ON

SERIES OR SEQUENCES

IN

THE GENOISE LOTTERY*

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Histoire de l'Academie des sciences et belles lettres de Berlin for 1769†
Vol. 25 (1771), pp. 234253.

This title announces a subject on which Mr. Euler senior has advanced his researches as far as one can expect it by him, I need not say more: it will be therefore necessary to begin with some words of justification.

Loving the calculus of probabilities, I have not failed to give some attention to the lottery which at my arrival I found established here in the imitation of that of Genoa, & which offered several problems, related to this calculus, to resolve. The one to find the probability of the sequences, to which it seemed to me that one could have attached also some prizes, appeared to me especially curious; & the difficulties that I found in the solution made me to decide myself that it could not be unworthy of being presented to you.

I have occupied myself with it therefore from time to time until I learned from Mr. Euler that he treated the same subject; this was enough for me to abandon my design, & I waited only to see from the memoir of this illustrious geometer if I had reasoned correctly; he had the kindness to communicate to me & I have seen that the little that I had made, being based on some arguments which, if they are not sublime, are at least not false.

It is however not immediate that I was convinced. A difference in the hypotheses had occasioned one in the results; likewise there was occasion to embrace some different ways in order to arrive to them; & here is that which emboldened me, Sirs, to present to you mine.

§1. I have supposed that the wheel of fortune contains the 90 numbers, & that they are the natural numbers from 1 to 90 inclusive, arranged in a circle in such a way that

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†This memoir had been read in 1765, after the memoir of Mr. Euler on this matter inserted in the Mémoires de l'Académie for that year. Since the memoirs of Mr. Beguelin printed after that of Mr. Euler refer to mine in several places, & since the lottery which has occasioned it is more in vogue than ever, I will not suppress it any longer. If my method does not lead as far as that of Messers Euler & Beguelin, it has at least, I believe, the advantage of being easier to understand.

one cannot say that there is no number which precedes 1, nor any which follows 90. One will sense at first the reason for this hypothesis; beyond that it encouraged my manner of proceeding, it seemed to me that it would not fail to be adopted by every director of a similar lottery, who would have designed to attach some prize to the events of which there is question; because, in making in this way that the numbers 90 & 1 form also a sequence, one avoids the small inconvenience that there would be to not render all the numbers indifferent to the players. It is this supposition that Mr. Euler has found more appropriate to reject, & whence is seen the difference in our results of which I speak.

§2. Before going further, I must warn again that this assembly of two or more numbers which follow themselves in the order of the natural numbers, that Mr. Euler calls a *sequence*, I have named a *series*, & that in order to distinguish the series among themselves, I named *binary series* the one formed of two numbers, *ternary series* the assembly of three numbers; *quaternary series* the one of 4 numbers &c. I will retain this definition; thus 30, 31 is a binary series just as 90, 1; 30, 31, 32 will be a ternary series just as 90, 1, 2 &c.

The simplest question, the one of which I have believed that it was acceptable to start, is this one.

PROBLEM 1.

§3. *Suppose that one draws two numbers from 90 numbers which are on a wheel; to find the expectation of the one who would wager that these two numbers will form a series.*

SOLUTION

I remark at first that as these two Numbers are equally able to be supposed drawn all at once or one after the other, I will adopt the second as this hypothesis. This put, since it is indifferent which number comes out in first place, the lot of the bettor will depend on the second number which one will draw, & which, if the first is $= m$, must be either $m - 1$ or $m + 1$ in order to win; now the number of all the cases being 89, we see that there are 2 which make the bettor win; therefore 87 cases make him lose, & his expectation will be expressed by $\frac{2}{89}$.

PROBLEM 2.

§4. *To find the probability that there will be binary series*[1] *in three numbers drawn from the wheel.*

SOLUTION.

One will consider firstly that the first No. being indifferent, there will be in the second drawing again 89 cases of which 2 will make at first the bettor win, & of which 87 will give to him expectation that the 3rd No. will form a series; but what is that

[1]I say *binary series*, & not *a binary series*, because I suppose that it is not important how many of these series the drawn numbers can form & that it is indifferent to the bettor how many he will draw of them.

expectation? It is evident that it will not be that which we have found in the preceding section; because it is necessary to pay attention: 1. That there are more than 88 Numbers on the wheel. 2. That indicating by m the number of Numbers drawn in the first place, if the one of the second Number is greater than $m + 2$ or less than $m - 2$, there will be 4 cases of the third drawing which will make the bettor win; but if the number of the second ticket is $m + 2$ or $m - 2$, there will be only 3. I will argue therefore in the following way. If the numbers of the tickets already drawn differ among themselves by more than 2, there will be 4 of the 88 remaining which will make the bettor win & his expectation will be

$$= \frac{2 + 87 \times \frac{4}{88}}{89} = \frac{176 + 348}{88 \times 89} = \frac{524}{88 \times 89};$$

but, if the said numbers surpass one another only by 2, the expectation of the bettor will be

$$= \frac{2 + 87 \times \frac{3}{88}}{89} = \frac{176 + 261}{88 \times 89} = \frac{437}{88 \times 89}.$$

Now, of 89 cases which can happen in the drawing of the second number, one knows already that there are 2 of them which have not taken place, namely those where the number of this ticket would have been marked by $m - 1$, or by $m + 1$; (otherwise there would already be a *series*, & the question here is of the cases where there had not been,) & of the 87 remaining cases there are 2 where the number of the second number is expressed by $m - 2$, or by $m + 2$, & which makes the lot of the bettor proportional to $\frac{437}{88 \times 89}$ as one has seen, & 85 where the second number is marked neither by $m - 2$, nor by $m + 2$, & where the lot of the bettor is proportional to $\frac{524}{88 \times 89}$; therefore finally the total expectation sought of the player will be expressed by

$$\frac{2 \times \frac{437}{88 \times 89} + 85 \times \frac{524}{88 \times 89}}{87} = \frac{2 \times \frac{437}{88 \times 89} + \frac{85 \times 437}{88 \times 89} + \frac{85 \times 87}{88 \times 89}}{87}$$
$$= \frac{\frac{87 \times 437}{88 \times 89} + \frac{85 \times 87}{88 \times 89}}{87} = \frac{437 + 85}{88 \times 89} + \frac{522}{88 \times 89} = \frac{2 \times 3 \times 87}{88 \times 89}.$$

I will remark that at the same time it is necessary to begin by supposing 2 numbers already drawn. We indicate a more convenient solution where one supposes that the second number is not yet drawn. It will be useless to observe that the result must be the same.

Let the number of the first ticket be $= m$. When one draws the second ticket, there will be 2 cases, namely $m - 1$, & $m + 1$, which will make the punter win at first: there are two others, $m - 2$ & $m + 2$, which will give to him, as we have seen, the expectation $\frac{3}{88}$. All the other cases, of which the number is 85, render the expectation $= \frac{4}{88}$. One has therefore

$$\frac{2 \times 1 + 2 \times \frac{3}{88} + 85 \times \frac{4}{88}}{89} = \frac{176 + 6 + 340}{88 \times 89} = \frac{522}{88 \times 89} = \frac{2 \times 3 \times 87}{89 \times 88},$$

as above.

FIRST LEMMA.

§5. *To find the probability, when drawing in sequence 3 numbers from among the 90 in the lottery, that the numbers of these 3 numbers follow themselves in the order of an arithmetic progression of which the difference is 2.*

SOLUTION

It is clear that the first number which will be drawn is indifferent; we will suppose therefore this number $= m$; & we will remark that of the 89 cases which can happen there are only four which with the third number can produce the progression, namely $m-2$, $m+2$, $m-4$, $m+4$; we imagine that one of the first two have taken place in the second drawing, there will be in the third, two cases of 88, which will make the progression succeed, & the probability will be $= \frac{2}{88}$. If it is $m-4$ or $m+4$ which is brought forth, there will be only one case which will produce the progression, & the probability will be expressed by $\frac{1}{88}$. Therefore in the second drawing there will be 2 cases which give $\frac{2}{88}$, 2 cases which give $\frac{1}{88}$, & 85 which give zero; the sought probability will be consequently

$$= \frac{2 \times \frac{2}{88} + 2 \times \frac{1}{88} + 85 \times 0}{89} = \frac{6}{88 \times 89}.$$

SECOND LEMMA.

§6. *To find the probability that in drawing in sequence three numbers from among the 90, there will be two which differ between themselves by 2 units, & that the third will differ by more than 2 from both of the preceding.*[2]

SOLUTION.

We suppose the number of the first ticket which one draws $= m$, & we see what will be the different probabilities that the 89 remaining cases will bear in the second drawing. I say at first that there will be 2 cases which give 0, these are those where the second number is $m+1$ or $m-1$; next, if this second number is $m+2$ or $m-2$, these two cases will give the expectation $\frac{83}{88}$; in third place, if the second number is $m+3$ or $m-3$, $m+4$ or $m-4$, the probability will be $\frac{2}{88}$, which is therefore produced by 4 cases; next the 81 remaining cases, where the difference between the first 2 numbers surpasses 4, will give $\frac{4}{88}$; consequently the sought probability is

$$\frac{2 \times 0 + 2 \times \frac{83}{88} + 4 \times \frac{2}{88} + 81 \times \frac{4}{88}}{89} = \frac{498}{88 \times 89}.$$

[2]One sees well by this second hypothesis the binary series are excluded, & would make the one lose who would wager on the event; it would be useless to attach this condition to the preceding Lemma, because a series would have rendered the probability of the event null, that which is not here when one demands simply that only 2 of the 3 drawn numbers differ by two units. To the rest one will not wait long to see the reasons for this hypothesis.

COROLLARY.

§7. One will find also the probability that the three numbers that one will draw will differ by more than 2, by adding together the results of the solutions of 2nd Problem, & the two preceding Lemmas, & by subtracting this sum from unity; because the said sum being $\frac{1026}{88\times 89}$, if one subtracts it from 1, the residue $\frac{6806}{88\times 89}$ will express the probability of which there is concern.

PROBLEM 3.

§8. *To find the probability that, in 4 numbers which one draws from 90, there will be found at least one binary series.*

SOLUTION.

The solution of this problem is deduced easily from that which precedes: we have only to remark that by virtue of §4. there is $\frac{522}{7832}$ probability that there will come forth a series in the first three numbers, & that thus there will be 522 cases which give expectation 1, & 7310 which allow hope that the fourth number will form a series with one of the preceding. But these 7310 cases do not give all the same probability; it varies according to the arrangement that the 3 numbers drawn in the first place maintain among themselves. Because *a*) if these 3 numbers follow in an arithmetic progression of which the difference is 2, there will be only 4 cases, of 87 possibles in the fourth drawing, which will make a winner of the one who wagers would wager for the series.

b) If 2 only of the three preceding numbers differ among themselves by 2 units, & if the difference of the three to those is greater, the expectation of the bettor will be proportional to $\frac{5}{87}$, having then 5 cases which make him a winner.

c) Finally, if the 3 numbers already drawn all differ by more than 2 from one another, 6 cases will turn to the advantage of the bettor, & his expectation will be expressed by $\frac{6}{87}$.

Now we have found in §6 that the probability of the event

	of *a* is	$= \frac{6}{88\times 89}$.
in §7. that of	b	$= \frac{498}{88\times 89}$,
in §8. that of	c	$= \frac{6806}{88\times 89}$.

Thus the probability that we seek will be

$$= \frac{522\times 1 + 6\times \frac{4}{87} + 498\times \frac{5}{87} + 6806\times \frac{6}{87}}{88\times 89} = \frac{88764}{87\times 88\times 89}$$
$$= \frac{3\times 4\times (86^2+1)}{87\times 88\times 89}.$$

PROBLEM 4.

§9. *To find the probability that in 5 numbers that one draws, there will be at least one binary series.*

SOLUTION.

In order to resolve this question, it is necessary to pay attention that of $89 \times 88 \times 87$ cases or 681384 cases, there are 8876 which give a series in the first four numbers which one draws, but, that if these 4 numbers do not form a sequence, they will be disposed necessarily among themselves in one of the 5 ways which follow.

α) Either the 4 numbers form an arithmetic progression of which the difference is 2.

β) Or 3 among them form this progression, & the fourth differs by more than 2 units from all the others.

γ) Or 2 of the 4 numbers will differ by two units from one another, & the two others likewise, but so that there is at least a difference of 3 between the two pairs.

δ) Or 2 numbers will differ between themselves by 2, & the difference of the two others so much between themselves as to each of those will be greater.

ϵ) Or all 4 numbers will differ at least by 3 units from one another.

Here are therefore 5 cases of which it is proper before all those seeking the probability, because, the third excepted, they render all the expectation of a series, at the fifth drawing, different. One could in truth be dispensed from making the calculation for the last arrangement, which is the most prolix, or else those that demand the cases β & γ, which give the same probability; but as in the enumeration of so many cases one can easily commit some error, I have preferred to make the calculation for all five arrangements, in order to assure myself of the correctness of my solution.

1. I remark at first, as for the first arrangement, that we have seen in §5 that there is $\frac{6}{7832}$ probability that the first 3 numbers will form an arithmetic progression when the difference is 2. Of 7832 cases there are therefore 6 which give $\frac{2}{87}$ probability. We see what expectation remains when the first three numbers are not in progression. Let the number of the first ticket be expressed by m, the expectation will be always null unless the one of the second ticket is one of the following six

$$m-2, \quad m+2, \quad m-4, \quad m+4, \quad m-6, \quad m+6.$$

Suppose, for example, that the second number is $m+2$; since the progression is excluded in the first 3 numbers, there are only $m+6$ & $m-4$ which can make it succeed, having then $\frac{1}{87}$ probability that the fourth number will form the progression. Now, for these two cases $m+6$ & $m-4$, there are $\frac{2}{88}$ of expectation; one sees therefore that of 89 cases which can take place in the second drawing, there are 6 which give $\frac{2}{88}$, & 82 which give 0, & that consequently the expectation for the case α is

$$= \frac{6 \times \frac{2}{87} + 12 \times \frac{1}{87} + 7814 \times 0}{88 \times 89} = \frac{24}{89 \times 88 \times 87}.$$

2. One could calculate the probability of the arrangement β by the method that I will employ in the following, but it is a difficulty that one can spare oneself by observing that there must be necessarily 81 times as many cases for this arrangement as

there are for the preceding, & that thus one can find only the probability $\frac{24\times 81}{89\times 88\times 87}$, or $\frac{1944}{89\times 88\times 87}$.

3. In order to answer the question: What is the probability that the first 4 numbers are arranged among themselves in the manner γ? I will suppose at first that m is the first number which comes forth, & I will say: there are two cases in the second drawing, where the probability is null, these are those where the second number that one draws is $m-1$ or $m+1$. I examine next what is the probability when the second number is $m-2$ or $m+2$. We suppose that this is $m+2$; I say that of 88 cases which can take place, there are 5 where the probability is 0, namely $m-2$, $m-1$, $m+1$, $m+3$, & $m+4$, because here there must be, neither a series, nor a progression, as in the arrangements α & β. I say moreover that there are 4 cases, $m+5$ & $m+6$, $m-5$ & $m-6$, which give $\frac{1}{87}$ probability that the 4 numbers will succeed to the arrangement in question; finally that the 79 remaining cases give $\frac{2}{87}$ probability for this arrangement of the numbers after the fourth drawing. Whence I conclude that, if the second number is $m-2$ or $m+2$, the probability for the arrangement γ is $\frac{162}{87\times 88}$, & reflecting in the same manner on all the cases, I find that if

the 2nd number is	$m-3$ or $m+3$	the probability is	$=\frac{2}{87\times 88}$,
	$m-4$ or $m+4$		$=\frac{2}{87\times 88}$,
	$m-5$ or $m+5$		$=\frac{6}{87\times 88}$,
	$m-6$ or $m+6$		$=\frac{6}{87\times 88}$,
	$m-7$ or $m+7$		$=\frac{8}{87\times 88}$.

All the other cases give the same probability $=\frac{8}{87\times 88}$; & therefore, being that one has drawn any number, the probability for the arrangement γ is

$$= \frac{2\times 0 + 2\times \frac{162}{87\times 88} + 4\times \frac{2}{87\times 88} + 4\times \frac{6}{87\times 88} + 77\times \frac{8}{87\times 88}}{89}$$
$$= \frac{972}{89\times 88\times 87}.$$

4. One will find likewise the probability for the case δ by examining scrupulously all the cases, as many of those which make vanish the expectation of the arrangement, as those which produce the arrangement, in the three drawings which follow the first, until that which one finds the probability after the second drawing constant. One will see that if

the 2nd number is	$m-2$ or $m+2$	the probability is	$=\frac{6480}{87\times 88}$,
	$m-3$ or $m+3$		$=\frac{480}{87\times 88}$,
	$m-4$ or $m+4$		$=\frac{474}{87\times 88}$,
	$m-5$ or $m+5$		$=\frac{788}{87\times 88}$,
	$m-6$ or $m+6$		$=\frac{782}{87\times 88}$,
	$m-7$ or $m+7$		$=\frac{776}{87\times 88}$,

The same probability that we just indicated for $m-7$ & $m+7$, is found for all the cases following the second drawing, such that the expectation that the 4 numbers will be arranged in the manner δ, will be expressed by this quantity

$$\begin{gathered}2\times 0+2\times \tfrac{6480}{87\times 88}+2\times \tfrac{480}{87\times 88}+2\times \tfrac{474}{87\times 88}\\ +2\times \tfrac{788}{87\times 88}+2\times \tfrac{782}{87\times 88}+77\times \tfrac{776}{87\times 88}\end{gathered}$$

which reduces to $\frac{77760}{89\times 88\times 87}$.

5. There remains for us to determine the probability that all 4 numbers that one draws in the first place will differ by more than 2 among themselves. One will note that here there are 4 cases which at first give zero to the second drawing, these are the numbers $m-1,\ m+1,\ m-2,\ m+2$. Passing next to the following, as we have made up to here, one will see that if

the 2nd number is	$m-3$ or $m+3$	the probability is	$=\frac{6320}{87\times 88}$,
	$m-4$ or $m+4$		$=\frac{6162}{87\times 88}$,
	$m-5$ or $m+5$		$=\frac{6006}{87\times 88}$,
	$m-6$ or $m+6$		$=\frac{6010}{87\times 88}$,
	$m-7$ or $m+7$		$=\frac{6012}{87\times 88}$.

It is again in these two cases that the probability becomes constant, although in the beginning, I want to say for $m-8$ & $m-9,\ m+8$ & $m+9$, the terms which constitute the probability will be different. The same thing happens also in the calculation for the other arrangements of 4 numbers, as one will see by verifying the results that I have only indicated here. One has therefore finally

$$\frac{4\times 0+2\times \frac{6320}{87\times 88}+2\times \frac{6162}{87\times 88}+2\times \frac{6006}{87\times 88}+2\times \frac{6010}{87\times 88}+2\times \frac{6012}{87\times 88}}{89}$$

$=\frac{511920}{89\times 88\times 87}$. Here is therefore all the different cases which influence on the probability that we seek, developed, & the sum of all these numerators being found equal to the product of $89\times 88\times 87$, proves that the enumeration has been exact, (unless by chance one error has canceled another.) There remains for us therefore only to pass to the complete solution of our problem; it no longer contains anything intricate; it will suffice to pay attention that the first four numbers being drawn without containing a sequence, if they are disposed in the manner α, there will be at the fifth drawing $\frac{5}{86}$ probability that this fifth number will form a series with one or two of the preceding. But

if the arrangement β or γ, takes place this probability will be	$=\frac{6}{86}$,
if it is δ	$\frac{7}{86}$,
ϵ	$\frac{8}{86}$.

And one will conclude thence that the probability which one seeks in the problem is

$$\begin{aligned}&=\frac{88674\times 1+24\times \frac{5}{86}+2916\times \frac{6}{86}+77760\times \frac{7}{86}+511920\times \frac{8}{86}}{89\times 88\times 87}\\ &=\frac{88674+120+17496+544320+4095360}{89\times 88\times 87\times 86}\\ &=\frac{12291000}{89\times 88\times 87\times 86}=\frac{4\times 5\times (85^3+5\times 85)}{89\times 88\times 87\times 86}.\end{aligned}$$

This is the most simple decomposition that the numerator of our fraction admits.

§10. In undertaking the solution of the preceding section I anticipated the prolixity, but I flattered myself that the result compared with those that I have found previously would indicate to me a law for the probabilities of the binary series contained in some number of tickets that one would draw, so that one could find some results without other calculations & to arrange them in the first vertical column of a table. But I have seen that it is necessary rather to seek this law in the terms which express these probabilities before one has formed them. In order to be more clear, I place beneath the eyes the results found hitherto, such as they were before the reduction

The solution of the 1st Problem has given $\frac{2\times 1+87\times 0}{89}$,

that of the 2nd Problem $\frac{2\times 1+2\times\frac{3}{88}+85\times\frac{4}{88}}{89}$,

that of the 3rd Problem $\frac{522\times 1+6\times\frac{4}{87}+6\times 83\times\frac{5}{87}+82\times 83\times\frac{6}{87}}{88\times 89}$,

that of the 4th Problem $\frac{88674\times 1+24\times\frac{5}{86}+6\times 6\times 81\times\frac{6}{86}+12\times 80\times 81\times\frac{7}{86}+79\times 80\times 81\times\frac{8}{86}}{89\times 88\times 87}$.

By comparing these quantities jointly, it seems to me that one could pronounce boldly that the probability that there would be a binary series in 6 numbers would be expressed by the sum of the following terms:

$$\begin{aligned}&\big(12291000\times 1+120\times\tfrac{6}{85}+6\times 6\times 6\times\, 79\times\tfrac{7}{85}\\&+12\times 12\times 78\times 79\times\tfrac{8}{85}+20\times 77\times 78\times 79\times\tfrac{9}{85}\\&+76\times 77\times 78\times 79\times\tfrac{10}{85} : 89\times 88\times 87\times 86.\end{aligned}$$

But I assert that the sum itself is not found the same as the result which the formulas of Mr. Euler gives, which I have said can be restored in my hypothesis[3]. Since however

[3]Here is how this is made: Mr. Euler finds that, if of any number n of tickets one draws from them 2 or 3 or 4 or 5 &c. the probabilities that there will be found no sequence at all in these drawn numbers will be

$$\begin{aligned}&\text{For 2 numbers} &&= \tfrac{n-2}{n},\\&3 &&= \tfrac{(n-3)\cdot(n-4)}{n\cdot(n-1)},\\&4 &&= \tfrac{(n-4)\cdot(n-5)\cdot(n-6)}{n\cdot(n-1)\cdot(n-2)},\\&5 &&= \tfrac{(n-5)\cdot(n-6)\cdot(n-7)\cdot(n-8)}{n\cdot(n-1)\cdot(n-2)\cdot(n-3)},\\&\&c. &&\&c.\end{aligned}$$

Consequently the probabilities that the drawn numbers will form at least one sequence will be

$$\begin{aligned}&\text{For 2 numbers} &&= 1-\tfrac{n-2}{n},\\&3 &&= 1-\tfrac{(n-3)\cdot(n-4)}{n\cdot(n-1)},\\&4 &&= 1-\tfrac{(n-4)\cdot(n-5)\cdot(n-6)}{n\cdot(n-1)\cdot(n-2)},\\&5 &&= 1-\tfrac{(n-5)\cdot(n-6)\cdot(n-7)\cdot(n-8)}{n\cdot(n-1)\cdot(n-2)\cdot(n-3)},\\&\&c. &&\&c.\end{aligned}$$

Now, in order to bring over these formulas to our Hypothesis, it suffices to subsitute everywhere $n-1$ in place of n; because one will have then the probabilities of one sequence at least in

$$\begin{aligned}&\text{For 2 numbers} &&= 1-\tfrac{n-3}{n-1},\\&3 &&= 1-\tfrac{(n-4)\cdot(n-5)}{(n-1)\cdot(n-2)},\\&4 &&= 1-\tfrac{(n-5)\cdot(n-6)\cdot(n-7)}{(n-1)\cdot(n-2)\cdot(n-3)},\\&5 &&= 1-\tfrac{(n-6)\cdot(n-7)\cdot(n-8)\cdot(n-9)}{(n-1)\cdot(n-2)\cdot(n-3)\cdot(n-4)},\\&\&c. &&\&c.\end{aligned}$$

the difference is small enough, one could attribute it to some error in the calculations which are not permitted to be sufficiently prolix.

§11. It is not that the results fail entirely in regularity; one has only to cast the eyes on the 4 which my solutions have given, & on the 2 which I have calculated according to the formulas of Mr. Euler, in order to be convinced of it; here they are:

Probabilities of at least one sequence:

In 2 numbers	$= \frac{2}{89}$,
3	$= \frac{2\times3\times87}{89\times88}$,
4	$= \frac{3\times4\times((86)^2+1)}{89\times88\times87}$,
5	$= \frac{4\times5\times((85)^3+5\times85)}{89\times88\times87\times86}$,
6	$= \frac{5\times6\times((84)^4+3\times5\times84^2)+8}{89\cdots85}$,
7	$= \frac{6\times7\times((83)^5+5\times7\times((83)^3-\frac{12}{5}\times83))}{89\cdots84}$,

But one will have difficulty to perceive therein an invariable law.

§12. I have thought that perhaps by supposing, as Mr. Euler has done, that the sequences conclude at No. 90 there will be more simplification of the results, & in order to clarify it myself I have calculated six of them according to the Memoir of Mr. Euler; but I have found yet an astonishing analogue between all these results and mine, as one sees it.

Number of tickets drawn.	Probabilities of a sequence.
2	$\frac{2}{90}$,
3	$\frac{2\times3\times88}{90\times89}$,
4	$\frac{3\times4\times((87)^2+1)}{90\times89\times88}$,
5	$\frac{4\times5\times((86)^3+5\times86)}{90\cdots87}$,
6	$\frac{5\times6\times((85)^4+3\times5\times(85)^2+8)}{90\cdots86}$,
7	$\frac{6\times7\times((84)^5+5\times7((84)^3+\frac{12}{5}\times84))}{90\cdots85}$,

The reason for this uniformity seems to me worthy of being approved[4].

§13. Be that as it may, this prolixity in which one is obliged to engage before arriving to some thing for certain, & the easier application of the formulas of Mr. Euler have prevented me from pushing farther the researches that I have begun on the probabilities of series of more than 2 numbers; & I have no plan to return to them, unless I think to see in this matter a more immediate use. I will content myself to remark again, that the first significant terms of the vertical columns of the table that I proposed to construct, is found more easily by the fundamental principle of the doctrine of combinations than by any other method. I suppose, for example, that one seeks the probability that 5 numbers which one draws will form a quinary series. One knows that

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And putting in these quantities $n = 90$, one attains to the results which the solutions of our problems have given.

[4]The Memoirs of Mr. Beguelin leave nothing to desire on the comparison of the two hypotheses. One finds there moreover a multitude of ingenious ideas, & many simplifications.

90 numbers can be combined 5 to 5 in

$$\frac{86 \times 87 \times 88 \times 89 \times 90}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$$

different ways: now there are 90 cases where the quinary series succeeds, (from $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5$ until $90 \cdot 1 \cdot 2 \cdot 3 \cdot 4$); the probability that one would seek would be therefore

$$= \frac{90}{\frac{86\times 87\times 88\times 89\times 90}{1\cdot 2\cdot 3\cdot 4\cdot 5}} = \frac{120}{86 \times 87 \times 88 \times 89}.$$

One could have found in the same manner the probability that one sought in §5 & that of the arrangement α in §9. Because one will not have difficulty to understand that in drawing 3 numbers from among 90, it must have the same probability that these three numbers will form an arithmetic progression different from that which they produce by forming a ternary series. And it is the same in the other case.