RECHERCHES

sur plusieurs points d'analyse relatifs à différens endroits des Mémoires précédens.

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1792–3 Berlin (1798), pp. 247–57.

FIRST MEMOIR.[1]

On the expression of the general term of the recurrent sequence, when the generating equation has some equal roots.

I have given in the Memoirs of 1775 (p. 185) a method & some very simple formulas in order to have the general term of a recurrent sequence, of which one knows the first terms. But these formulas have, as all those which are some functions of the different roots of a similar equation, the inconvenience to be able to serve only when all the roots are unequal. The case of the equality of two or many roots, demands some deductions & some transformations based on this principle of the differential calculus that some equal quantities can be supposed to differ among themselves by infinitely small quantities; but the application of this principle to the formulas of which there is question requires some particular attention, & gives place to some new & remarkable results for their simplicity; this is that which has engaged me to make it the matter of this Memoir.

1. I will commence by recalling the principal formulas from the place cited.

Let y_0, y_1, y_2, y_3, &c. y_x, y_{x+1}, y_{x+2}, &c. be the sequence in which one has constantly this equation among n consecutive terms.

$$Ay_x + By_{x+1} + Cy_{x+2} \text{ \&c.} + Ny^{x+n} = 0 \qquad \text{(A)}$$

A, B, C &c. being some constant coefficients any whatsoever. The expression of the general term y_x will be of this form

$$y_x = a\alpha^x + b\beta^x + c\gamma^x + \text{\&c.}$$

the quantities α, β, γ &c. being the different roots of the equation

$$A + By + Cy^2 = dY^3 = \text{\&c.} + Ny^n = 0 \qquad \text{(B)}$$

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[1]There are five memoirs in this sequence all published in the Mémoires de l'Acad. . .Berlin. 2 ° Sur les Sphéroides elliptiques, 1792/3 pp. 258–70. 3 ° Sur la méthode d'interpolation, 1792/3 pp. 271–88. 4 ° Sur l'équation séculaire de la Lune, 1792/3 pp. 289–99. 5 ° Sur une loi génerale d'Optique, 1803, pp. 3–12.

that I call the generating equation, & the coefficients $a,\ b,\ c$, &c. being of this form

$$a = \frac{y_{n-1} - (\beta+\gamma+\delta+\text{\&c.})y_{n-2} + (\beta\gamma+\beta\delta+\gamma\delta+\text{\&c.})y_{n-2} - \text{\&c.}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)\dots}$$
$$b = \frac{y_{n-1} - (\alpha+\gamma+\delta+\text{\&c.})y_{n-2} + (\alpha\gamma+\alpha\delta+\gamma\delta+\text{\&c.})y_{n-2} - \text{\&c.}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)\dots}$$

and thus in sequence.

I remark first that one can give to these expressions a more simple & more commodious form for the calculus, by observing that if in the product $(y-\beta)(y-\gamma)(y-\delta)\dots$ one changes according to the development of the powers $y^0, y^1,\ y^2, y^3$ etc. y^{n-1} into $y_0,\ y_1,\ y_2,\ y_3$ &c. y_{n-1} one will have the numerator of the expression of a; as likewise one will have the one of the expression of b by making the same change in the product $(y-\alpha)(y-\gamma)(y-\delta)\dots$; & thus of the others. So that with this condition one will be able to suppose first

$$a = \frac{(y-\beta)(y-\gamma)(y-\delta)\dots}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)\dots}, \qquad b = \frac{(y-\alpha)(y-\gamma)(y-\delta)\dots}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)\dots}$$
$$c = \frac{(y-\alpha)(y-\beta)(y-\delta)\dots}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}, \qquad \text{\&c.}$$

2. This put let $\beta = \alpha$, the first two terms $a\alpha^x$, $b\beta^x$ of the expression of y_x will become infinite.

We make for abridgment $\frac{a^x}{(\alpha-\gamma)(\alpha-\delta)\dots} = f.\alpha$ (function of α), one will have $\frac{\beta^x}{(\beta-\gamma)(\beta-\delta)\dots} = f.\beta$, & the two terms $a\alpha^x + b\beta^x$ will become $\frac{(y-\beta)(y-\gamma)(y-\delta)}{\alpha-\beta}f.\alpha + \frac{(y-\alpha)(y-\gamma)(y-\delta)\dots}{\beta-\alpha}f.\beta$.

We make now $\beta = \alpha + \omega$, ω being an infinitely small quantity, one will have $\alpha - \beta = -\omega$, $\beta - \alpha = \omega$,

$$\begin{aligned}(y-\beta)(y-\gamma)(y-\delta)\dots =&(y-\alpha)(y-\gamma)(y-\delta)\dots\\ &-\omega(y-\gamma)(y-\delta)\dots,\\ f\beta = f(a+\omega) =& fa + \frac{d.f\alpha}{d\alpha}\omega + \text{\&c.}\end{aligned}$$

Substituting these values into the two terms of which there is question, erasing that which is destroyed, & making next $\omega = 0$, one will have for result

$$(y-\gamma)(y-\delta)\dots \times f.\alpha + (y-\alpha)(y-\gamma)(y-\delta) \times \frac{d.f\alpha}{d\alpha}.$$

This is the value of the first two terms of the expression of y_x. And the third term $c\gamma^x$ of the same expression will become then because of $\alpha = \beta$,

$$\frac{(y-\alpha)^2(y-\delta)\dots}{(\gamma-\alpha)^2(\gamma-\delta)\dots}\gamma^x.$$

The value of the others will not be subject to any difficulty.

If besides $\beta = \alpha$ one has yet $\gamma = \alpha$, this which is the case of three equal roots, then $f\alpha$ would become infinite, in this way the value of the 3rd term; the first three terms would be therefore infinite, & it would be necessary to make anew $\gamma = \alpha + \omega$.

Let $\frac{\alpha^x}{(\alpha-\delta)(\alpha-\epsilon)\ldots} = f'.\alpha$, one will have $f\alpha = \frac{f'.\alpha}{\alpha-\gamma}$, & differentiating according to α,

$$\frac{d.f\alpha}{d\alpha} = \frac{\frac{d.f'\alpha}{d\alpha}}{\alpha-\gamma} - \frac{f'.\alpha}{(\alpha-\gamma)^2}.$$

Therefore making $\gamma = \alpha + \omega$, one will have

$$f\alpha = -\frac{f'\alpha}{\omega}, \qquad \frac{d.f\alpha}{d\alpha} = -\frac{d.f'.\alpha}{\omega\, d\alpha} - \frac{f'\alpha}{\omega^2}.$$

Moreover

$$(y-\gamma)(y-\delta)\ldots \text{ will become } (y-\alpha)(y-\delta)\ldots - \omega(y-\delta)\ldots,$$

&

$$(y-\alpha)(y-\gamma)(y-\delta)\ldots \text{ will become } (y-\alpha)^2(y-\delta)\ldots - \omega(y-\alpha)(y-\delta)\ldots$$

Finally the third term being represented by

$$\frac{(y-\alpha)^2(y-\delta)\ldots}{(\gamma-\alpha)^2} f'.\gamma$$

will become, by setting $\alpha+\omega$ for γ,

$$\frac{(y-\alpha)^2(y-\delta)\ldots}{\omega^2} f'(\alpha+\omega)$$

namely

$$\frac{(y-\alpha)^2(y-\delta)}{\omega^2}\left(f'\alpha + \omega\,\frac{d.f\alpha}{d\alpha} + \frac{\omega^2\, d^2.f'\alpha}{2d\alpha^2} + \right).$$

Making all these substitutions, erasing that which is destroyed, & making next $\omega = 0$, one will find for the value of the first three terms of y_x the quantity

$$(y-\delta)\ldots \times f'\alpha + (y-\alpha)(y-\delta)\ldots \\ \times \frac{d.f'\alpha}{d\alpha} + (y-\alpha)^2(y-\delta)\ldots \times \frac{d^2.f'\alpha}{2d\alpha^2}.$$

If one has still $\delta = \alpha$, so that the four roots $\alpha, \beta, \gamma, \delta$ make equals among themselves, one would find, by following the same march, that the first four terms of the expression of y_x namely $a\alpha^x + b\beta^x + c\gamma^x + d\delta^x$, of which each would be infinite, taken together would be reduced to the following quantity

$$(y-\epsilon)\ldots \times f'.\alpha + (y-\alpha)(y-\epsilon)\ldots \times \frac{d.f''\alpha}{d\alpha} \\ + (y-\alpha)^2(y-\epsilon)\ldots \frac{d^2.f''\alpha}{2d\alpha^2} \\ + (y-\alpha)^3(y-\epsilon)\ldots \times \frac{d^3.f''\alpha}{2.3\, d\alpha^3},$$

by making

$$f''.\alpha = \frac{\alpha^x}{(\alpha-\epsilon)(\alpha-\zeta)\dots}.$$

And thus in sequence, the law of the progression being visible by itself.

In order to employ these expressions, it will be necessary to develop the different products $(y-\gamma)(y-\delta)\dots$, $(y-\alpha)(y-\gamma)(y-\delta)\dots$, $(y-\delta)\dots$, $(y-\alpha)(y-\delta)\dots$, $(y-\alpha)^2(y-\delta)\dots$ and thus in sequence in powers of y, & to change next into these powers the exponents into indices, that is to say to change y^0, y^1, y^2 &c. into y_0, y_1, y_2 &c.; by converting the coefficients of these powers.

3. The difficulty which results from the equal roots is therefore resolved in a general manner; but the expressions which one comes to find being given by functions of all the roots α, β, γ &c., one can desire to have them as functions of the single root α, this which will give likewise to us simpler formulas.

For this we will remark, that since α, β, γ &c. are the roots of equation (B), one will have

$$A + By + Cy^2 + \text{&c.} + Ny^n = n(y-\alpha)(y-\beta)(y-\gamma)\dots$$

By making $y = \alpha$ one will have $A + B\alpha + C\alpha^2$+&c.+$N\alpha^n = 0$; subtracting this quantity from the first member of the preceding equation, & dividing next by $y-\alpha$ one will have

$$Q + Ry + Sy^2 + Ty^3 + \text{&c.} + Ny^{n-1} = N(y-\beta)(y-\gamma)(y-\delta)\dots$$

by making as in N$^\circ$ 2 of the memoir cited

$$\begin{aligned} Q =& B + C\alpha + D\alpha^2 + \text{&c.} \\ R =& C + D\alpha + \text{&c.} \\ S =& D + \text{&c.} \\ & \text{&c.} \end{aligned}$$

We make in the preceding equation $y = \beta$, one will have

$$Q + R\beta + S\beta^2 + T\beta^3 + \text{&c.} = 0;$$

subtracting this quantity from the first member of the same equation, & dividing the whole by $y-\beta$, one will have

$$Q' + R'y + S'y^2 + T'y^3 + \text{&c.} = N(y-\gamma)(y-\delta)\dots$$

by making

$$\begin{aligned} Q' &= R + S\beta + T\beta^2 + \text{&c.} \\ R' &= R + S\beta + T\beta^2 + \text{&c.} \\ S' &= S + \text{&c.} \end{aligned}$$

Similarly one will find

$$Q'' + R''y + S''y^2 + \text{&c.} = N(y-\delta)\dots$$

by making

$$Q'' = R' + S'\beta + T'\beta^2 + \text{&c.}$$
$$R'' = R' + S'\beta + T'\beta^2 + \text{&c.}$$
$$S'' = S' + \text{&c.}$$

& thus in sequence.

4. Let now 1°. $\beta = \alpha$ one will have

$$N(y-\alpha)(y-\gamma)(y-\delta)\ldots = Q + Ry + Sy^2 + \text{&c.}$$
$$N(y-\gamma)(y-\delta)\ldots = Q' + R'y + S'y^2 + \text{&c.}$$

Making in Q', R', S' &c. $\beta = \alpha$, & substituting the values of Q, R &c. into α, one finds

$$Q' = C + 2D\alpha + 3E\alpha^2 + \text{&c.} = \frac{dQ}{d\alpha},$$
$$R' = D + 2E\alpha + \text{&c.} = \frac{dR}{d\alpha},$$
$$S' = E + \text{&c.} = \frac{dS}{d\alpha}.$$

Therefore

$$N(y-\gamma)(y-\delta) = \frac{dQ}{d\alpha} + \frac{dR}{d\alpha}y + \frac{dS}{d\alpha}y^2 + \text{&c.}$$

Let 2°. $\gamma = \beta = \alpha$, one will have first

$$N(y-\alpha)^2(y-\delta)\ldots = Q + Ry + Sy^2 + \text{&c.}$$
$$N(y-\alpha)(y-\delta)\ldots = \frac{dQ}{d\alpha} + \frac{dR}{d\alpha}y + \frac{dS}{d\alpha}y^2 + \text{&c.}$$
$$N(y-\gamma))\ldots = Q'' + R''y + S''y^2 + \text{&c.}$$

Making in Q'', R'', S'' &c. $\gamma = \alpha$, & substituting the values above of Q', R', S' &c., one finds

$$Q'' = D + 3E\alpha + \text{&c.} = \frac{d^2Q}{2d\alpha^2},$$
$$R'' = \frac{d^2R}{2d\alpha^2},$$
$$S'' = \frac{d^2S}{2d\alpha^2}\ \text{&c.}$$

So that one will have

$$N(y-\delta)\ldots = \frac{d^2Q}{2d\alpha^2} + \frac{d^2R}{2d\alpha^2}y + \frac{d^2S}{2d\alpha^2}y^2 + \text{&c.}$$

& thus in sequence.

We make these substitutions in the formulas found above for the case of the equal roots, & we change, as we have prescribed, the powers y^0, y^1, y^2 &c. into y_0, y_1, y_2 &c. one will find these results quite simple.

1 ˚. In the case where $\beta = \alpha$, the quantity

$$\frac{d.(Qy_0 + Ry_1 + Sy_2 + \text{\&c.})f.\alpha}{Nd\alpha}$$

for the value of the two terms $a\alpha^x + b\beta^x$.

2 ˚. In the case of $\gamma = \beta = \alpha$, the quantity

$$\frac{d^2.(Qy_0 + Ry_1 + Sy_2 + \text{\&c.})f'.\alpha}{2Nd\alpha^2}$$

for the value of the three terms $a\alpha^x + b\beta^x + c\gamma^x$. And thus in sequence.

5. By considering these results, it is clear that one had been able to find them more easily, by substituting into the expression of the coefficient a in place of

$$y_{n-1} - (\beta + \gamma + \delta +)y_{n-2} + (\gamma + \beta\delta +)y_{n-3} - \text{\&c.}$$

its value

$$\frac{Qy_0 + Ry_1 + Sy_2 + \text{\&c.}}{N},$$

and considering this quantity as a function of α; for by designating it by $F\alpha$, one had had, likewise, by the coefficient b, the quantity

$$y_{n-1} - (\alpha + \gamma + \delta +)y_{n-2} + (\alpha\gamma + \alpha\delta +)y_{n-3} - \text{\&c.} = F.\beta;$$

so that one had had for the two terms $a\alpha^x + b\beta^x$ the expression

$$\frac{F\alpha \times f\alpha}{\alpha - \beta} + \frac{F\beta \times f.\beta}{\beta - \alpha},$$

which had given immediately, by making $\beta = \alpha + \omega$, $\frac{d.(F\alpha \times f\alpha)}{d\alpha}$. One had found in the same manner for the case of three equal roots, by making $f\alpha = \frac{f'\alpha}{\alpha - \gamma}$, that the first three terms $a\alpha^x + b\beta^x + c\gamma^x$ had given

$$\frac{\frac{d.F\alpha \times f'\alpha}{d\alpha}}{\alpha - \gamma} - \frac{F\alpha \times f'\alpha}{(\alpha - \gamma)^2} + \frac{F\gamma \times f'\gamma}{(\gamma - \alpha)^2};$$

this which, by making $\gamma = \alpha + \omega$, is reduced to $\frac{d^2.(F\alpha \times f'\alpha)}{2d\alpha^2}$. It is also in this manner that I myself had taken first in order to resolve the case of equal roots; but although it leads to some exact results, it seems to me that one can not adopt it without precaution; because it is remarkable that the quantity what one takes for a simple function of α, contains all the other roots β, γ &c. without α; that, likewise, that which one took for a function of β, would contain the other roots without β, & thus in sequence; this which must at least leave some doubt on the goodness of the method; but according to that which we have followed, there must remain nothing on the exactitude of our results.

6. But these results have not yet all the simplicity of which they are susceptible; because the quantities which we have designated by $f.\alpha$, $f'.\alpha$ &c. depend at once on the different roots α, γ, δ &c., & it is necessary to reduce them to be only some functions of the single root α.

For this I make, as in article 2, of the memoir cited,

$$P = A + B\alpha + C\alpha^2 + D\alpha^3 + \text{\&c.} + N\alpha^n;$$

I change for a moment α into y; I will have $P = 0$ for equation (A) of N$^\circ$ 1 above, of which the roots are α, β, γ &c. So that by the nature of the equations I will have

$$P = N(y-\alpha)(y-\beta)(y-\gamma)\dots$$

an identical equation.

Therefore 1°. by differentiating & making next $y = \alpha$, one will have

$$\frac{dP}{d\alpha} = N(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)\dots$$

2°. If $\alpha = \beta$, one has

$$P = N(y-\alpha)^2(y-\gamma)(y-\delta)\dots$$

Let $P' = N(y-\gamma)(y-\delta)\dots$ one will have $P = (y-\alpha)^2 P'$; therefore differentiating & making next $y = \alpha$, one will have

$$\frac{dP}{d\alpha} = 0, \quad \frac{d^2P}{d\alpha^2} = 2P', \quad \frac{d^3P}{d\alpha^3} = 2.3\frac{dP'}{d\alpha}, \quad \frac{d^4P}{d\alpha^4} = 3.4\frac{d^2P'}{d\alpha^2}$$

& thus in sequence; whence one draws

$$P' = \frac{d^2P}{2d\alpha^2}, \quad \frac{dP'}{d\alpha} = \frac{d^3P}{2.3\,d\alpha^3}, \quad \frac{d^2P'}{d\alpha^2} = \frac{d^4P}{2.4\,d\alpha^4} \quad \text{\&c.}$$

3°. If $\alpha = \beta = \gamma$, one has

$$P = N(y-\alpha)^3(y-\gamma)(y-\delta)\dots$$

Let $P'' = N(y-\delta)(y-\epsilon)\dots$ one will have $P = (y-\alpha)^3 P'$. Differentiating & making next $y = \alpha$, one will have

$$\frac{dP}{d\alpha} = 0,\ \frac{d^2P}{d\alpha^2} = 0,\ \frac{d^3P}{d\alpha^3} = 2.3\frac{dP''}{d\alpha},\ \frac{d^4P}{d\alpha^4} = 2.3.4\frac{dP''}{d\alpha},\ \frac{d^5P}{d\alpha^5} = 3.4.5\,\frac{d^2P''}{d\alpha^2}\ \text{\&c.}$$

whence one draws

$$P'' = \frac{d^3P}{2.3\,d\alpha^3},\ \frac{dP''}{d\alpha} = \frac{d^4P}{2.3.4\,d\alpha^4},\ \frac{d^2P''}{d\alpha^2} = \frac{d^5P}{3.4.5\,d\alpha^5},\ \frac{d^3P''}{d\alpha^3} = \frac{d^6P}{4.5.6\,d\alpha^6}\ \text{\&c.}$$

And thus in sequence.

One will have therefore by these substitutions, by supposing that one had put α in the place of y in P', P'' &c., $f\alpha = \frac{N\alpha^x}{P'}$, $f'\alpha = \frac{N\alpha^x}{P''}$ & thus in sequence (N$^\circ$ 2.). Therefore finally, substituting these values into the formulas of N$^\circ$ 3, one will find

1°. That when $\alpha = \beta$, the two terms $a\alpha^x + b\beta^x$ of the expression of the general term y_x, will be reduced to this expression

$$\frac{d.\left(\frac{Qy_0 + Ry_1 + Sy_2 + \text{\&c.}}{P'}\,\alpha^x\right)}{d\alpha}$$

by making $P' = \frac{d^2 P}{2d\alpha^2}$, $\frac{dP'}{d\alpha} = \frac{d^3 P}{2.3\, d\alpha^3}$ &c.

2°. That when $\alpha = \beta = \gamma$, the three terms $a\alpha^x + b\beta^x + c\gamma^x$ will be reduced to

$$\frac{d^2.\left(\frac{Qy_0+Ry_1+Sy_2+\&c.}{P''}\,\alpha^x\right)}{2d\alpha^2}$$

by making $P'' = \frac{d^3 P}{2.3\, d\alpha^3}$, $\frac{dP''}{d\alpha} = \frac{d^4 P}{2.3.4\, d\alpha^4}$, $\frac{d^2 P''}{d\alpha^2} = \frac{d^5 P}{2.4.5\, d\alpha^5}$.

3°. That when $\alpha = \beta = \gamma = \delta$, the four terms $a\alpha^x + b\beta^x + c\gamma^x + d\delta^x$ will be reduced to

$$\frac{d^3.\left(\frac{Qy_0+Ry_1+Sy_2+\&c.}{P'''}\,\alpha^x\right)}{2.3\, d\alpha^3}$$

by making $P''' = \frac{d^4 P}{2.3.4\, d\alpha^4}$, $\frac{dP'''}{d\alpha} = \frac{d^5 P}{2.3.4.5\, d\alpha^5}$, $\frac{d^2 P'''}{d\alpha^2} = \frac{d^6 P}{3.4.5.6\, d\alpha^6}$, $\frac{d^3 P'''}{d\alpha^3} = \frac{d^7 P}{4.5.6.7\, d\alpha^7}$. And thus in sequence.

7. These formulas are a little different from those that I had given without demonstration in the memoir cited for the case of the equality of the roots.

I myself had perceived their inexactitude after the impression of the memoir; but seduced by other objects, I had always deferred returning on to that which I regarded as less important; & I have been prevented in this regard by a member of the Italian society, Jean François Malfatti, who has given on this subject a scholarly memoir in the third volume of the compilation of the society. As the analysis of this author is quite long & leads to some results a little complicated, I have believe I must seek to resolve this question in a more direct & a manner more conformed to the simplicity of the general method exposed in my memoir of 1775; it is this which has occasion the preceding researches; but although the formulas to which I am arrived appear to leave nothing to desire for the simplicity & the generality; nevertheless, as these formulas are different for the different cases of equality of two roots, of three, of four &c. one could desire again a formula which contained all these cases; & here is that which I have found, & which I present to the geometers by inviting them to demonstrate it directly.

By converting the values of P, Q, R &c. of N° 3. & 6., namely by making

$$P = A + B\alpha + C\alpha^2 + D\alpha^3 + E\alpha^4 + \&c.$$
$$Q = B + C\alpha + D\alpha^2 + E\alpha^3 + \&c.$$
$$R = C + D\alpha + E\alpha^2 + \&c.$$

& thus in sequence, I have for abridgment

$$(Qy_0 + Ry_1 + Sy_2 + Ty_3 + \&c.)\alpha^x = F\alpha$$

$F\alpha$ denoting, as one sees, a given function of α.

I consider next the formula

$$\frac{F\alpha + \omega\,\frac{d.F\alpha}{d\alpha} + \frac{\omega^2}{2} \times \frac{d^2.F\alpha}{d\alpha^2} + \&c.}{\frac{dP}{d\alpha} + \frac{\omega}{2} \times \frac{d^2 P}{d\alpha^2} + \frac{\omega^2}{2.3} \times \frac{d^3 P}{d\alpha^3} + \&c.}$$

& after having developed it in a series according to the ascending powers of ω, I retain only the terms where the quantity ω is not found at all, by rejecting those which will be found divided or multiplied by some powers of ω; I say that these terms will be those of the expression of the general term y_x, which will arise from the root α, be it that this root is a simple root or double or triple &c.

Thus, if α is a simple root, one will have immediately $\frac{F\alpha}{\frac{dP}{d\alpha}}$ for the term due to the root.

If α is a double root, then $\frac{dP}{d\alpha} = 0$, & the formula will be reduced to

$$\frac{F\alpha + \omega\,\frac{d.F\alpha}{d\alpha} + \text{\&c.}}{\frac{\omega}{2}\,\frac{d^2P}{d\alpha^2} + \frac{\omega^2}{2.3} \times \frac{d^3P}{d\alpha^3} + \text{\&c.}} = \frac{F\alpha}{\omega\,\frac{d^2P}{2\,d\alpha^2}} + \frac{\frac{d.F\alpha}{d\alpha}}{\frac{d^2P}{2d\alpha}} - \frac{F\alpha \times \frac{d^3P}{2.3\,d\alpha^2}}{\left(\frac{d^2P}{2d\alpha^2}\right)^2} + \omega \times \dots$$

Therefore the terms due to the double root α will be

$$\frac{\frac{d.F\alpha}{d\alpha}}{\frac{d^2P}{2d\alpha}} - F\alpha \times \frac{\frac{d^3P}{2.3\,d\alpha^3}}{\left(\frac{d^2P}{2d\alpha^2}\right)^2},$$

or else

$$\frac{d.F\alpha}{P'} - F\alpha\,\frac{dP'}{(P')^2} \quad (\text{N}^\circ\ 6.)$$

or else again, $\frac{d.\frac{F\alpha}{P'}}{d\alpha}$, as one has found it in the N$^\circ$ cited.

If α is a triple root, then one will have $\frac{dP}{d\alpha} = 0$ & $\frac{d^2P}{d\alpha^2} = 0$, this which will reduce the formula to this one:

$$\frac{F\alpha + \omega\,\frac{d.F\alpha}{d\alpha} + \frac{\omega^2}{2} \times \frac{d.F\alpha}{d\alpha^2} + \text{\&c.}}{\frac{\omega^2}{2.3}\left(\frac{d^3P}{d\alpha^3} + \frac{\omega}{4} \times \frac{d^4P}{d\alpha^4} + \frac{\omega^2}{4.5} \times \frac{d^5P}{d\alpha^5} + \text{\&c.}\right)}$$

Making the development according to the ordinary methods one will find that the terms independent of ω will be the same as those which result from formulas given above for the case of three equal roots. And thus in sequence.