MÉMOIRE

SUR UN PROBLEME

DE

LA DOCTRINE DU HAZARD

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§ 1.

On the occasion of certain calculations of political Arithmetic in which I had engaged myself, the following Problem, which the calculus of probabilities claims entirely, offered itself to my mind.

Any number of persons of one same age, half men, half women, being married together the same year, to find the probability that, the half of this complete number of married persons being dead, all the marriages will be broken.

§ 2. In reflecting on this problem, I see that it could be considered under more than one point of view: That first it was more appropriate to treat separately the case where the total number of persons is even, & the one where this number is odd.

Next: that one can seek the solution of the problem for each case, by supposing the possibility that in the number of persons who die there is a relative disparity in the sexes & that perhaps there is found even absolutely only men & *vice versa*; or else, by regarding as more natural & as certain that death will have removed an equal number of men & of women.

I will begin by admitting the first assumption, & for the case where the number of persons of whom one half is married with the other, is even; & I will serve myself of the easy method to seek the law which the progression of the probabilities follows that one wishes to determine, by carrying up from the question which contains the fewer cases. In order to aid also to the imagination I will indicate the men by the capital letters of the alphabet & their wives by the lower case which correspond to them.

§ 3. Our question is the simplest when there is only one couple, $A + a$, & the probability that one seeks is here certitude, that is to say, according to the ordinary fashion to express itself, $= 1$.

§ 4. When there are two couples $A + a$, $B + b$, it can happen in two ways that one of them dies, & then the marriages not being broken, these two cases give 0; but there are 4 cases where the death of two persons break the marriages & give 1 probability: thus that which one demands is $= \frac{4}{6} = \frac{2}{3}$.

§ 5. We suppose that there are 3 couples: $A+a$, $B+b$, $C+c$, one will be able since at present to be saved from the enumerations of cases, which would become tedious, by considering that by the Doctrine of combinations one finds first the number of cases which take place in all, when of any number of things one removes from them the half, & that next one has only to follow the reasoning that is here. I say: when there was a question of two couples there were 4 cases in which these marriages were broken by the death of the half of the persons; or, in order that this event take place also in the present state of the question, it is absolutely necessary that these 4 preceding cases are combined with those where one person dies from the third couple, that is to say, C or c, this which gives 8 cases. But 6 things can be taken three by three in 20 different ways, thus the probability for the dissolution of 3 marriages is $\frac{9}{20}$ or $\frac{4}{10}$.

§ 6. One will reason in the same manner for 4 marriages, by combining the 8 cases of the preceding question with the two that the new couple furnish, & by dividing the product 16 by the total number of cases which is 70, so that the probability that the 4 marriages will be broken by the death of 4 persons is $= \frac{16}{70}$.

§ 7. We have no need to go further. One will perceive without doubt already that the number of marriages being whatever $= m$, the number of cases where the persons who remain alive are each of a different couple, can only be constantly expressed by the number 2 raised to the power m. This is that which makes the numerator of the fraction. As for the denominator it is always equal to the quantity of times of which one can combine m by m the total number of persons $2m$. Finally one sees that the general formula of the probability that we seek is

$$\frac{2^m}{(2m.(2m-1).(2m-2)\cdots(2m-m+1)):(1.2.3\cdots m)}.$$

§ 8. On could divide this formula by 2, but the construction of it would be discovered less easily. Moreover one could have then instead of the progression

$$\frac{2}{2}, \quad \frac{4}{6}, \quad \frac{8}{20}, \quad \frac{16}{70}, \quad \frac{32}{252}, \quad \text{\&c.}$$

this one

$$\frac{1}{1}, \quad \frac{2}{3}, \quad \frac{4}{10}, \quad \frac{8}{35}, \quad \frac{16}{126}, \quad \text{\&c.}$$

This is seen at first glance.

But one would have done poorly to begin by reducing thus the progression, because then it would have been necessary to compare some terms with the table of figurate numbers in order to find the general law that it follows.

§ 9. If am going to consider at present the Problem under the hypothesis that there die only an equal number of men & of women. I do not see that one may acknowledge half with reason between this hypothesis & the preceding; because, if on the one hand in this one finds that it is beyond all truth & of physical possibility that death remove only the men or the women out of many thousands of persons of the two sexes in equal number, of which the half die, I say that this case also enters in line accounting only for the degree of probability that it has, which is very small. Where would one wish to cease to take account of these slightly probable cases? They enter not at all into the

calculation for more than mathematical rigor & physical possibility does not permit it. We conclude therefore that it is necessary in our question to regard all the cases as possible, or to admit the hypothesis that in the number of deaths there will be found as many men as women.

§ 10. I will indicate anew the men by capital letters & women by the lower case letters of the alphabet. And considering first the simplest case, the one of a single marriage, I see that the probability that one demands is without contradiction $= 1$. But, one will perhaps object to me, it is impossible here that an equal number of men & of women die because there must die only one person. I would respond that without doubt one does not suppose that there die the half of a man & the half of a woman, but that it happens here but one alternative of the single case can take place, which makes nothing at the foundation of the question & does not change the result.

It is necessary to make the following reasoning: One person only before dying, the risk is by hypothesis equal for one & for the other sex. Thus when by enumeration of the cases for the probability that I seek, I will have found a result, I should say next: One half (risk of A) gives *so much* probability; a half (risk of a) gives or multiplies the same probability: adding these products together & dividing by two halves or one, does one not see that the result remains the same, & that thus the remark which seems to raise a contradiction between the hypothesis & the process is baseless? I myself am extended on this subject because the same difficulty, if there is one of it, returns all the time when the number of couples is odd, & with more force yet when there is a total number of persons which is odd.

§ 11. But we go further & see always what is the total number of cases which can happen through the death of the half of the married persons; we subtract those of them that the condition excludes, & out of those who remain we seek how many break all the unions.

When there are two marriages $A + a$, $B + b$, one has 6 combinations for the death of two persons; but it is necessary to exclude two, namely that of two men & that of two women; there remain of them 4; out of these 4 two, $A + a$ & $A + b$, do not break the marriages & give 0; the probability that one seeks is therefore $= \frac{2 \times 1 + 2 \times 0}{4} = \frac{2}{4}$.

§ 12. Let there be 3 marriages; the number of combinations three by three is 20; it is necessary to exclude first 2 for the cases where 3 men or 3 women would die. There remain 18 of which 9 are those of $2H^{\mathrm{s}}$. & $1F$. & 9 those of $2F^{\mathrm{s}}$. & $1H$.[1] Now this case falls into the one of § 10; because it is indifferent to regard the first 9 or the last 9 combinations as possibles, because there is $\frac{1}{2}$ probability for each kind. We suppose therefore for example that $2H^{\mathrm{s}}$. & $1F$. die, it will suffice to consider that, out of these 9 cases, those which break the marriages are the 3 combinations by two of the three H^{s}. with the wife of the one who has been omitted in the combination, that is to say ABc, ACb & BCa, in order to have immediately the sought probability $= \frac{3}{9}$.

§ 13. When there are 4 unions, one has the total number of cases $= 70$. Those which it is necessary to reject are the dead of the $4H^{\mathrm{s}}$. those of the $4F^{\mathrm{s}}$., the combinations of $3H^{\mathrm{s}}$. with $1F$. & those of $3F^{\mathrm{s}}$. with $1H$. This which makes $2 + 2$ times 16, in all 34 cases. There remain of them 36 which can take place; but we demand only the

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[1]*Translator's note*: Throughout F abbreviates *Femme* and H abbreviates *Homme*. Rather than replace these by W and M for *Woman* and *Man* respectively, the notation has been retained.

combinations of two men with the two wives of the two others; which are in number 6; thus the result is $= \frac{6}{36}$.

§ 14. Let the couples be in the number of five: $A+a$, $B+b$, $C+c$, $D+d$, $E+e$. The number of cases is 252; we take off of them first the $2ABCDE$ & $abcde$; next the 25 cases where $4H^{\mathrm{s}}$. die & $1F$. & 25 others of $4F^{\mathrm{s}}$. & $1H$. It will be necessary now according to the principles established higher to regard as possibles the half of the 200 cases which remain & to seek the probability of which there is question by supposing at will, or that $3H^{\mathrm{s}}$. & $2F^{\mathrm{s}}$., or else $3F^{\mathrm{s}}$. & $2H^{\mathrm{s}}$. die. If one regards as certain that $3H^{\mathrm{s}}$. & $2F^{\mathrm{s}}$. will die, one sees that the marriages are found broken by the combinations of $5H^{\mathrm{s}}$. three by three with the 2 other women. This which makes therefore 10 cases to divide by 100 or $\frac{10}{100}$ probability.

§ 15. For 6 couples one has in total 924 cases. It is necessary to subtract 524 which could not take place under our hypothesis; & out of the 400 which remain, there are 20 which break marriages: therefore the probability that we seek is here $= \frac{20}{400}$.

§ 16. The 6 cases which we just developed, were necessary for the construction of a general formula, although one has been able to note already rather a great harmony in the expressions. One has been able to see that the number of possible cases is always the square of the one of the cases which are that all the unions are found broken by the death of the half of the persons who have contracted them; that thus the numerator of the fraction which one determines is reduced always to unity, & the denominator to the square root of the cases in which there is found as many persons of one sex as of the other in the number of those who die; finally that the progression

for	I,	II,	III,	IV,	V,	VI,	&c.	marriages
is	$\frac{1}{1}$,	$\frac{2}{4}$,	$\frac{3}{9}$,	$\frac{6}{36}$,	$\frac{10}{100}$,	$\frac{20}{400}$,	&c.	
or	$\frac{1}{1}$,	$\frac{1}{2}$,	$\frac{1}{3}$,	$\frac{1}{6}$,	$\frac{1}{10}$,	$\frac{1}{20}$,	&c.	

There must be some attention to the Table of the figurate numbers to continue these progressions, & especially in order to construct the general term. If one consults this Table, one will see first a property which concerns already the formation of the terms which express our probabilities: it is that in the enumeration of all the cases, as many of those which cannot take place, as of the others, for any number M of marriages, one passes successively through the squares of all the figurate numbers which is found in the $M-1^{\mathrm{st}}$ transversal column, or, that which returns to the same through the squares of the coefficients of all the terms of a binomial raised to M. Thus, for example, for 5 couples one has

1 non-admissible case that the	$5H^{\mathrm{s}}$. — they die
25 non-admissible cases that	$4H^{\mathrm{s}}$. & $1F$.
100 admissible by one half for	$3H^{\mathrm{s}}$. & $2F^{\mathrm{s}}$.
100 — — —	$2H^{\mathrm{s}}$. & $3F^{\mathrm{s}}$.
25 non-admissible cases for	$1H$. & $4F^{\mathrm{s}}$.
1 non-admissible case for	— $5F^{\mathrm{s}}$.

This property would suffice to express immediately the general term of the progression, because one knows what is the coefficient of the term or of the mean terms of the Mth power of a binomial. I am going however to indicate how I have found this formula by considering the terms of the series formed & reduced, & respectively to our question.

§ 17. It is always that of the denominators of our fraction there is question. Now one sees that for I & II pairs they express the 2nd & 3rd terms of the natural numbers; for the III & IV pairs, the 4th & the 5th of the triangular numbers; for the V & VI pairs, the 6th & 7th of the pyramidal numbers, &c.

If therefore the number of couples is in general, m or $m - 1$, (where m signifies an even number,) it will be necessary to seek the denominator of the fraction which expresses the probability that one demands, in the figurate numbers of order $\frac{m}{2}$; & the law of the progression of these probabilities requires that one substitute for n the number of couples m or $m - 1$.[2] The general term of the figurate numbers of order $\frac{m}{2}$ is

$$\frac{n \times (n - 1).(n - 2) \cdots n - \frac{m}{2} + 1}{1.2.3 \cdots \frac{m}{2}};$$

consequently the formula is for m marriages

$$\frac{\frac{1}{m(m-1)(m-2)\cdots m - \frac{m}{2} + 1}}{1.2.3 \cdots \frac{m}{2}} = \frac{1.2.3 \cdots \frac{m}{2}}{m(m - 1)(m - 2) \cdots \left(m - \frac{m}{2} + 1\right)}$$

& for $m - 1$ marriages

$$\frac{1.2.3 \cdots \frac{m}{2}}{(m - 1)(m - 2) \cdots \left(m - \frac{m}{2}\right)}$$

m signifying always an even number.

If one wishes to avoid the inconvenience of having recourse to a different formula according as the number of marriages is even or odd, one can make it, but by introducing two letters in the general expression that one demands. Because we name the number of marriages M & we substitute this letter in the place of n in the general term

$$\frac{n(n - 1) \cdots n - \frac{m}{2} + 1}{1.2.3 \cdots \frac{m}{2}},$$

we arrive to the single general expression

$$\frac{1.2.3 \cdots \frac{m}{2}}{M(M - 1)(M - 2) \cdots \left(M - \frac{m}{2} + 1\right)}$$

in which m is always an even number, $= M$ if M is even, & $= M + 1$ if M is odd.

§ 18. The order that I myself have proposed demands that I continue these researches by the examination of the cases where the number of persons who are married

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[2]It is not, as it seems first, the number of couples augmented by unity which it is necessary to substitute, but only the number of couples, by reason that in the general terms of the figurate numbers one finds the I^{st} term of the sequence, by making $n = 0$.

together is odd. I will stop myself to defend the apparent absurdity of this hypothesis; in the calculation of mortality one can be led there, & the one of the probabilities admits it in its principles.

I will make rather some other more necessary considerations to take precedence.

1° When in an odd number of persons the equality of the number is so great between the one & the other sexes as it is possible, there can never be but one celibate person. And as *in abstracto* this person can be equally a man or a woman, I will suppose in the following that the number of men has been one unit greater that the one of the women, & that thus it is a man who has been condemned to the celibate when the marriages have been contracted.

2° I will suppose also that this man remains a bachelor all his life; this would be an embarrassing condition for him to make marry one of the women who become widows. Besides one sets aside the length of the life of each individual, & those who die are counted to die together.

3° The half of an odd number of persons ought to die, if this number is $2m + 1$, one will suppose in truth that $\frac{2m+1}{2}$ persons die, but in a fashion that the assumption rolls out of the two equivalent cases of death of $\frac{2m+1}{2} - 1$ & $\frac{2m+1}{2} + 1$ persons.

4° It appears to me more advantageous to make the enumeration of the cases by separating those where the celibate is found from with those where he is not found.

These things being posed, I am going to enter into material as in the beginning of this Memoir, under the hypothesis that all the ratios between the deaths of the two sexes are possibles.

§ 19. As it is of odd numbers that there is question, one can not pass under silence the case where the number of persons is $= 1$; but the probability in question being unity in this case, one has not need to stop there.

§ 20 When there are three persons $A + a, B$, & when one supposes that there die two of them, the probability is $= 1$, since all three combinations break marriages. But, if one supposes that only one person dies, of 3 cases which can happen there are only 2 of them, A & a, which break marriages: therefore the probability is then only $\frac{2}{3}$: now it is equally probable that 2 persons will die, or that only one of them will die: therefore the entire expression for our probability is

$$\frac{1 \times 1 + 1 \times \frac{2}{3}}{2} = \frac{5}{6}.$$

This result is seen also from first glance under the disposition following from the calculation, which will serve me principally in the following for more order & brevity.

Possible combinations of dying			Cases which break marriages		
Men		Women	Without the celibate	With the celibate	Sum
2	&	0	0	1	3
1	—	1	1	1	3
1	—	0	1	0	2
0	—	1	1	0	2

The transversal stripe separates, as one sees, the calculation for the assumption of the death of the grand half, of the one that one makes by supposing that the small half

of the total number of persons ceases to live. One sees at the end the sums of the cases, that it suffices to add together & to divide by the double of the sum of the possible combinations for the death of the great or of the small half. This sum is always the same in the 2 cases by a known property.

§ 21. Let there be now 5 persons $A + a$, $B + b$, C, one will have

Possible combinations of dying			Cases which break marriages		
H^s.		F^s.	Without the celibate	With the celibate	Sums
3	&	0	0	1	8
2	—	1	2	2	8
1	—	1	2	1	8
2	&	0	1	0	4
1	—	1	2	2	4
0	—	2	1	0	4

Now five things are combined 3 by 3, or 2 by 2 in 10 ways: the probability that one demands is therefore

$$\frac{1 \times \frac{8}{10} + 1 \times \frac{4}{10}}{2} = \frac{12}{20}.$$

§ 22. For 7 persons $A + a$, $B + b$, $C + c$, D, one has

Possible combinations of dying			Cases which break marriages		
H^s.		F^s.	Without the celibate	With the celibate	Sums
4	&	0	0	1	20
3	—	1	3	3	20
2	—	2	6	3	20
1	—	3	3	1	20
3	&	0	1	0	8
2	—	1	3	0	8
1	—	2	3	0	8
0	—	3	1	0	8

But the number of the combinations 4 by 4, or 3 by 3, is 35: therefore the sought probability $= \frac{28}{70}$.

§ 23. If there are 9 persons $A + a$, $B + b$, $C + c$, $D + d$, E, one has

Possible combinations of dying			Cases which break marriages		
H^s.		F^s.	Without the celibate	With the celibate	Sums
5	&	0	0	1	48
4	—	1	4	4	48
3	—	2	12	6	48
2	—	3	12	4	48
1	—	4	4	1	48
4	&	0	1	0	16
3	—	1	4	0	16
2	—	2	6	0	16
1	—	3	4	0	16
0	—	4	1	0	16

The number of combinations is 126: consequently the probability that one demands $= \frac{64}{252}$.

§ 24. It is nearly superfluous to push these enumerations further: here is now yet the Table for 11 persons.

Possible combinations of dying			Cases which break marriages		
H^s.		F^s.	Without the celibate	With the celibate	Sums
6	&	0	0	1	112
5	—	1	5	5	112
4	—	2	20	10	112
3	—	3	30	10	112
2	—	4	20	5	112
1	—	5	5	1	112
5	&	0	1	0	32
4	—	1	5	0	32
3	—	2	10	0	32
2	—	3	10	0	32
1	—	4	5	0	32
0	—	5	1	0	32

The number of the possible Combinations is 462; thus the probability sought $= \frac{144}{924}$.

§ 25. There is no longer means to doubt at present the law that follows the numbers of the cases for the different combinations, under the assumption of such number of persons as one will wish. One sees that for the combinations following the grand half, 1o the cases which break marriages without that the bachelor die, are expressed by the products of the number which indicates the small half or the number of marriages, multiplied with the coefficients of the binomial raised to this number of marriages which have taken place at the beginning, diminished by unity. 2o That if the bachelor is supposed dead, the cases in question are expressed by the coefficients of a binomial raised to the number of the small half. One sees next that, for the combinations following the small half, the cases without the celibate follow precisely the law of the cases with the celibate for the grand half; & that if one supposes that the bachelor dies, there is never

a case which breaks marriages. Now, if one indicates the odd number of persons in general by $2m + 1$, & if one pays attention that the sum of the coefficients of a binomial raised to any power whatsoever is always equal to the number 2 raised to the same power, one will conclude without pain that our probability is expressed by a fraction which has for numerator

$$m \times 2^{m-1} + 2 \times 2^m \quad \text{or} \quad 2^{m-1}m + 2^{m+1},$$

& for denominator

$$2 \times \left(\frac{(2m + 1).2m.(2m - 1) \cdots (2m - m + 1)}{1.2.3.4 \cdots m + 1} \right)$$

or

$$2 \times \left(\frac{(2m + 1).2m.(2m - 1) \cdots (2m - m + 2)}{1.2.3.4 \cdots m} \right)$$

or

$$\text{(by another property)}$$

$$\frac{(2m + 2).(2m + 1)\, 2m(2m - 1) \cdots (2m - m + 2)}{1.2.3.4 \cdots m + 1}$$

that is to say therefor, by the fraction

$$\frac{2^{m-1}m + 2^{m+1}}{((2m + 2).(2m + 1)\, 2m(2m - 1) \cdots (2m - m + 2)) : (1.2.3.4 \cdots m + 1)}$$

§ 26. I will make remark that it was not necessary, in order to find the numerator of the fraction, to consider the order that the cases observe which break marriages before they were reduced to sums; these sums making to arrive, to the same end. Indeed, if one considers always the sum of the cases under the assumption that the small half of the persons die, & the ratio of the other sum to that there, one will see that our numerator, or the sum of the two sums

$$\text{is for 3 persons} \ldots \qquad 2^1 + \frac{3}{2}.2^1 = 2^1 + 3.2^{1-1}$$

$$\ldots 5 \ldots \qquad 2^2 + \frac{4}{2}.2^2 = 2^2 + 4.2^{2-1}$$

$$\ldots 7 \ldots \qquad 2^3 + \frac{5}{2}.2^3 = 2^3 + 5.2^{3-1}$$

$$\ldots 9 \ldots \qquad 2^4 + \frac{6}{2}.2^4 = 2^4 + 6.2^{4-1}$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$\text{\& for } 2m + 1 \ldots \qquad 2^m + \frac{m + 2}{2}.2^m = 2^m + (m + 2).2^{m-1}$$

Now it is easy to be convinced that this numerator $2^m + 2^{m-1}(m + 2)$ is $= 2^{m-1}m + 2^{m+1}$, or to the one that we have found previously.

§ 27. The general formula of § 25. can be drawn also from the comparison of the probabilities for an odd number of persons with those that we have found for an even

number; & one has need then only of the results such that as we have reduced them, without passing by the considerations of §§ 25 & 26.

As besides this comparison is in its place here, I will indicate this new route. Our numerical results follow the order that is here:

Number of P.	1	2	3	4	5	6	7	8	9	10	&c.
Number of M.	0	I	I	II	II	III	III	IV	IV	V	&c.
Probab.	$\frac{2}{2}$	$\frac{2}{2}$	$\frac{5}{6}$	$\frac{4}{6}$	$\frac{12}{20}$	$\frac{8}{20}$	$\frac{28}{70}$	$\frac{16}{70}$	$\frac{64}{252}$	$\frac{32}{252}$	&c.

A slight attention to this Table suffices to demonstrate 1○ That the denominators for an odd number of persons are always the same as for the even number following. 2○ That the numerators for the odd number passes constantly the numerators for the following even number, by the product of the quantity of the marriages which are found in this odd number, with the binary number raised to this number of marriages, diminished by unity.

Thus therefore to find the probability in question for any odd number whatsoever $2m + 1$, it is necessary to seek the probability for the even number $2m + 2$, by substituting into the formula of § 7. $m + 1$ instead of m, & to add to this quantity a fraction which has the same denominator & of which the numerator is $2^{m-1}m$. The substitution of $m + 1$ for m in the general expression of § 7 gives

$$\frac{2^{m+1}}{((2m + 2).(2m + 1)\, 2m \cdots (2m - m + 2)) : (1.2.3.4 \cdots m + 1)}$$

& the addition of which I just spoke being made, one has

$$\frac{2^{m+1} + 2^{m-1}m}{((2m + 2).(2m + 1)\, 2m \cdots (2m - m + 2)) : (1.2.3.4 \cdots m + 1)}$$

as in § 25.

§ 28. This formula & that of § 7 are able to be rendered from one application again more easily by a slight transformation, which introduces into the formulas the letter which indicates the number of persons. Because if I make in the formula of § 7. $2m = N$, & in that of § 25, $2m + 1 = N$, the first is changed into this one

$$\frac{2^{\frac{N}{2}}}{\left(N.(N - 1).(N - 2) \cdots (N - \frac{1}{2}N + 1)\right) : \left(1.2.3 \cdots \frac{N}{2}\right)}$$

& the second into this one

$$\frac{2^{\frac{N+1}{2}} + 2^{\frac{N-3}{2}}\left(\frac{N-1}{2}\right)}{\left((N + 1)N(N - 1) \cdots \left(N - \left(\frac{N+1}{2}\right)\right)\right) : \left(1.2.3 \cdots \frac{N}{2}\right)}$$

so that the formulas contain the same letter which expresses the number of persons, & that there is no longer question but to know if this number is even or not even.

§ 29. I have yet one observation to make before going further, it is that our formulas satisfy exactly to all the corollaries that one draws from them; that the assumption of $N = 1$ in the second of § 28 or of $m = 0$ in the same before its transformation, gives

exactly the sought probability $= \frac{2}{2} = 1$. Perhaps one would not be surprised to find this probability $= \infty$, since it seems that one thing already done must have an infinite advantage over a thing yet to do, (when just as it is certain that it will be done,) such as is the dissolution of the marriage contract by two persons, for which the probability is also $= 1$. But it is that, *in abstracto*, there is not in time in the calculation of the probabilities.

§ 30. I am going to engage myself in seeking also the solution of the Problem for an odd number of persons, & under the hypothesis that there die an equal number of men & of women. I believe myself to be sufficiently explicated on the sense in which I take this condition when the number of the persons is odd; but there could remain some doubts if the alternative of the case which I would employ to express precisely the equality that one demands. I will try to prevent them by the preliminary consideration that is here.

Let $A + a$, B be the persons of whom the half is counted dead. There should be in rigor to die $1H$ & $\frac{1}{2}F$. but $\frac{1}{2}$ is only the result of the two equally possible cases of 1 & of 0. I say therefore that it is equally probable that there will die $1H$ & $1F$ or $1H$ & $0F$. & that this alternative fulfills perfectly the condition of equality. Likewise, if there is a question of 5 persons $A + a$, $B + b$, C, there must die $2\frac{1}{2}$, namely $1\frac{1}{2}H$ & $1F$, & this case returns to the alternative of $2H^\text{s}$ & $1F$ or of $1H$ & $1F$. & thus in sequence: when the number of marriages is even, it is the half of H^s. & when it is odd it is the half of the F^s if it is necessary thus to decompose.

This put, one is going to see in the small Tables the principal elements: I suppose that the enumeration of the cases is made mentally.

§ 31. For 3 persons $A + a$, B, one has

| Deaths | | Poss. combs. | Poss. combs. | No. of cases broken mar. | | Sums |
Men	Women	without celib.	with the celib.	without celib.	with the celib.	of these cases
1	1	1	1	1	1	2
1	0	1	1	1	0	1

Adding the two sums together & dividing by the number of possible combinations, one has the probability in question $= \frac{3}{4}$.

§ 32. For 5 persons $A + a$, $B + b$, C, one has

| Deaths | | Poss. combs. | Poss. combs. | No. of cases broken mar. | | Sums |
Men	Women	without celib.	with the celib.	without celib.	with the celib.	of these cases
2	1	2	4	2	2	4
1	1	4	2	2	0	2

Therefore the Prob.$= \frac{6}{12}$.

§ 33. For 7 pers. $A + a$, $B + b$, $C + c$, D.

| Deaths | | Poss. combs. | Poss. combs. | No. of cases broken mar. | | Sums |
Men	Women	without celib.	with the celib.	without celib.	with the celib.	of these cases
2	2	9	9	6	3	9
2	1	9	9	3	0	3

Therefore the prob.$= \frac{12}{36}$.

§ 34. For 9 pers. $A + a$, $B + b$, $C + c$, $D + d$, E.

Deaths		Poss. combs.	Poss. combs.	No. of cases broken mar.		Sums
Men	Women	without celib.	with the celib.	without celib.	with the celib.	of these cases
3	2	24	36	12	6	18
2	2	36	24	6	0	6

Therefore the prob.$= \frac{24}{120}$.

§ 35. For 11 pers. $A + a$, $B + b$, $C + c$, $D + d$, $E + e$, F.

Deaths		Poss. combs.	Poss. combs.	No. of cases broken mar.		Sums
Men	Women	without celib.	with the celib.	without celib.	with the celib.	of these cases
3	3	100	100	20	10	30
3	2	100	100	10	0	10

Therefore the prob.$= \frac{40}{400}$.

§ 36. For 13 pers. $A + a$, $B + b$, $C + c$, $D + d$, $E + e$, $F + f$, G.

Deaths		Poss. combs.	Poss. combs.	No. of cases broken mar.		Sums
Men	Women	without celib.	with the celib.	without celib.	with the celib.	of these cases
4	3	300	400	80	20	100
3	3	400	300	20	0	20

Therefore the prob.$= \frac{120}{1400}$.

§ 37. For 15 pers. $A + a$, $B + b$, $C + c$, $D + d$, $E + e$, $F + f$, $G + g$, H.

Deaths		Poss. combs.	Poss. combs.	No. of cases broken mar.		Sums
Men	Women	without celib.	with the celib.	without celib.	with the celib.	of these cases
4	4	1225	1225	140	35	175
4	3	1225	1225	35	0	35

Therefore the prob.$= \frac{210}{4900}$.

§ 38. These results suffice for the construction of one formula which can express them; & one sees among the elements of each kind the relation which serves to continue them.

1° The possible combinations for the grand half & with the celibate are always the square of the coefficient of the mean term of a binomial raised to the number of marriages.

2° The possible combinations for the grand half without the celibate are to the first in the ratio of the number of women who die to the one of the men who die.

3° The possible combinations for the small half are equal in number to the ones which take place for the grand half, but they have among them the inverse ratio relatively to the consideration of the celibate.

4° The cases which break the marriages are found in the figurate numbers, namely

for	3 & 5	pers. in those of the	1 order
for	7, 9 & 11	— — — —	2 —
for	13, 15, 17, 19	— — — —	3 —
	&c.		&c.

The cases for the small half without the celibate are the same in quantity as those for the grand half with the celibate; & are found purely in the figurate numbers: but this number is multiplied by a coefficient in the cases for the grand half without the celibate.

§ 39. I am going to give at present the necessary indications in order to find in general this number of cases for the death of the grand half with the celibate, that for an instant I will name P. One has just seen

1°	That the	2	Nos. P prev	1 & 2 Marriage. are taken in the	1$^{\text{st}}$ Order of the Fig. Nos.
	— —	3	N^s fol. P prev.	3, 4, 5, — — —	2$^{\text{nd}}$ — —
	— —	4	— —	6, 7, 8, 9 — — —	3 — —
	— —	5	— —	10, 11, 12, 13, 14 —	4 — —

It will be necessary therefore to seek the m Nos. P for $\frac{m(m-1)}{1.2}$, $\frac{m(m-1)}{1.2}+1$, $\cdots$ $\frac{m(m-1)}{1.2}+m-1$, of the $(m-1)^{\text{st}}$ order.

One remarks moreover

2° That this number m is always the natural number which is immediately at the left of the triangular number, equal to the number of the marriages, or that approaches it more.

Finally

3° That in order to obtain the value of P, it is precisely the number of the marriages that it is necessary to substitute for n in the general term

$$\frac{n(n-1)(n-2)\cdots n-m+2}{1.2.3\cdots m-1}$$

of the order $m-1$, to which on is sent back by the number m.

§ 40. After these remarks one sees easily that, if one expresses by M the number of marriages, P becomes

$$= \frac{M(M-1)(M-2)\cdots(M-m+2)}{1.2.3\cdots m-1}.$$

The determination of the letter m does not render the application of this expression difficult, because: the natural number which is at the left of any triangular number whatsoever $\frac{n(n-1)}{1.2}$ is n; now if $\frac{n(n-1)}{1.2}$ is in general the triangular number which is equal or more approachably $= M$, it is necessary that n become equal, or at least yet most nearly equal, to $\frac{1}{2}+\sqrt{2M+\frac{1}{4}}$, & consequently m will be always the greatest whole number less than $\frac{1}{2}+\sqrt{2M+\frac{1}{4}}$, or else absolutely $=\frac{1}{2}+\sqrt{2M+\frac{1}{4}}$, in the case where M is a triangular number.

§ 41. The complete determination of the numerator of the general formula that I seek, requires yet the evaluation of the coefficient of which I have spoken at the end of § 38. In order to arrive to it I observe that in the examples that I have developed, these

coefficients follow themselves in the order that is here:

Persons	3	5	7	9	11	13	15	17	19	21	&c.
Marriages	1	2	3	4	5	6	7	8	9	10	&c.
Coefficients	1	1	2	2	2	4	4	4	4	8	&c.

Thus, in the probabilities for 1 & 2 couples, the coefficient is 2°. For the following 3 probabilities, it is 2^1; for the 4 following it is 2^2; & it will arrive one time that in m results consecutively it will be 2^{m-2}, where m is the same as we have employed in the 2 §§ preceding.

§ 42. It follows thence that P being by § 39 the general term of the figurate numbers of order $m - 1$ or (§ 40)

$$= \frac{M(M-1)(M-2)\cdots(M-m+2)}{1.2\cdots(m-1)}$$

the numerator in question becomes $= 2^{m-2}P + 2P$, or

$$(2^{m-2} + 2) \times \frac{M(M-1)(M-2)\cdots(M-m+2)}{1.2\cdots(m-1)}.$$

§ 43. We pass to the general denominator that it remains to determine. And for this effect we examine the denominators of the results found in §§ 31, 32 & following.

Persons	$2^H.1^F$	$3^H.2^F$	$4^H.3^F$	$5^H.4^F$	$6^H.5^F$	$7^H.6^F$	$8^H.7^F$	&c.
Marriages	1	2	3	4	5	6	7	&c.
Denominators	4	12	36	120	400	1400	4900	&c.
Den. decomposed.	2(1.2)	2(2.3)	2(3.6)	2(6.10)	2(10.20)	2(20.35)	2(35.70)	&c.

§ 44. One sees that I have not made enter separately into this little Table the 4 numbers of the possible combinations, or which comprise the denominators. The reason for it is easy to sense; I would not be able to proceed here in the same manner as for the numerators because these 4 numbers could not be contained in one same formula. One would have, when M is odd, 4 times the square of the combinations according to $\frac{M+1}{2}$; & when M is even, 2 times the number of the combinations according to $\frac{M+2}{2}$, plus 2 times the square of the number of the combinations according to $\frac{M}{2}$.

§ 45. I remark moreover that the reduced denominators 4, 12, 36, 120 &c. follow a law too long hidden so that one must stop to put them into the formula such as they are, & that it is this which has engaged me to decompose them anew in the manner that one sees in the 4th transverse column.

§ 46. These decomposed numbers have the property that is here: They are the double of the product of the combinations of the numbers of persons of one & of the other sex, following the half of the even number which expresses alternately the one of the men & the one of the women. That is to say that these combinations are made alternately according to $\frac{M+1}{2}$ & according to $\frac{M}{2}$. For example for 4^H & 3^F one has 2 times (3.6): in this product 3 is the number of the combinations according to the half

of 4 (or two by two) of $3F$. & 6 is the number of the combinations two by two of $4H$. For $5^H.4^F$ one has $2(6.10)$, that is to say 2 times the number of combinations two by two of $4F$ multiplied with the number of combinations two by two of $5H$.

I speak of this property only because it merits in itself to be remarked & that it can have its utility in some particular cases. Because besides it has also the inconvenience to not be able to give one same formula for all the cases. One would obtain a different general denominator, according as the number of Men or else the one of the Women would be even. It is in the following manner that I consider these numbers in order to arrive to the end that I myself propose.

§ 47. I set apart the number 2 which multiplies all these products, & I note in the sequence of these products for 1, 2, 3 &c. Marriages

$$(1.2),\ (2.3),\ (3,6),\ (6.10),\ (10.20),\ (20.35),\ (35.70)\ \&c.$$

that the first members of each product

$$1,\quad 2,\quad 3,\quad 6,\quad 10,\quad 20\quad \&c.$$

not only follow the same law as we have made to remark above (§ 39. 1°), but that these are also the same numbers as P for the same number of marriages; whence it follows that the general expression for these first members is also

$$\frac{M(M-1)(M-2)\cdots(M-m+2)}{1.2\cdots(m-1)}.$$

As for the second members of the products in question, which are

$$2,\quad 3,\quad 6,\quad 10,\quad 20,\quad 35,\quad 70\quad \&c.$$

they observe a law which is of the same nature, but with a difference that one seized easily by considering 1°. that of the members or numbers in question

The 1st for	1 Marriage is taken in the	1st	Order of the Fig. Nos.
The 2 following prev.	2, 3 — are taken in the	2nd	— — —
The 3	4, 5, 6 — — —	3rd	— — —
The 4	7, 8, 9, 10 — — —	4th	— — —
. . .			
The μ Nos. prev.	$\frac{\mu(\mu-1)}{2}+1,\ \frac{\mu(\mu-1)}{2}+2,\cdots\frac{\mu(\mu-1)}{2}+\mu$ in the	μ^{th}	— — —

2° That μ signifies here the natural number which is at left, immediately above from the triangular number equal to M or from the first triangular number greater than M. Or else: that μ is the whole number $= \sqrt{2M+\frac{1}{4}} - \frac{1}{2}$, or the first whole number which surpasses $\sqrt{2M+\frac{1}{4}} - \frac{1}{2}$.

3° That knowing in what order of the figurate numbers one must seek each second member of the products, there remains in order to find this number only to substitute $M+1$ instead of n into the general term of that order.

§ 48. We conclude thence that the products of which we speak, multiplied with the number 2 that we had set to the side, are generally expressed by

$$2\left(\left(\frac{M.(M-1).(M-2)\cdots M-m+2}{1.2\cdots m-1}\right)\times\left(\frac{(M+1)M(M-1)\cdots(M-\mu+2)}{1.2.3\cdots\mu}\right)\right)$$

& finally that the general formula of the probability that we sought under the hypothesis of § 30 is

$$\frac{(2^{m-2}+2)\left(\frac{M.(M-1).(M-2)\cdots M-m+2}{1.2\cdots m-1}\right)}{2\left(\left(\frac{M.(M-1).(M-2)\cdots M-m+2}{1.2\cdots m-1}\right)\times\left(\frac{(M+1)M(M-1)\cdots(M-\mu+2)}{1.2.3\cdots\mu}\right)\right)}$$

or

$$\frac{(2^{m-3}+1)(1.2.3\cdots\mu)}{(M+1)M(M-1)\cdots(M-\mu+2)}.$$

One could dispense with introducing the letter μ in this general expression: but it would be necessary then to express the result that one would seek by two different analytic expressions of which the one would serve when M is a triangular number, & the other when this number is not triangular. This comes from this that in the 1$^{\text{st}}$ case (where M is a triangular number) μ is always $= m - 1$, & that in the others μ is constantly $= m$.

One sees easily hence which of the two cases takes place when one has substituted for M its value into $\sqrt{2M + \frac{1}{4}}$: because if M is a triangular number, this quantity becomes rational; if it is not, it remains a surd.

§ 49. If I had not myself stopped perhaps already too long on this material, I would give yet here the comparison of the numerical results of §§ 31, 32 &c. with those of §§ 11, 12 & following which would not permit being interesting; but the formula of § 48 being never in default in the numerical applications, & being very simple, I will content myself to have proved thence that there is no objection to make against the application that I have made in this Memoir of the calculus of the probabilities & of many good properties of figurate numbers.

Orders of the figurate Numbers

	0	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	&c.
1	1	0	0	0	0	0	0	0	0	0	0	0	
2	1	1	0	0	0	0	0	0	0	0	0	0	
3	1	2	1	0	0	0	0	0	0	0	0	0	
4	1	3	3	1	0	0	0	0	0	0	0	0	
5	1	4	6	4	1	0	0	0	0	0	0	0	
6	1	5	10	10	5	1	0	0	0	0	0	0	
7	1	6	15	20	15	6	1	0	0	0	0	0	
8	1	7	21	35	35	21	7	1	0	0	0	0	
9	1	8	28	56	70	56	28	8	1	0	0	0	
10	1	9	36	84	126	126	84	36	9	1	0	0	
11	1	10	45	120	210	252	210	120	45	10	1	0	
12	1	11	55	165	330	462	462	330	165	55	11	1	

&c.