which the general term is given by a linear equation in the differences, with constant coefficients. N° 5.

The formula is arrested when the relation of the terms is that of a similar series, and then it gives the integral of the linear equations in finite differences, of which the coefficients are constants. General integration of these equations, in the case even where they have a last term a function of the index. N° 6.

Formula of interpolation of the same series, ordered with respect to the successive differences of the principal variable. N° 7.

Passage of this formula, from the finite to the infinitely small. Interpolation of the series of which the last ratio of the terms is that of an equation in the infinitely small linear differences, with constant coefficients. Integration of this kind of equations, when also they have a last term. N° 8.

On the transformation of series. N° 9.

Theorems on the development of functions and of their differences into series.

We deduce from the calculus of generating formulas the formulas

$$'\Delta^n y_x = [(1+\Delta y_x)^i - 1]^n, \qquad '\Sigma^n y_x = [(1+\Delta y_x)^i - 1]^{-n},$$

Δ and Σ corresponding to the case where x varies by unity and $'\Delta$ and $'\Sigma$ corresponding to the case where x varies with i. We deduce from these formulas the following:

$$'\Delta^n y_x = \left(c^{\alpha \frac{dy_x}{dx}} - 1\right)^n, \qquad '\Sigma^n y_x = \left(c^{\alpha \frac{dy_x}{dx}} - 1\right)^{-n},$$

in which c designates the number of which the hyperbolic logarithm is unity, and $'\Delta$ and $'\Sigma$ correspond to the variation α of x. We transform the expression of $'\Delta_n y_x$ into this here

$$\left(c^{\frac{\alpha}{2}\frac{dy_{x+\frac{n\alpha}{2}}}{dx}} - c^{-\frac{\alpha}{2}\frac{dy_{x+\frac{n\alpha}{2}}}{dx}}\right)^n.$$

We arrive to these formulas

$$\begin{aligned} \frac{d^n y_x}{dx^n} &= [\log(1+\Delta y_x)]^n, \\ \int^n y_x dx^n &= [\log(1+\Delta y_x)]^{-n}. \end{aligned}$$

Analogy between the positive powers and the differences and between the negative powers and the integrals, based on this that the exponents of the powers, in the generating functions, are transported to the characteristics corresponding to the variable y_x. Generalization of the preceding results. N° 10.

Theorem analogous to the previous on the products of the many functions of one same variable and especially with respect to the product $p^x y_x$. N° 11.

Chapter II. — CONCERNING GENERATING FUNCTIONS IN TWO VARIABLES

u being a function of two variables t and t', and $y_{x,x'}$ being the coefficient of $t^x t'^{x'}$ in the development of this function, u is generating function of $y_{x,x'}$. If we multiply u by a function s of $\frac{1}{t}$ and $\frac{1}{t'}$, the coefficient of $t^x t'^{x'}$ in the development of this product will be a function of $y_{x,x'}$, $y_{x+1,x'}$, $y_{x,x'+1}$, etc.; by designating it by $\nabla y_{x,x'}$, us^i will be the generating function of $\nabla^i y_{x,x'}$. N° 12.

On the interpolation of the series in two variables and on the integration of linear equations in partial differences.

General formula of the interpolation of series of which the last ratio of the terms is that of a series of which the general term is given by a linear equation in partial differences, with constant coefficients. N° 13.

The formula is arrested when the relation of the terms is that of a similar series, and then it gives the integral of the linear equations in the partial finite differences, of which the coefficients are constants. This integral supposes that we know or that we can deduce from the conditions of the problem n arbitrary values of $y_{x,x'}$, by giving, for example, to x the n values 0, 1, 2, $\ldots$, $n-1$, x' being moreover unspecified. A very simple expression of $y_{x,x'}$, when these arbitrary functions in x' are given by some linear equations in the differences, with constant coefficients. N° 14.

General expression of $y_{x,x'}$ under the form of definite integral; important remark on the number of arbitrary functions which the integral of the equations in partial differences contains. N° 15.

Examination of some cases which escape from the general formula of integration given in that which precedes; in this case, the characteristics of the finite differences which the integrals contain have for exponents the variable indices of the equations in the partial differences. N° 16.

Integration of the equation

$$0 = \Delta^n y_{x,x'} + \frac{a}{\alpha}\Delta^{n-1\prime}\Delta y_{x,x'} + \frac{b}{\alpha^2}\Delta^{n-2} y_{x,x'} + \cdots ,$$

Δ corresponding to the variability of x of which the unit is the difference, and $'\Delta$ corresponding to the variability of x' of which α is the difference. We deduce from it the integral of the equation in the infinitely small and finite partial differences, that we obtain by changing, in the preceding, α into dx', and the characteristic $'\Delta$ into d. N° 17.

Theorems on the development into series of the functions of many variables.

These theorems are analogous to those who have been given previously with respect to the functions in one variable alone, and we recover the observed analogy between the positive powers and the differences, and between the negative powers and the integrals. N$^{\text{o}}$ 18.

Considerations on the passages from the finite to the infinitely small.

The consideration of these passages is very proper to clarify the most delicate points of the infinitesimal Calculus. It shows evidently that the quantities neglected in this Calculus remove nothing from its rigor. By applying it to the problem of the vibrating cords, it proves the possibility to introduce some arbitrary discontinuous functions into the integrals of the equations in the finite and infinitely small partial differences, and it gives the conditions of this discontinuity. N$^{\text{o}}$ 19.

General considerations on the generating functions.

To find the generating function of a given quantity by a linear equation in the finite differences, of which the coefficients are some rational and entire functions of the index. N$^{\text{o}}$ 20.

Expressions of the integrals of these equations as definite integrals. The functions under the integral sign $\int$ are of the same nature as the generating functions of the quantities given by these equations. Thus all the theorems deduced previously from the analogy of the powers and the differences are applied to these integrals. Their principle advantage is to furnish an approximation as handy as convergent of these quantities, when their index is a very great number. This method of approximation acquires a great extension by the passages from the positive to the negative and from the real to the imaginary, passages of which I have given the first traces in the *Mémoires de l'Académie des Sciences* of 1782. It seems, by the posthumous Works of Euler, that, toward the same time, this great geometer occupied himself with the same object. N$^{\text{o}}$ 21.

SECOND PART

THEORY OF THE APPROXIMATION OF FORMULAS WHICH ARE FUNCTIONS OF VERY LARGE NUMBERS

Chapter I. — ON THE INTEGRATION BY APPROXIMATION OF THE DIFFERENTIALS WHICH CONTAIN SOME FACTORS RAISED TO GREAT POWERS

Expression, in convergent series, of their integral taken between two given limits: the series ceases to be convergent near to the *maximum* of the function under the integral sign. Nº 22.

Expression, in convergent series, of the integral in this last case. Nº 23.

That which this series becomes when the integral is taken between two limits which render null the function under the integral sign. Its value depends then on integrals of the form $\int t^r \, dt \, e^{-t^n}$ and taken from t null to t infinity. We establish this theorem

$$n^2 \int t^{r-2} dt \, c^{-t^n} \int t^{n-r} dt \, c^{-t^n} = \frac{\pi}{\sin\left(\frac{r-1}{n}\right)\pi},$$

π being the semi-circumference of which the radius is unity. We deduce from it this remarkable result

$$\int dt \, c^{-t^2} = \frac{1}{2}\sqrt{\pi}.$$

Nº 24.

This last result gives, by the passage from the real to the imaginary,

$$\int dx \cos rx \, c^{-a^2 x^2} = \frac{\sqrt{\pi}}{2a} c^{-\frac{r^2}{4a^2}},$$

the integral being taken from x null to x infinity; a direct method which leads to this equation and from which we deduce the value of the integral when the quantity under the sign $\int$ is multiplied by x^{2n}: value of the integral

$$\int x^{2n\pm 1} dx \; \sin rx \, c^{-a^2 x^2}.$$

Nº 25.

We arrive to the formulas

$$\int \frac{dx \cos rx}{1+x^2} = \int \frac{x dx \sin rx}{1+x^2} = \pi \, c^{-r},$$

the integrals being taken from $x = -\infty$ to $x = +\infty$; and we deduce from it the integrals $\frac{M}{N} dx \begin{matrix}\cos\\ \sin\end{matrix} rx$, taken within the same limits, N being a rational and entire function of x, of a degree superior to M, and not having a real factor of first degree. Nº 26.

Expression of the integral $\int dt\, c^{-t^2}$ taken between the given limits, either as series, or as continued fraction. N° 27.

Approximation of the double, triple, etc. integrals of the differentials multiplied by some factors raised to high powers. Formulas in convergent series in order to integrate, within some given limits, the double integral $\iint y dx dx'$, y being a function of x and of x'. Examination of the case where the integral is taken very near the *maximum* of y. Expression of the integral as convergent series. N° 28.

Chapter II. — ON THE INTEGRATION BY APPROXIMATION OF LINEAR EQUATIONS IN FINITE AND INFINITELY SMALL DIFFERENCES

Integration of the equation in the finite differences

$$S = \mathrm{A}y_x + \mathrm{B}\Delta y_x + \mathrm{C}\Delta^2 y_x + \cdots ,$$

A, B, C being some rational and entire functions of s. If the variable y_s is expressed by the definite integral $\int x^s \phi\, dx$ or by this here $\int c^{-sx}\phi\, dx$, ϕ being function of x, we have, by the formulas of the preceding Chapter, the value of y_s in very convergent series, when the index s is a large number. In order to determine ϕ, we substitute for y_s, its expression as definite integral in the equation with the difference in y_x, which is partitioned into two others, of which the one is a differential equation in ϕ, which serves to determine this unknown; the other equation gives the limits of the definite integral. N° 29.

Integration of any number of linear equations in one index alone and having a last term, the coefficients of these equations being some rational and entire functions of this index. This method is able to be extended to the linear equations in differences either infinitely small, or into finite parts and into infinitely small parts. N° 30.

The principal difficulty of this analysis consists in integrating the differential equation in ϕ, which is integrable generally only in the case where the index s is only to the first power in the equation in the differences in y_s, which then is of the form $0 = \mathrm{V} + s\mathrm{T}$, V and T being some linear functions of y_s and of its differences, either finite, or infinitely small. Integral of this last equation, by a very convergent series, when s is a large number. Important remark on the extent of this series, which is independent of the limits of the definite integral by which y_s is expressed, and which subsists in the same case where the equation in the limits has only imaginary roots. When, in the equation in y_s, s surpasses the first degree, we can sometimes decompose it into many equations which contain only the first power of s. We can further, in many cases, integrate, by a very convergent approximation, the differential equation in ϕ. N° 31.

Integration of the equation

$$0 = \mathrm{V} + s\mathrm{T} + s'\mathrm{R},$$

V, T, R being unspecified linear functions of $y_{s,s'}$ and of its ordinary and partial differences, finite and infinitely small. N° 32.

Chapter III.— APPLICATION OF THE PRECEDING METHODS TO THE APPROXIMATION OF DIVERSE FUNCTIONS OF VERY GREAT NUMBERS

On the approximation of the products composed of a great number of factors and of the terms of polynomials raised to great powers.

The integral of the equation $0 = (s+1)y_x - y_{s-1}$, approximated by the methods of the preceding Chapter and compared to its finite integral, given, by a very convergent series, the product $(\mu+1)(\mu+2)\dots s$. By making s negative and passing from the positive to the negative and from the real to the imaginary, we arrive to this remarkable equation

$$\frac{2\pi(-1)^{\frac{1}{2}-\mu}}{\int x^{\mu-1}dxe^{-x}} = \int \frac{dx\, c^{-x}}{x^{\mu}},$$

the first integral being taken from x null to x infinity, and the last integral being taken between the imaginary limits of x which render null the function $\frac{c^{-x}}{x^{\mu}}$; that which gives an easy means to have the integral $\int \frac{dx \begin{matrix}\cos\\ \sin\end{matrix} x}{x^{\mu}}$, taken from x null to x infinity. This equation gives further the value of the integrals

$$\int \frac{d\varpi\ \cos\varpi}{1+\varpi^2}, \qquad \int \frac{d\varpi\ \sin\varpi}{1+\varpi^2}$$

taken from ϖ null to ϖ infinity. One finds $\frac{\pi}{2c}$ for these integrals; their accord with the results of the n° 26 proves the justice of these passages from the positive to the negative and from the real to the imaginary: these diverse results have been given in the *Mémoires de l'Académie des Sciences* for the year 1782. N° 33.

The approximate integral of the equation $0 = (a'+b')y_{s+1} - (a+bs)y_x$, whence we deduce, by a simple and very convergent series, the middle term or independent of a of the binomial $\left(a+\frac{1}{a}\right)^{2s}$. N° 34.

General method in order to have, by a convergent series, the middle term or independent of a, in the development of the polynomial

$$a^{-n} + a^{-n+1} + a^{-n+2} + \cdots + a^{n-1} + a^n$$

raised to a very high power. N° 35.

Expressions, in convergent series, of the coefficient of $a^{\pm l}$, in the development of this power, and of the sum of its coefficients, from the one of a^{-l} to the one of a^l. N° 36.

Integration by approximation of the equation in the differences $p^s = sy_s + (s-i)y_{s+i}$. One deduces from it the expression of the sum of the terms of the very high power of a binomial, by arresting its development at any term quite distant from the first. N° 37.

On the approximation of the very elevated differences infinitely small and finite of functions

Approximation of the very elevated infinitely small differences of the powers of a polynomial. Very approximate expression of the very elevated differential of an angle, taken with respect to its sine. N° 38.

Expressions by definite integrals of the finite and infinitely small differences of y_s, when we are arrived to give to it either of the forms $\int x^s \phi\, dx$, $\int c^{-sx} \phi\, dx$. N° 39.

Approximation by very convergent series of $\Delta^n \frac{1}{s}$, n being a large number. We deduce, by means of the passages from the positive to the negative and from the real to the imaginary, the approximation of $\Delta^n s$. The convergence of the series requires that i surpass n and that the difference $i - n$ is not too small with respect to $s + \frac{n}{2}$. Expression in series of $\Delta^n s^i$, in the last case. N° 40.

Expression of the difference $\Delta^n s^i$ when i is smaller than n. N° 41.

Expression of the sum of the terms of $\Delta^n s^i$, by arresting its development at the term in which the quantity raised to the power i commences to become negative. Approximation, by very convergent series, of the function

$$(n + r\sqrt{n})^{n \pm l} - n(n + r\sqrt{n} - 2)^{n \pm l} + \frac{n(n-1)}{1.2}(n + r\sqrt{n} - 4)^{n \pm l} - \cdots$$

in which we reject the terms where the quantity raised to the power $n \pm l$ is negative, l being a very considerable whole number with respect to n. N° 42.

Extension of the preceding methods to the very elevated finite differences of the form $\Delta^n (s+p)^i (s+p')^{i'} (s+p'')^{i''} \cdots$ N° 43.

General remarks on the convergence of the series. N° 44.