

NO. CLIV
DE ALEA, SIVE ARTE CONJECTANDI
PROBLEMATA QUÆDAM

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PROBLEM I.

*Several Gamesters play with dice, of whom the first has a certain number of casts; it is being asked how many casts should be assigned to the second, to the third, to the fourth, to the fifth before the game, so that the lot of the ones become equal?*¹

Let the number of faces of the die by which one wins = a , & the number of faces by which one loses = b , the number of casts of the first Gamester = c , the number of Gamesters = n , & the number of casts, the ones associated with the casts of the previous = x . In one cast are a cases to obtain, & b to lose; therefore the lot of one cast = $\frac{a}{a+b}$. In the first of two casts there are a cases to obtain, & b to lose to chance $\frac{a}{a+b}$; therefore the lot of two casts = $\frac{ab}{(a+b)^2} + \frac{a}{a+b}$. In the first of three casts are a cases to 1, & b to $\frac{ab}{(a+b)^2} + \frac{a}{a+b}$; therefore the lot of three casts = $\frac{abb}{(a+b)^3} + \frac{ab}{(a+b)^2} + \frac{a}{a+b}$. In the same way the lot of four casts has = $\frac{ab^3}{(a+b)^4} + \frac{abb}{(a+b)^3} + \frac{ab}{(a+b)^2} + \frac{a}{a+b}$. And the lot of c casts

$$= \frac{ab^{c-1}}{(a+b)^c} + \frac{ab^{c-2}}{(a+b)^{c-1}} \cdots \frac{a}{a+b}$$

(because it is in geometric progression) = $1 - b^c : (a+b)^c$ = to the lot of the first Gamester. The lot of x casts will be in ratio equal = $1 - b^x : (a+b)^x$; because moreover the lot of the Gamesters is set equal, it will be

$$1 - b^x : (a+b)^x = n - nb^c : (a+b)^c,$$

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Translated by Richard J. Pulskamp, Department of Mathematics and Computer Science, Xavier University, Cincinnati, OH. The title may be read as “Certain problems concerning chance, or the art of conjecturing.” No date is attributed to it. Volume IV of the collected works appeared in 1742.

¹In modern terminology. The are n gamesters in total. Let the probability of a success on a given trial be p and failure be $q = 1 - p$. The first gamester has c opportunities to obtain his point. The probability of obtaining his point on the first or the second or the third. . . to the c^{th} trial is

$$p + pq + pq^2 + \cdots + pq^{c-1} = p \frac{1 - q^c}{1 - q}.$$

Now each of the other players will have x opportunities to obtain the point. This x is to be determined so that $p \frac{1 - q^x}{1 - q} = np \frac{1 - q^c}{1 - q}$ or equivalently, $1 - q^x = n(1 - q^c)$. It is certainly easy enough to solve for x . We obtain $x = \frac{\ln(nq^c - n + 1)}{\ln q}$. His final step is to remove the first Gamester from the set, by replacing n with $n - 1$. This gives

$$x = \frac{\ln(q^c(n - 1) - n + 2)}{\ln q}$$

or

$$1 - n + nb^c : (a + b)^c = b^x : (a + b)^x;$$

and therefore

$$\text{Log.}(1 - n + nb^c : (a + b)^c) = x lb - x l(a + b);$$

and therefore

$$x = l(1 - n + nb^c : (a + b)^c) : (lb - l(a + b)),$$

by which if the number of previous casts is removed [which is had by replacing $n - 1$ for n] there will remain

$$\frac{l(2 - n + (n - 1)b^c : (a + b)^c) - l(1 - n + nb^c : (a + b)^c)}{l(a + b) - lb}$$

=to the demanded number of casts.

DIFFERENTLY

Because it is equivalent, & the same expectation is had, if a single die is found cast as a single cast with just as many dice has been made; put in the place of the numbers of casts of the first Gamester is the number of dice = c , & in place of the numbers of casts of the sequence of Gamesters the ones with the casts, the number of dice = x . It follows from the art of combinations, because c dice (on account of $a + b$ faces of one die) are able to be varied in $(a + b)^c$ cases, & in b^c cases in which no faces of a itself fall, it is, in which it is lost; and therefore there are $(a + b)^c - b^c$ cases in which it is obtained: Therefore the lot of the first Gamester is found

$$= ((a + b)^c - b^c) : (a + b)^c = 1 - b^c : (a + b)^c,$$

as before. Equally the lot of the x casts will be = $1 - b^x : (a + b)^x$; the others are completed as before.

PROBLEM II.

Given $a + b$ faces on one die; it is demanded how many changes with one die, or, because it is just as much, how many dice with one turn someone is able to undertake that he casts one, two, 3, 4, &c., out of a faces.²

The number of dice shall be = x , the cases will be $(a + b)^x$ in which x dice are able to be varied, b^x cases in which no face falls of themselves a , $\frac{x}{1}b^{x-1}a^1$ cases in which one, $\frac{x(x-1)}{1.2}b^{x-2}a^2$ cases in which two, $\frac{x(x-1)(x-2)}{1.2.3}b^{x-3}a^3$ in which three &c. If therefore one out of the a faces must be thrown; the cases will be $(a + b)^x - b^x$ in which he wins: if

²The assumption here is to construct a fair game. In modern terminology the idea is this. Given that the player needs to achieve at least n successes, how many dice should be cast so that he has an even chance to do so. Let p and $q = 1 - p$ denote the probability of success and failure in the throw of one die. The probability that k successes occur in the toss of x dice is given by this term in the expansion of the binomial $(p + q)^x$:

$$\binom{x}{k} p^k q^{x-k}$$

whence we seek x so that

$$\sum_{k=0}^{n-1} \binom{x}{k} p^k q^{x-k} = \frac{1}{2}.$$

Here is an example. Let $p = \frac{1}{6}$ and $n = 5$. In this case we find $x \approx 27.7$. Rounding to 28 gives a slight favor to the player so that his probability of winning is about 51%.

If the number of dice is any odd, it is proven the lots between each other to be equals. If indeed it is even, it is that = $2n$, therefore that the number of cases, in which all casts are able to vary, is 6^{2n} ; among these the cases will be

$$\frac{(7n-1)(7n-2)(7n-3)\cdots(5n+1)}{1.2.3.4\dots(2n-1)},$$

in which the arithmetic mean between the extremes is able to fall; and exactly the lot of Peter is to the lot of Paul as

$$6^{2n} + \frac{(7n-1)(7n-2)(7n-3)\cdots(5n+1)}{1.2.3.4\dots(2n-1)}$$

to

$$6^{2n} - \frac{(7n-1)(7n-2)(7n-3)\cdots(5n+1)}{1.2.3.4\dots(2n-1)}$$

PROBLEM V.

*Two Gamesters A & B play with one die, with this condition, that A makes three successive throws, and he adds the points which he has cast in that three throws into one sum; B indeed makes so many casts as how many points A has made in the first cast, all cast points equally being collected into one: Moreover who will have held the greatest sum of points, that one will be victor; But if indeed for both the number of points are equal, then the stake will be divided into two parts. It is demanded the ratio of both lots?*⁵

R. The lot of A is to the lot of B as 4200563 to 5877133.

PROBLEM VI.

With the others put as before, let it be they both have had an equal sum of points, then further A will win. It is demanded the ratio of the lots?

R. The lot of A is to the lot of B as 282571 to 347285.

PROBLEM VII.

*With number of Electors given, who are multiples of three, yet not smaller than six; But two out of the Electors A & B will have declared themselves favoring someone C out of the candidates. It is demanded how much expectation C has, or what probability be, that A & B be arranged in one same class of three Electors by lot.*⁶

SOLUTION.

The number of ways to obtain $7n$ is given by the coefficient of x^{7n} in the expansion of

$$(x + x^2 + x^3 + x^4 + x^5 + x^6)^{2n}$$

so that the probability p is the ratio of this coefficient to 6^{2n} . For example, we have

n	Term	Coefficient	6^{2n}	$\frac{1+p}{1-p}$
1	x^7	6	36	$\frac{7}{5}$
2	x^{14}	146	216	$\frac{721}{575}$
3	x^{21}	4332	46656	$\frac{4249}{3527}$
4	x^{28}	135954	1679616	$\frac{100865}{85759}$

⁵The solutions given by Bernoulli for this problem and for the next are correct. Although Todhunter complains that the computations are lengthy, they can be expressed rather succinctly. See the addendum.

⁶Todhunter claims that this is unintelligible. Pearson, on the other hand, offers an interpretation which does make sense. It would seem that Bernoulli has made another error here.

Let the number of Electors be $6 + 3n$. I say the sought probability to be $= \frac{1+n}{5+3n}$; This is the probability that A & B are united in the same class, is to the probability of the contrary event as $1 + n$ to $4 + 2n$.

COROLLARY I.

If there should be 6 Electors, the ratio of expectations to the fear of adverse success is, as 1 to 4.

COROLLARY II.

If the number of Electors was infinite, that ratio should be as 1 to 2.

COROLLARY III.

Hence to any extent the greater be the number of Electors, therefore C has more favorable expectation. What had been seen a paradox.

Addendum on Problems V and VI by the Translator.

Herein the solutions to Problems V and VI are derived in modern notation. The first roll of Player A determines the number of rolls of Player B. Hence Player B may roll from 1 to 6 dice each of these outcomes occurring with probability $\frac{1}{6}$. The distributions of the outcomes of Player B are consequently governed by some probability distributions p_k for $k = 1, 2, \dots, 6$ to which k corresponds to the number of dice cast. On the other hand, Player A, given the outcome of the first roll, always has a sum distributed as p_2 . We therefore proceed by conditioning on the 1st roll of Player A.

Given the 1st roll yields 1, Player A can obtain any value from $2 + 1$ to $12 + 1$ but B can only obtain values from 1 to 6. A tie occurs if B rolls k and A rolls the sum $k - 1$ on the remaining two tosses. Therefore, the probability that *Player A tie Player B* is the $\sum_{k=3}^6 p_1(k)p_2(k-1)$. To see where Player A beats B, suppose Player B obtains the outcome k , then Player A wins with any outcome from $k + 1$ to 13. This requires that for the remaining two throws Player A obtain at least k . The probability that *Player A beat Player B* is clearly $\sum_{j=k}^{12} p_1(k)p_2(j)$ where $k = 1$ to 6. Finally, we note Player B has only three potentially winning throws. He cannot win with a 1 or 2 and he can do no more than tie with a 3. If Player B throws k , where $k \geq 4$, then Player A will lose if he throws at most $k - 2$. Therefore *Player A loses to Player B* with probability $\sum_{j=1}^{k-2} p_1(k)p_2(j)$ where $k = 4$ to 6.

Consider now that the 1st row produces a three. Player A can obtain any value from $2 + 3$ to $12 + 3$ and Player B from 3 to 18. Given B throws k , a tie occurs if Player A throws a total of $k - 3$ on the remaining two tosses. The probability *Player A ties Player B* is therefore $p_3(k)p_2(k-3)$ for $k = 5$ to 18. To see where A beats B, suppose B threw a $k = 12$, say. A would need at least a 10 or $k - 2$ from the remaining 2 dice. The probability that Player A beats Player B is $\sum_{j=k-2}^{12} p_3(k)p_2(j)$ for $k = 1$ to 6. Finally, Player B needs at least a 6 to win since 3 and 4 always lose and 5 only ties. If Player B casts, say, $k = 11$, then Player A will lose if his remaining two throws produce at most $k - 4 = 7$. In general, if Player A loses to Player B with probability $\sum_{j=1}^{k-4} p_3(k)p_2(j)$ where $k = 6$ to 18.

As Bernoulli, let the number of electors be $6 + 3n$. There are clearly $\binom{6+3n}{3}$ ways to select 3 electors from among $6 + 3n$. However there are only $4 + 3n$ sets of size 3 containing the two electors A and B. We have

$$\frac{4 + 3n}{\binom{6+3n}{3}} = \frac{2}{(2+n)(5+3n)}$$

The ratio of this event to the contrary is $\frac{2}{8+11n+3n^2}$. If there are 6 electors, the ratio of expectations is $\frac{1}{4}$ just as Bernoulli noted. However, as n increases without bound the ratio tends to 0. Hence there is no paradox.

Let the outcome of the first roll be n , the sum produced by Player B be k and the sum produced by Player A be j . If further we define $p_k(j)$ for $j = 1$ to 36 for each k , then we have the following formulas.

$$\Pr(\text{Player A ties Player B}) = \frac{1}{6} \sum_{n=1}^6 \sum_{k=2+n}^{6n} p_n(k) p_2(k-n)$$

$$\Pr(\text{Player A beats Player B}) = \frac{1}{6} \sum_{n=1}^6 \sum_{k=n}^{6n} \sum_{j=k}^{36} p_n(k) p_2(j)$$

$$\Pr(\text{Player A loses to Player B}) = \frac{1}{6} \sum_{n=1}^6 \sum_{k=3+n}^{6n} \sum_{j=1}^{k-n-1} p_n(k) p_2(j)$$

Using the distributions as presented below, we obtain:

$$\Pr(\text{Player A ties Player B}) = \frac{320573}{5038848}$$

$$\Pr(\text{Player A beats Player B}) = \frac{215555}{559872}$$

$$\Pr(\text{Player A loses to Player B}) = \frac{347285}{629856}$$

Hence the expectation of A is $\frac{4200563}{10077696}$ and the expectation of B is $\frac{5877133}{10077696}$ when the stakes are divided equally in the case of tie. On the other hand, if Player A receives the stake in a tie, the expectation of A is $\frac{282571}{629856}$ and that of B is $\frac{347285}{629856}$.

x	$p_1(x)$	$p_2(x)$	$p_3(x)$	$p_4(x)$	$p_5(x)$	$p_6(x)$
1	$\frac{1}{6}$	0	0	0	0	0
2	$\frac{1}{6}$	$\frac{1}{36}$	0	0	0	0
3	$\frac{1}{6}$	$\frac{1}{18}$	$\frac{1}{216}$	0	0	0
4	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{72}$	$\frac{1}{1296}$	0	0
5	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{1}{36}$	$\frac{1}{324}$	$\frac{1}{7776}$	0
6	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{5}{108}$	$\frac{5}{648}$	$\frac{5}{7776}$	$\frac{1}{46656}$
7	0	$\frac{1}{6}$	$\frac{5}{72}$	$\frac{5}{324}$	$\frac{5}{2592}$	$\frac{1}{7776}$
8	0	$\frac{5}{36}$	$\frac{7}{72}$	$\frac{35}{1296}$	$\frac{35}{7776}$	$\frac{7}{15552}$
9	0	$\frac{1}{9}$	$\frac{25}{216}$	$\frac{7}{162}$	$\frac{35}{3888}$	$\frac{7}{5832}$
10	0	$\frac{1}{12}$	$\frac{1}{8}$	$\frac{5}{81}$	$\frac{7}{432}$	$\frac{7}{2592}$
11	0	$\frac{1}{18}$	$\frac{1}{8}$	$\frac{13}{162}$	$\frac{205}{7776}$	$\frac{7}{1296}$
12	0	$\frac{1}{36}$	$\frac{25}{216}$	$\frac{125}{1296}$	$\frac{305}{7776}$	$\frac{19}{1944}$
13	0	0	$\frac{7}{72}$	$\frac{35}{324}$	$\frac{35}{648}$	$\frac{7}{432}$
14	0	0	$\frac{5}{72}$	$\frac{73}{648}$	$\frac{5}{72}$	$\frac{43}{1728}$
15	0	0	$\frac{5}{108}$	$\frac{35}{324}$	$\frac{217}{2592}$	$\frac{833}{23328}$
16	0	0	$\frac{1}{36}$	$\frac{125}{1296}$	$\frac{245}{2592}$	$\frac{749}{15552}$
17	0	0	$\frac{1}{72}$	$\frac{13}{162}$	$\frac{65}{648}$	$\frac{119}{1944}$
18	0	0	0	$\frac{5}{81}$	$\frac{65}{648}$	$\frac{3431}{46656}$
19	0	0	0	$\frac{7}{162}$	$\frac{245}{2592}$	$\frac{217}{2592}$
20	0	0	0	$\frac{35}{1296}$	$\frac{217}{2592}$	$\frac{469}{5184}$
21	0	0	0	$\frac{5}{324}$	$\frac{5}{72}$	$\frac{361}{3888}$
22	0	0	0	$\frac{5}{648}$	$\frac{35}{648}$	$\frac{469}{5184}$
23	0	0	0	$\frac{1}{324}$	$\frac{305}{7776}$	$\frac{217}{2592}$
24	0	0	0	$\frac{1}{1296}$	$\frac{205}{7776}$	$\frac{3431}{46656}$
25	0	0	0	0	$\frac{7}{432}$	$\frac{119}{1944}$
26	0	0	0	0	$\frac{35}{3888}$	$\frac{749}{15552}$
27	0	0	0	0	$\frac{35}{7776}$	$\frac{833}{23328}$
28	0	0	0	0	$\frac{5}{2592}$	$\frac{43}{1728}$
29	0	0	0	0	$\frac{5}{7776}$	$\frac{7}{432}$
30	0	0	0	0	$\frac{1}{7776}$	$\frac{19}{1944}$
31	0	0	0	0	0	$\frac{7}{1296}$
32	0	0	0	0	0	$\frac{7}{2592}$
33	0	0	0	0	0	$\frac{7}{5832}$
34	0	0	0	0	0	$\frac{7}{15552}$
35	0	0	0	0	0	$\frac{1}{7776}$
36	0	0	0	0	0	$\frac{1}{46656}$