

General Solution to the Problem of Points¹

Suppose there are m players engaged in a game for which the winner must achieve n points. These players wish to terminate the game prematurely after each player has achieved some number of points less than n . What is sought is the share each player is due to the stakes at this time.

There is no attempt here to prove that the following is the correct solution to the Problem of Points. Rather we just present a form of the solution which lends itself to easy computation. The solution incorporates the Gamma function which, when x is a positive integer, satisfies $\Gamma(x) = (x - 1)!$. All of its instances may be replaced accordingly.

Rather than write out this general solution for m players, we suppose there are 3 players. This easily generalizes to m players by imitating the pattern developed below.

Assume Player i wins at each trial with probability p_i . Let a_i be the number of points lacking to Player i . Finally, let $\Phi_i(a_1, a_2, a_3)$ denote the probability that Player i wins the game.

It may be shown that we have this elegant general solution.

$$\begin{aligned}\Phi_1(a_1, a_2, a_3) &= \sum_{u_3=0}^{a_3-1} \sum_{u_2=0}^{a_2-1} \frac{\Gamma(a_1 + u_2 + u_3) p_1^{a_1} p_2^{u_2} p_3^{u_3}}{\Gamma(a_1) \Gamma(u_2 + 1) \Gamma(u_3 + 1)} \\ \Phi_2(a_1, a_2, a_3) &= \sum_{u_3=0}^{a_3-1} \sum_{u_1=0}^{a_1-1} \frac{\Gamma(u_1 + a_2 + u_3) p_1^{u_1} p_2^{a_2} p_3^{u_3}}{\Gamma(u_1 + 1) \Gamma(a_2) \Gamma(u_3 + 1)} \\ \Phi_3(a_1, a_2, a_3) &= \sum_{u_2=0}^{a_2-1} \sum_{u_1=0}^{a_1-1} \frac{\Gamma(u_1 + u_2 + a_3) p_1^{u_1} p_2^{u_2} p_3^{a_3}}{\Gamma(u_1 + 1) \Gamma(u_2 + 1) \Gamma(a_3)}\end{aligned}$$

To illustrate its use, we obtain the solution to the questions posed by Huygens in Appendix I. Here each player has the same probability of winning a point, namely $p_i = 1/3$ for all i .

$$\begin{aligned}\Phi(1, 1, 2) &= \left(\frac{4}{9}, \frac{4}{9}, \frac{1}{9} \right) \\ \Phi(1, 2, 3) &= \left(\frac{19}{27}, \frac{6}{27}, \frac{2}{27} \right) \\ \Phi(1, 2, 2) &= \left(\frac{17}{27}, \frac{5}{27}, \frac{5}{27} \right)\end{aligned}$$

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