Explanations on the Memoir of Mr. de la Grange inserted in the Vth Volume of the Melanges of Turin concerning the Method of taking the Mean among the Results of several Observations etc.[*]

Leonhard Euler[†]

SUMMARY

One knows that the astronomers, after having made a certain number of observations which give different results, take the sum of these results and divides it by the number of observations. The author of this memoir has proposed to determine the probability that this sum become equal either to zero or to any other number, positive or negative. He supposes, for this result, that in order to know the elevation of a star, or its declination, or finally whatever this be, one has made a observations which give correctly that which one seeks, b observations which give it by one minute too great, and c observations which give it too small by one minute, so that the number of all the observations made on this subject is $N = a + b + c$; and the question is to see what will be the probability that the sum of the results becomes either 0 or ± 1 or ± 2 or ± 3 etc.

This question is reduced to the following: Of N tickets, of which a are marked by 0, b by $+1$ and c by -1, one draws at random successively n tickets, replacing into it each time the drawn ticket; and one demands the probability that the sum of all the numbers drawn be either 0 or ± 1 or ± 2 etc., a problem in which the author considers first some particular cases, where $n = 1$, $n = 2$, $n = 3$ and $n = 4$, of which the resolution is followed by the general problem solved by the known principles of the

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[*]J.L. Lagrange (17361813), "Mémoire sur l'utilité de la méthode de prendre le milieu entre les résultats de plusieurs observations, dans lequel on examine les avantages de cette méthode par le calcul des probabilits, et où l'on résout différents problèmes relatifs cette matière," *Mélanges de philosophie et de mathématiques de la société royale de Turin* 5, 1770-1773, p. 167232; *Oeuvres de* LAGRANGE, published under the care of M.J.-A. SERRET, t. II, Paris 1868, p. 171-234.

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theory of combinations and of the calculus of probabilities, by the development of the power $N^n = (a+b+c)^n$, where the general form of each term is $Ma^\alpha b^\beta c^\gamma$, the sum of the exponents being $\alpha + \beta + \gamma = n$ and the coefficient

$$M = \frac{1 \cdot 2 \cdot 3 \cdots n}{1 \cdot 2 \cdot 3 \cdots \alpha \times 1 \cdot 2 \cdot 3 \cdots \beta \times 1 \cdot 2 \cdot 3 \cdots \gamma}.$$

These operations are long and disagreeable, if one makes them in the ordinary manner; M. EULER shows how one can find the affected terms of the same power $\alpha + \beta + \gamma = n$, without recourse to the actual development, and he ends his memoir by solving the general problem, where, after having made $N = a + b + c + d+$ etc. observations of which

a have the same error α,

b have the same error β,

c have the same error γ,

etc.,

one wishes to know the probability that the mean becomes any number $\frac{\lambda}{n}$. For this result, it is necessary to consider the power $(ax^\alpha + bx^\beta + cx^\gamma +$ etc.$)^n$ and to take the sum of all the terms affected with the same power x^λ, which, divided by N^n, will give the probability that the mean be $\frac{\lambda}{n}$.

Let one imagine a quarter circle such that, when one avails oneself to observe, for example, the altitude of the pole, after having made a great number of such observations, there are found a observations which give this altitude exactly, b observations which give it by one minute too great and finally c observations which give it too small by one minute; so that the number of all these observations are

$$N = a + b + c,$$

among which there are a [where the error is 0, b]where the error is $= +1$ and c where the error is $= -1$.

Such a quarter circle being supposed, when one will have n observations in order to determine the elevation of the pole, the declination of a star, or whatever it be, among which one will have taken a mean, according to the ordinary manner, by adding all the results together and by dividing the sum by the number of observations, n, one demands: what will be the probability that this mean gives exactly the true sought height, or that the sum of all the results give exactly zero? and next: what will be the probability that the sum of all the results becomes either $+1$ or -1 or $+2$ or -2 or $+3$ or -3 etc.?

In order to put more light on this question, one has only to imagine several tickets, of which the number is $N = a + b + c$ and which a are marked by 0, b by $+1$ and c by -1. If from these tickets contained in a box, one draws, at random, successively n tickets, replacing each time the ticket drawn from the box, and one could demand: what will be the probability that the sum of all the numbers drawn successively be either 0 or $+1$ or -1 or $+2$ or -2 etc., where it is first clear that this sum would never be known to surpass either $+n$ or $-n$. In this manner, the question proposed could easily be resolved by the theory of combinations, by supposing first $n = 1$, next $n = 2$, $n = 3$ etc.

Let therefore $n = 1$ or let one draw from the box only a single ticket; and it is evident that there will be a cases where the number drawn can be 0, b cases where it can be $+1$ and c cases where it can be -1 ; therefore, since the number of all the cases is

$$a + b + c = N,$$

the probability that the number drawn be 0 will be $\frac{a}{N}$, the probability that it is $+1$ will be $\frac{b}{N}$ and this finally that it is -1 will be $\frac{c}{N}$; and since here one draws only a single ticket, there is no mean to take.

Let $n = 2$ or let one draw from the box two tickets successively; and whatever be the number of the first, the second could be one of each ticket in the box, so that each ticket drawn first admits N variations, whence one sees that the number of all possible cases will be

$$N^2 = (a + b + c)^2.$$

Now, this formula being developed gives

$$aa + bb + cc + 2(ab + ac + bc),$$

of which the first term aa, expresses the number of cases where the two tickets drawn are $0 + 0$, and the probability of this case will be $\frac{aa}{N^2}$. Next, the second term bb, marks the number of cases where the two tickets drawn are $+1 + 1 = +2$, of which the probability will be $\frac{bb}{N^2}$. The third term cc, marks the number of cases where the numbers of the tickets drawn are $-1 - 1 = -2$, of which the probability $= \frac{cc}{N^2}$. In the same manner, the fourth term, $2ab$, expresses the number of cases where one of the numbers drawn is 0 and the other $+1$ and hence their sum $= +1$, of which the probability $= \frac{2ab}{N^2}$. The term $2ac$ is the number of cases where it happens that the numbers of the two tickets are 0 and -1 and hence the sum $= -1$, of which the probability $= \frac{2ac}{N^2}$. Finally, the last term marks the number of cases where these numbers drawn are $+1$ and -1, their sum hence $= 0$, and the probability of them is $\frac{2bc}{N^2}$.

Since here we have two tickets, the sum of the two numbers drawn in each case being divided by 2, gives the mean of which we have spoken above, and hence the probability that this mean be $= 0$ will be $\frac{aa+2bc}{N^2}$. If this mean is $+1$, the probability will be $\frac{bb}{N^2}$; if it is -1, it will be $\frac{cc}{N^2}$. Now, it also can happen that this mean be $+\frac{1}{2}$, and the probability of this case will be $\frac{2ab}{N^2}$; and finally, for the mean $-\frac{1}{2}$ the probability is $\frac{2ac}{N^2}$.

We suppose now $n = 3$; and since each of the preceding cases, of which the number was N^2, can be followed by each of N tickets, the number of all the possible cases will be

$$N^3 = (a + b + c)^3,$$

of which we must consider all the terms which result from the development of this formula, and each will express the number of cases which produce the three drawn numbers, of which one will have consequently so much the sum as the mean by dividing it by three, to which it will be easy to add the probability, which is found by dividing

each term by the number of all the possible cases, that is to say by N^3. We will represent all the cases in the following table:

Terms	Numbers	Sum	Mean	Probability
a^3	$0+0+0$	0	0	$\frac{a^3}{N^3}$
b^3	$1+1+1$	3	1	$\frac{b^3}{N^3}$
c^3	$-1-1-1$	-3	-1	$\frac{c^3}{N^3}$
$3aab$	$0+0+1$	$+1$	$+\frac{1}{3}$	$\frac{3aab}{N^3}$
$3aac$	$0+0-1$	-1	$-\frac{1}{3}$	$\frac{3aac}{N^3}$
$3abb$	$0+1+1$	$+2$	$+\frac{2}{3}$	$\frac{3abb}{N^3}$
$3acc$	$0-1-1$	-2	$-\frac{2}{3}$	$\frac{3acc}{N^3}$
$6abc$	$0+1-1$	0	0	$\frac{6abc}{N^3}$
$3bbc$	$+1+1-1$	$+1$	$+\frac{1}{3}$	$\frac{3bbc}{N^3}$
$3bcc$	$+1-1-1$	-1	$-\frac{1}{3}$	$\frac{3bcc}{N^3}$

One sees from this table that the mean $=0$ is met twice, the probability of it will be consequently $=\frac{a^3+6abc}{N^3}$. Next, it shows that the mean $\frac{1}{3}$ is met twice, and hence the probability of it will be $\frac{3aab+3bbc}{N^3}$. Now, if this mean becomes $-\frac{1}{3}$ the probability will be $\frac{3aac+3bcc}{N^3}$. The probability for the mean $+\frac{2}{3}$ will be $\frac{3abb}{N^3}$; for $-\frac{2}{3}$ it will be $\frac{3acc}{N^3}$. In order that the mean becomes $+1$, the probability is $\frac{b^3}{N^3}$, and in order that it be -1, there will be the probability $\frac{c^3}{N^3}$.

In the same manner, we examine the case where $n=4$ and where consequently one draws four tickets; and it is clear that the number of all the possible cases will be

$$N^4 = (a+b+c)^4,$$

Developing therefore this formula, one will see easily so much these four drawn numbers, corresponding to each number, as their sum and their mean, which, with the

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probability, will be represented in the following table:

Terms	Numbers	Sum	Mean	Probability
a^4	$+0+0+0+0$	0	0	$\frac{a^4}{N^4}$
$4a^3b$	$+0+0+0+1$	$+1$	$+\frac{1}{4}$	$\frac{4a^3b}{N^4}$
$4a^3c$	$+0+0+0-1$	-1	$-\frac{1}{4}$	$\frac{4a^3c}{N^4}$
$6aabb$	$+0+0+1+1$	$+2$	$+\frac{1}{2}$	$\frac{6aabb}{N^4}$
$6aacc$	$+0+0-1-1$	-2	$-\frac{1}{2}$	$\frac{6aacc}{N^4}$
$12aabc$	$+0+0+1-1$	0	0	$\frac{12aabc}{N^4}$
$4ab^3$	$+0+1+1+1$	$+3$	$+\frac{3}{4}$	$\frac{4ab^3}{N^4}$
$12abbc$	$+0+1+1-1$	$+1$	$+\frac{1}{4}$	$\frac{12abbc}{N^4}$
$12abcc$	$+0+1-1-1$	-1	$-\frac{1}{4}$	$\frac{12abcc}{N^4}$
$4ac^3$	$+0-1-1-1$	-3	$-\frac{3}{4}$	$\frac{4ac^3}{N^4}$
b^4	$+1+1+1+1$	$+4$	$+1$	$\frac{b^4}{N^4}$
$4b^3c$	$+1+1+1-1$	$+2$	$+\frac{1}{2}$	$\frac{4b^3c}{N^4}$
$6bbcc$	$+1+1-1-1$	0	0	$\frac{6bbcc}{N^4}$
$4bc^3$	$+1-1-1-1$	-2	$-\frac{1}{2}$	$\frac{4bc^3}{N^4}$
c^4	$-1-1-1-1$	-4	-1	$\frac{c^4}{N^4}$

In this table, one finds 9 different means of which some are met two and even three times. We will mark therefore in the following table for each of these means the probability which corresponds to it:

Mean	Probability
0	$\frac{a^4+12aabc+6bbcc}{N^4}$
$+\frac{1}{4}$	$\frac{4a^3b+12abbc}{N^4}$
$-\frac{1}{4}$	$\frac{4a^3c+12abcc}{N^4}$
$+\frac{1}{2}$	$\frac{6aabb+4b^3c}{N^4}$
$-\frac{1}{2}$	$\frac{6aacc+4bc^3}{N^4}$
$+\frac{3}{4}$	$\frac{4ab^3}{N^4}$
$-\frac{3}{4}$	$\frac{4ac^3}{N^4}$
$+1$	$\frac{b^4}{N^4}$
-1	$\frac{c^4}{N^4}$

The consideration of these four cases opens to us the route to the general question, where the number of the drawn tickets is $= n$; because first it is evident that the number of all the possible cases is here

$$N^n = (a+b+c)^n,$$

of which the development has not any difficulty. But for us to dispense of the consideration of each of the terms which result from it, we will consider here the general form

of each of these terms, which is

$$Ma^\alpha b^\beta c^\gamma,$$

where the sum of the exponents $\alpha + \beta + \gamma$ must be $= n$; and in order to find the coefficient M of this term, the theory of combinations furnishes us this formula

$$M = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdots n}{1 \cdot 2 \cdots \alpha \times 1 \cdot 2 \cdots \beta \times 1 \cdot 2 \cdots \gamma}.$$

Now, the drawn numbers which correspond to this term will be

$$0 \cdot \alpha + 1 \cdot \beta - 1 \cdot \gamma$$

and hence the mean of these drawn numbers will be $\frac{\beta - \gamma}{n}$, to which corresponds the probability

$$\frac{Ma^\alpha b^\beta c^\gamma}{N^n}.$$

Since the sum of all the drawn numbers which correspond to this term is $0 \cdot \alpha + 1 \cdot \beta - 1 \cdot \gamma$, one sees that if we wrote in the place of the letters a, b, c these formulas ax^0, bx^{+1}, cx^{-1}, the term that we consider would take this form $Ma^\alpha b^\beta c^\gamma x^{\beta - \gamma}$, so that the exponent of x will give us at first the sum of all the drawn numbers. For this result, one has only to develop this power

$$(ax^0 + bx^1 + cx^{-1})^n,$$

and each of these terms, which will have, as we show, the form $Ma^\alpha b^\beta c^\gamma x^{\beta - \gamma}$, gives us for the mean

$$\frac{\beta - \gamma}{n},$$

and the probability will be

$$\frac{Ma^\alpha b^\beta c^\gamma}{N^n}.$$

Now, one understands easily that this same mean $\frac{\beta - \gamma}{n}$ can result from several different terms, whence consequently it will be necessary to obtain the probability which corresponds to each, and the sum of all these probabilities will give the probability for the same mean $\frac{\beta - \gamma}{n}$. In order to find all the affected terms of the same power of x, I will endeavor to give a method which will excuse us from collecting all these terms by the actual development; but before, I will examine successively the cases where the exponent of x becomes either 0 or ± 1 or ± 2 or ± 3.

And at first for the first, where the mean $= 0$, it will be necessary to put $\beta = \gamma$, so that the following terms which produce this same mean will be

$$a^n, \ a^{n-2}bc, \ a^{n-4}b^2c^2, \ a^{n-6}b^3c^3, \ a^{n-8}b^4c^4 \quad \text{etc.}$$

which it is necessary to continue until the exponents of a become negatives; and from this which we have observed, it will be easy to assign to each of these terms its coefficient; consequently, the sum of all these terms divided by N^n gives the entire probability in order that this mean takes place.

In the same manner, in order that the mean in question becomes $\frac{1}{n}$, where the exponent of x, namely $\beta - \gamma$, becomes $= 1$, all the terms which lead to it, because $\beta = \gamma + 1$ and $\alpha = n - 2\gamma - 1$, will be expressed by the following formulas

$$a^{n-1}b, \ a^{n-3}b^2c, \ a^{n-5}b^3c^2, \ a^{n-7}b^4c^3, \ a^{n-9}b^5c^4 \quad \text{etc.,}$$

of which the sum, after having joined their coefficients, divided by N^n gives the probability for this mean $\frac{1}{n}$. Likewise, in order that the mean becomes $-\frac{1}{n}$ and hence $\beta - \gamma = -1$, one will have $\gamma = \beta + 1$ and $\alpha = n - 2\beta - 1$, whence the terms which produce this mean will be

$$a^{n-1}c, \ a^{n-3}bc^2, \ a^{n-5}b^2c^3, \ a^{n-7}b^3c^4, \ a^{n-9}b^4c^5 \quad \text{etc.}$$

If one wishes that the mean be found $= +\frac{2}{n}$, or that there be $\beta - \gamma = 2$ and hence $\beta = \gamma + 2$ and $\alpha = n - 2\gamma - 2$, all the terms which produce this mean will be

$$a^{n-2}b^2, \ a^{n-4}b^3c, \ a^{n-6}b^4c^2, \ a^{n-8}b^5c^3, \ a^{n-10}b^6c^4 \quad \text{etc.}$$

Now, in order that the mean be $-\frac{2}{n}$, the terms which produce it, because $\beta - \gamma = -2$ and hence $\alpha = n - 2\beta - 2$, will be

$$a^{n-2}c^2, \ a^{n-4}bc^3, \ a^{n-6}b^2c^4, \ a^{n-8}b^3c^5, \ a^{n-10}b^4c^6 \quad \text{etc.,}$$

where one must continue these formulas as long as the exponent of a remains positive; and it is very easy to continue this operation for all the different means which can take place.

Now it will be very easy to resolve this question in general, when the quarter circle is supposed such that among a very great number N of observations, there are a of them which have the same error $= \alpha$, b observations which produce the same error $= \beta$, c observations which have the same error $= \gamma$ and d observations of which the error is $= \delta$ etc., where $\alpha, \beta, \gamma, \delta$ etc. mark in general some numbers any positives or negatives, such that one has

$$N = a + b + c + d + e + \text{ etc.}$$

This put, if one wishes to know the probability that having made n observations, the sum of all the numbers drawn (supposing that to each observation corresponds a ticket, as above) is either 0 or 1 or 2 or 3 or in general λ, one has only to develop this power

$$(ax^\alpha + bx^\beta + cx^\gamma + dx^\delta + \text{ etc.})^n$$

and to take the sum of all the terms affected by the same power x^λ, which, being divided by N^n, will give the probability which agrees with the sum $= \lambda$ of all the drawn numbers, or else to their mean $\frac{\lambda}{n}$. All these operations are made, without any difficulty, in the same manner as we have taught above.