MÉMOIRE
sur une question concernant les annuités

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This Memoir has been read to the Academy more than ten years ago. As it has not been printed within the time, I have believed I can present it anew, because of the utility of which the formulas & the tables that it contains can be in different occasions.

1. Here is the object of the question. One demands the present value of an annuity constituted on one or many heads of which the ages are givens, on condition that it begins to run only after the death of another person of a given age, & that it ceases as soon as all the persons on which the annuity is constituted, will have passed a given age.

In order to give a clearer idea of the state of the question, one has only to suppose that a father wishes to assure to his children an annual pension payable only after his death, & until the youngest has attained a given age, for example the one of the majority; the concern is to determine the sum which he must pay in order to achieve such a pension, the age of the father, the number & the ages of the children being given.

2. One could suppose also that instead of paying first a certain sum, the father is committed to pay annually, but only during his life & the minority of all his children, a sum given in order to assure them after this death an annuity which would endure only until the majority of all the children.

The question presented in this manner is a little more difficult because there are two annuities to estimate; the one constituted conjointly on the heads of the father & of the minor children, & the other constituted only on the heads of the children & on their minority, but which must commence only after the death of the father; it is clear that the state of the question requires that the absolute or the present value of each of these two annuities are equal, provided that one can exchange in the pair the one against the other.

3. Although the principles necessary in order to resolve these sorts of questions are known, the application of it is nevertheless so much the more difficult as the questions are complicated; and the one that we are going to propose is enough in order that this application not br presented easily. As the solution of this question could be useful in other occasions, I have believed that one could see with pleasure the method that I have imagined in order to arrive to it, & which reduces the difficulty to the calculus of the ordinary annuities & constituted on one or many heads. I will give besides some

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particular applications of this method, & I will present some tables which will serve to resolve, with a sufficient exactitude, the greater part of the cases which one could propose.

4. I remark first that one can much simplify the question of which there is concern through the following consideration.

Let x be the annuity that the father must pay, & of which one seeks the value, & a the annuity which he wishes to assure to the children, after his death, it is clear that one can suppose without changing anything in the state of the question, that the annuity x is increased by a, so that the annuity to pay by the father is $x + a$, & that the annuity due to the children commences immediately; because in this manner that which the father pays too much is immediately rendered to the children; but by envisioning the question thus, one has the advantage that the two annuities commence at the same epoch, & are similar, excepting that the annuity $a + x$ depends on the life of the father, & that the annuity a depends on it not at all.

5. We denote in general by M the present value of an annuity of one unit (for example an écu, or one hundred écus, or &c.) constituted uniquely on the minority of the children, that is to say payable while there are some minor children; & we denote by N the present value of an equal annuity but constituted conjointly on the head of the father & on the minority of the children, that is to say payable only while the father lives & while he has some minor children. The absolute value of the annuity $a + x$ that the father is supposed to pay will be therefore $(a + x)N$, & the value of the annuity a that the children receive will be aM.

Therefore in order that these two values are equals, it will be necessary that one has the equation $(a + x)N = aM$, which gives $x = \frac{a(M - N)}{N}$; it is the real annuity that the father must pay. And all the difficulty will be reduced to determine the quantities M & N.

6. In order to restore this question to the ordinary notions, & to render that which I just said more simple & more intelligible, I will call the father P, & the different minor infants A, B, C &c.

Next I will designate by $\overline{A}$, $\overline{B}$, $\overline{C}$ &c. the value of an annuity of one unit constituted uniquely on the minority of the child A, or B, or C &c.; I will designate moreover by $\overline{AB}$ the value of a similar annuity, but constituted conjointly on the minority of the two children A & B, that is to say payable while they are both minors; I will designate likewise by $\overline{ABC}$ the value of an equal annuity, but constituted on the minority of the children A, B, C, that is to say payable while they are all three alive & in minority; & thus of the rest.

Finally I will denote likewise by $\overline{AP}$ the value of an annuity constituted on the minority of the infant A & on the life of the father P; that is to say payable while the child is a minor & while the father is alive; by $\overline{ABP}$ I will denote similarly the value of an annuity constituted on the minority of the children A & B, & on the head of the father, that is to say payable while the children A & B are minors all at once & while the father will be living. And thus in sequence.

7. This put & well understood, I am going to examine successively the case of a child A, of two children A & B &c., & I will determine for each case the values of the quantities M & N by means of the quantities $\overline{A}$, $\overline{AP}$, $\overline{B}$, $\overline{AB}$, $\overline{BP}$ &c. of which the

signification is now known, & of which the determination can be drawn from the tables of annuities.

And first, if there is only a single child A, it is clear that the value of M is equal to $\overline{A}$, & that of N is equal to $\overline{AP}$. One has therefore in this case

$$M = \overline{A}, \quad N = \overline{AP}.$$

8. In second place, if there are two children A & B; then the value of M must be that of an annuity constituted on the longest of the minorities of these two children, & the value of N must be that of an annuity constituted on the longest minority of the children & at the same time on the head of the father. I suppose that the minority of A is the shortest, either because A dies or because he attains the age of majority before B. It is clear that the value of the annuity M must be equal to $\overline{A}$, plus to the value of an annuity constituted on the minority of B, but payable only at the majority of A. The question is therefore to find the value of this last annuity. I call it X, & I consider that if I add the value of an annuity constituted on the minority of the two children A & B, the value that I have designated by $\overline{AB}$, I will have then the value of an annuity payable during the common minority of the children A & B, & next after the extinction of the minority of A, continued to the majority of B; this which is evidently the same thing as an annuity constituted on the single minority of B, of which the value has been designated by $\overline{B}$. I will have therefore $X + \overline{AB} = \overline{B}$; & thence $X = \overline{B} - \overline{AB}$; therefore since $M = \overline{A} + X$ I will have

$$M = \overline{A} + \overline{B} - \overline{AB}.$$

I have supposed that the minority of A was the first to be extinguished but if one supposed that it was that of B, one will arrive to the same result.

9. It remains now to find the value of N. For this, it is necessary to make a reasoning similar to the preceding, but by combining the life of the father with the minority of the children.

I consider therefore that the value of N is that of an annuity constituted on the head of the father & on the longest of the minorities of the two children A & B; & supposing that the minority of B is longer than that of A, I conclude from it that the value of N must be equal to the value of an annuity constituted on the head of the father & on the minority of the child A, a value that we have denoted by $\overline{AP}$, plus to the value of an annuity constituted on the head of the father & on the minority of the child B, but which commences only after the minority of A. Naming this last value X, I observe that if I add the value of an annuity constituted on the head of the father & on the common minority of the children A & B, a value which we have denoted by $\overline{ABP}$, I will have the value of an annuity payable while the life of the father & the total minority of the child B, but begins only after the minority of A. Naming this last value X, I observe that if I add the value of an annuity constituted on the head of the father & on the minority common to the children A & B, the value that we have denoted by $\overline{ABP}$, I will have the value of an annuity payable during the life of the father & the total minority of the child B, that is to say of an annuity constituted on the head of the father & on the minority of the child B, a value expressed according to

our denominations by $\overline{BP}$. Therefore $X + \overline{ABP} = \overline{BP}$, & thence $X = \overline{BP} - \overline{ABP}$. Therefore, since $N = \overline{AP} + X$, one will have finally

$$N = \overline{AP} + \overline{BP} - \overline{ABP}.$$

And one will find the same expression, if one supposed that the minority of the child B extinguished before that of the child A.

10. In third place, if there are three children A, B, C, one will find by some analogous reasonings that I will suppress in order to not be too long,

$$M = \overline{A} + \overline{B} + \overline{C} - \overline{AB} - \overline{AC} - \overline{BC} + \overline{ABC}$$

& likewise

$$N = \overline{AP} + \overline{BP} + \overline{CP} - \overline{ABP} - \overline{ACP} - \overline{BCP} + \overline{ABCP}.$$

And thus in sequence if there are a greater number of children.

11. Thence I conclude in general, that whatever be the number of minor children, the value of M is always equal to the sum of the values of the annuities constituted of the minority of each child in particular, less the sum of the values of the annuities constituted on the minority common to each pair of children taken two by two in all the possible ways, plus the sum of the values of the annuities constituted on the minority common to each trio of children taken three by three in all possible ways, less &.

And the value of N will be similarly equal to the sum of the values of the annuities constituted on the life of the father & on the minority of each child in particular, less the sum of the values of the annuities constituted on the life of the father & on the common minority of each pair of children taken two by two in all possible ways, plus the sum of the values of the annuities constituted on the life of the father & on the common minority of each trio of children taken three by three in all possible ways, less &c.

12. The question is therefore reduced now to find the values of these different annuities; this is what one can attain by the known rules for the evaluation of old age pensions. I will observe only that between an ordinary annuity constituted on the life of one or many persons, & the same annuity constituted on the life of some of these persons & and on the minority of the others, there is no other difference, but it is only the first must be counted continued until the last term of life, & that the second must be continued only to the time where the most age of the minors would become major; because then this person, becoming major, is with respect to the annuity in the same case as if she died immediately when she attained the age of majority.

Here are the general formulas for the calculation of the annuities.

13. I designate by (1), (2), (3), &c. the numbers of persons born at the same time & who have attained the age of one year, of two years, of three years &c. These numbers are given by the known tables of mortality, & vary according to these different tables. According to the table of the late Sussmilch, given in the first edition of his work, one has $(0) = 1000$, $(1) = 740$, $(2) = 660$, $(3) = 620$ &c. Thus these numbers are supposed known.

I suppose, moreover, that the interest on the money is at m percent, and I make, for brevity, $1 + \frac{m}{100} = r$.

This put, the present value of an annuity in life, constituted on one person of age a, & payable at the beginning of each year, is, counting the first year, of

$$\frac{(a) + \frac{(a+1)}{r} + \frac{(a+2)}{r^2} + \frac{(a+3)}{r^3} + \&\text{c.}}{(a)}.$$

The present value of an annuity in life, constituted on two persons of whom the ages are a and b, is, counting the first year, of

$$\frac{(a) \times (b) + \frac{(a+1)(b+1)}{r} + \frac{(a+2)(b+2)}{r^2} + \frac{(a+3)(b+3)}{r^3} + \&\text{c.}}{(a) \times (b)}.$$

The present value of an annuity in life, constituted on three persons of whom the ages are a, b, c, is, counting always the first year, of

$$\frac{(a)(b)(c) + \frac{(a+1)(b+1)(c+1)}{r} + \frac{(a+2)(b+2)(c+2)}{r^2} + \&\text{c.}}{(a) \times (b) \times (c)};.$$

& thus in sequence.

And if one wishes that these annuities depend on the minority of some of the persons on which they are constituted, then if a is the age of the minor most aged, it will be necessary to take only as many terms of the series as there are of units in $26 - a$, by supposing that the minority ceases at 25 years; so that it will be necessary to stop at the term which will have r^{25-a} in the denominator.

14. The application of these formulas has no more, as one sees, other difficulty than the length of the calculation, but one can abridge it by considering that, as the tables of mortality are not rigorously exact & as they have even been constructed only by the means taken between different years, it will suffice to take the years by four to four, or by five to five, & to suppose that the intermediary terms in the formula are in arithmetic progression.

Now if one has the series a, b, c, d, e &c. u, & if among the consecutive terms of this series it is necessary to place m other terms which are in arithmetic progression with the given terms, by denoting by a', a'' &c. the terms between a & b, by b', b'' &c. the terms between b & c, & thus in sequence, it is clear that one will have by the known property of the arithmetic progressions

$$a + a' + a'' + \&\text{c.} + b = (a+b)\frac{m+2}{2}$$
$$b + b' + b'' + \&\text{c.} + c = (b+c)\frac{m+2}{2}$$
$$c + c' + c'' + \&\text{c.} + d = (c+d)\frac{m+2}{2}$$
$$\&\text{c.}$$

therefore adding

$$a + a' + a'' + \&\text{c.} + 2b + b' + b'' + \&\text{c.}$$
$$+ 2c + c' + c'' + \&\text{c.} + 2d + \&\text{c.} + u$$
$$= (a + 2b + 2c + 2d + \&\text{c.} + u)\frac{m+2}{2};$$

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& consequently the entire sum of the series

$$a + a' + a'' + \&\text{c.} + b + b' + b'' + \&\text{c.} + c + c' + \&\text{c.} + u$$

will be

$$= (a + 2b + 2c + \&\text{c.} + u)\frac{m+2}{2} - b - c - d - \&\text{c.}$$

$$= (a + b + c + \&\text{c.} + u)\frac{m+2}{2} + (b + c + d + \&\text{c.})\frac{m}{2}$$

$$= (a + b + c + \&\text{c.} + u)(m+1) - (a + u)\frac{m}{2}.$$

Whence it follows that in order to have the sum of the interpolated series, there will be only to multiply the sum of the first series by $m + 1$, & by subtracting the sum of the two extreme terms multiplied by $\frac{m}{2}$.

If one takes the years only by four to four, one will have then $m = 3$, & it will be necessary to quadruple the sum of the series, & to subtract from it $\frac{3}{2}$ of the sum of the extreme terms.

15. I have calculated in this manner two tables for the case of a single minor child, & by taking successively for the age of the child, 1, 5, 9, 13, 17, 21 years, & for the age of father 30, 34, 38, 42 &c. to 90 years; but in one of these tables I have taken account of the mortality of the child conformably to the state of the question; in the other, on the contrary, I have set it aside, that is to say I have supposed that the child arrives surely to the age of majority. Here is the reason that has engaged me to calculate this second table conjointly to the first.

16. It is clear, in general, that the more the number of heads on which any annuity whatever is constituted is great, the more also must be great the present value of this annuity, that is to say that I would be necessary to pay in order to achieve it, because the risk of the loss by the death of all the persons on which it is constituted is so much less. But, on the other hand, however great be the number of these persons, the value of the annuity will be always less than if one had no regard at all to their mortality, & that one supposed that the youngest attained surely a given age.

Thence it follows that if an annuity is constituted on many persons, its value, whatever it be, will be always necessarily contained between these two limits, of which the one will be the value of the same annuity constituted only on the youngest of these persons, by having regard to his mortality, & the other will be the value of the annuity, constituted likewise on this person, but having no regard to his mortality.

And if it happens that these two limits are little different between themselves, then one will be assured that the value of the annuity is very nearly the same, whatever be the number of heads on which it is constituted.

17. One must therefore regard the two tables of which we just spoke, as the limits of all the similar tables that one could construct for the case of two, of three &c. or of any number whatsoever of minor children.

In these tables we have taken for base the table of mortality which is found in the new edition of the work of Sussmilch (Tome 3, Table 22. No. 4) & which has been prepared particularly for this country; according to this table, one has $(0) = 1000$, $(1) = 759$, $(5) = 603$, $(9) = 552$ &c.

In regard to the interest on the money, we have supposed 4 p. 100, this which gives $m = 4$, & $r = \frac{104}{100} = \frac{26}{25}$.

These tables give immediately the annual sum or the annuity that the father must pay during his life & the minority of his child, in order to assure to him after his death an annuity of one unit which would endure only until he had attained his twenty-fifth year. In the first table one has set aside the mortality of the child, & one sees that the numbers are all a little greater than in the second, where one has taken account of this mortality, but one sees simultaneously that the differences of the corresponding numbers in the two tables are in general quite small. So that when there will be many children, one will not be deceived much by taking the mean among the numbers given by these two tables & relative to the age of the father & to the one of the youngest of the children. But one could perhaps in this case approach further to exactitude by the following formula:

Let $1 + n$ be the number of children, A & B be the numbers given by the first & by the second table, for the case of the youngest of these children, one will take for the annuity that the father must pay, the quantity $\frac{nA+B}{n+1}$; this formula becomes B when $n = 0$, & $= A$ when $n = \infty$, this which must be.

Table I.

By setting aside the mortality of the child.

	Age of father	Age of the child.					
		1.	5.	9.	13.	17.	21.
1	30	0.1699	0.1399	0.1108	0.0815	0.0527	0.0251
2	34	1957	1597	1264	0940	0617	0300
3	38	2253	1808	1403	1033	0681	0332
4	42	2674	2114	1601	1144	0736	0358
5	46	3327	2616	1958	1361	0840	0389
6	50	4358	3446	2590	1805	1105	0504
7	54	5705	4555	3754	2412	1489	0676
8	58	7360	5945	4522	3171	1962	0903
9	62	9289	8002	5830	4094	2498	1122
10	66	1.1777	9760	7604	5417	3331	1510
11	70	4505	1.2187	9622	6908	4334	1950
12	74	6987	4342	1.1473	8346	5219	2337
13	78	9516	6499	3213	9788	6150	2728
14	82	2.1946	8681	4871	1.0913	7090	3166
15	86	3560	2.0131	6128	1445	7015	3359
16	90	5770	2113	7848	2856	7018	2261

Table II.

By setting aside the mortality of the child.

	Age of father	Age of the child.					
		1.	5.	9.	13.	17.	21.
1	30	0.1513	0.1324	0.1069	0.0792	0.0514	0.0246
2	34	1740	1511	1220	0914	0602	0293
3	38	1994	1707	1354	0979	0665	0325
4	42	2353	1989	1543	1111	0718	0350
5	46	2906	2455	1881	1320	0820	0381
6	50	3783	3225	2488	1748	1076	0492
7	54	4917	4251	3298	2335	1449	0660
8	58	6286	5531	4329	3066	1910	0882
9	62	7851	7035	5568	3952	2429	1095
10	66	9844	9009	7251	5220	3236	1446
11	70	1.1991	1.1206	9161	6686	4206	1899
12	74	3898	3145	1.0905	8022	5059	2273
13	78	5794	5066	2551	9393	5956	2653
14	82	7954	6983	4101	1.0481	6861	3076
15	86	8719	8214	5214	0957	6802	3262
16	90	2.0001	9838	6761	2195	6732	2200