Problems posed by Fermat to Chr. Huygens[1]

Problem 1 A and B play with two dice so that if A casts 6 points with his two dice before B casts a 7, then Player A wins. And if B casts a 7 before A casts his 6, then player B wins. Moreover Player A has the lead.

Solution: Player A casts a 6 with probability $p = 5/36$. Player B casts a 7 with probability $r = 6/36$. The turns of the players proceed as `ABABAB...` Player A wins if a success (`S` occurs on an odd turn with all previous casts being failures (`F`).

Put $q = 1 - p$ and $s = 1 - r$. The sequences `S, FFS, FFFFS, ...` occur with probability $p, qsp, q^2s^2p, \ldots$ It follows that the probability that A wins is

$$\sum_{k=0}^{\infty} q^k s^k p = \frac{30}{61}.$$

Consequently the probability that B wins is $31/61$ and hence the advantage of A to that of B is as 30 to 31.

Problem 2 Player A wins if he casts a 6 with two dice before Player B casts a 7, just as before. However the turns of the players proceed as `ABBAABBAA...`

Solution: This problem was incorporated as **Exercise 1** into the Treatise. Player A casts a 6 with probability $p = 5/36$. Player B casts a 7 with probability $r = 6/36$. Put $q = 1-p$ and $s = 1-r$. Player A wins for the following sequences `S, FFFS, FFFFS, FFFFFFFS, FFFFFFFFS, ...` which occur with probabilities p, qs^2p, q^2s^2p, q^3s^4p, q^4s^4p,... respectively. It is apparent that there are two subsequences (one where the exponents of q and s differ by 1, and the other where their exponents are equal) which may be summed easily simultaneously. Thus the probability that A wins is given by

$$p + p(qs^2 + q^2s^2) \times \sum_{k=0}^{\infty} (qs)^{2k} = \frac{10355}{22631}.$$

The probability that B wins is thus $\frac{12276}{22631}$. The advantage of A to B is 10355 to 12276.

Problem 3 This problem differs from the previous only in that the sequence of turns proceeds as `AABBBAABBB...`

Solution: There are again two subsequences formed from the events which give a win to A. Retaining the same notation as in Problem 2, it follows that the probability that A win is given by

$$p + qp + p(q^2s^3 + q^2s^3) \sum_{k=0}^{\infty} (q^2s^3)^k = \frac{72360}{159811}.$$

Consequently the probability that B wins is $\frac{87451}{159811}$. The advantage of A to that of B is as 72360 to 87451.

Problem 4 Three players A, B, C draw in order without replacement from a deck of 52 cards. The one who first draws a heart will win. What is the probability of success of each player?

Solution: The turns proceed as `ABCABCABC...`

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In order that a player win on turn k, it is necessary that the previous $k-1$ drawings take cards from the non-hearts and the kth drawing choose from the hearts. Thus, for example, the probability that the win occurs on turn $k=3$ is

$$\frac{39}{52}\cdot\frac{38}{51}\cdot\frac{13}{50}.$$

Now A wins if the first heart appears on the draws numbered 1, 4, 7, ... 40. It is easy to see that A wins with probability

$$13\sum_{k=0}^{13}\frac{39!(51-3k)!}{52!(39-3k)!}=\frac{1006853859}{2334608675}\approx 0.4313$$

Player B wins if the first heart appears on the draws numbered 2, 5, 8, ... 38. The probability of this event is

$$13\sum_{k=0}^{12}\frac{39!(49-3k)!}{52!(38-3k)!}=\frac{6072680801}{18676869400}\approx 0.3251$$

Finally, Player C wins if the first heart appears on the draws which are multiples of 3 from 1 to 39. The probability of this event is

$$13\sum_{k=0}^{12}\frac{39!(49-3k)!}{52!(37-3k)!}=\frac{4549357727}{18676869400}\approx 0.2436$$

Problem 5 A deck consists of 40 cards with 4 suits of 10 each. We seek the part of the one who would wager that the first four cards drawn will be from different suits to that who would wager the contrary.

Solution: This problem was incorporated as **Exercise III** of the Treatise. Let $C(n,k)$ denote the binomial coefficient "n choose k." The probability that 4 distinct suits are represented in the four cards drawn is clearly

$$\frac{C(10,1)^4}{C(40,4)}=\frac{10000}{91390}.$$

Therefore the part of the one is to the other as 1000 to 8139.

Problem 6 The first of two players of Piquet undertakes to have 3 aces in his first twelve cards. What is the part of the one here against the other who wagers that he will not have the three aces.

Solution: There are 36 cards used in Piquet of which 4 are aces. It should be noted that the problem can be interpreted in two ways: the player has exactly 3 aces among the 12 or at least 3 aces among the 12 cards.

Adopting the notation of the previous problem, the probability of having exactly 3 aces among the 12 is

$$\frac{C(4,3)C(32,9)}{C(36,12)}=\frac{32}{357}.$$

Hence the shares of the players are as 32 to 325. The probability of having as least 3 aces among the 12 is

$$\frac{C(4,4)C(32,8)}{C(36,12)}+\frac{32}{357}=\frac{35}{357}.$$

Hence the shares are as 35 to 322.