SUR LES RENTES VIAGERES

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Having established the true principle on which it is necessary to base the calculus of life annuities, I believe that the development of this calculus will not lack to be quite interesting, as much for those who will wish to undertake one such establishment as for those who will wish to profit from them. I have outlined this material in my "General Researches on mortality and the multiplication of the human race," where I have exposed the correct method to determine by the calculus, how much a man of a certain age must pay, in order to enjoy during all his life a given annual annuity. But, since the calculus appeared to me then quite troublesome, I could not resolve myself to execute it. Now a certain occasion obliged me lately to undertake this work, of which, for some artifices in order to abridge the calculation, I am happily come to end.

2. There are two things, on which the determination of the life annuities must be founded: one is a good list of mortality, which shows us, for each age, how many there will die of them probably during the course of one or many years: the other is the manner by which the entrepreneur can turn the money to account which he will have received from the annuitants: or at what interest he is in a state to invest it. These two articles compete very essentially to determine the annuities to which the entrepreneur could engage himself, as much with respect to the sum which has been paid to him first, as with respect to the age of the annuitant. Because it is evident, that the more the entrepreneur can withdraw of profit from the capital, that he has between his hands, the more he will be in a state to pay higher annuities.

3. For the list of mortality, the entrepreneur would risk without doubt much, if he wished to regulate himself on the mortality of men in general, as one concludes from the observations made in a great city, or in an entire country, where one holds equally account of all the men as much vigorous as infirm. Now, when there is a question to get for oneself some life annuities, it is very natural that it is necessary to exclude from the all those of whom the constitution does not seem to promise a long life, thus one has reason to regard the annuitants as a more robust kind. It is also under this view that I have chosen in my cited Memoir the list of Mr. Kerseboom, which he has drawn from the observations made uniquely on some persons who have enjoyed life annuities: & hence also this same list will serve me as foundation in the following calculations.

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4. If the entrepreneur was not in a state to place rather well the capital which is paid to him by the annuitants, he would know how to accord only some annuities so mediocre, that no person would wish to acquire them. Another time the city of Amsterdam has paid ten per cent of annuities to all the persons below twenty years, or else for 1000 florins it has paid 100 per year to them; this which is an annuity so rich that the city would have suffered a very considerable loss from it if it had not won nearly 10 percent per year from the funds that this enterprise had procured for it. Thus, if one could rely only on 5 percent interest, the annuities must become considerably much less; however it is thereupon that it seems that it is necessary at present to regulate the life annuities, expecting that those who will have occasion to make from them a greater profit, will be scarcely troubled from one such enterprise, which would know how to be achieved only after a great number of years.

5. In order to determine the price of these annuities, one fixes for each age a mean term of life, that it is as probable to survive as to die before having reached it; or else this term is taken such, that from a great number of men of the same age there die as many of them before this term as after. Then one supposes that all the men of this age attain precisely this term, & that they die next all at once; thereupon one believes to be able to fix surely the price of the annuities, since it is a question to find the present value of an annual annuity, payable during a certain number of consecutive years: & one estimates that the profit that the entrepreneur removes from the side of those who die before the said term, is precisely recompensed by the loss that those annuitants who survive to this term cause to him. But one will understand easily that this reasoning lame, since one does not take account of the diminution of the present price of one annuity which will be paid only after many years. Because of this circumstance, it will be necessary to base the calculus on some true principles, as I have taught in my mentioned Memoir, without serving myself of another reasoning which could appear suspect.

6. For this effect, we consider the number of 1000 infants born at once, & that the characters (1), (2), (3), (4), &c. mark the numbers of those who will live yet to the end of 1, 2, 3, 4, &c years, so that in general (m) represents the number of those, who will attain the age of m years. Let now r be the annual annuity that a man aged m years would wish to acquire, & x the price that he must pay for it at present to the entrepreneur; which must be a just equivalent to the expense of which the entrepreneur invests himself by this convention. In order to determine this price x, it is necessary to consider many men of the same age of m years, & who enter into the same condition. Let (m) be the number of these men, & the sum which they will pay at present to the entrepreneur will be $= (m)x$, which must be sufficient in order to furnish all the annuities, which he will have to pay in the future.

7. Now of these (m) men there will be of them living after one year $(m+1)$, after two years $(m+2)$, after three years $(m+3)$, & thus in sequence: therefore the entrepreneur will have to pay after one year $(m+1)r$, after two years $(m+2)r$, after three years $(m+3)r$, &c. until all these annuitants will be extinguished. Now one has therefore only to reduce each of these payments to the present time at rate of five per cent, & to equate the sum to $(m)x$ in order to conclude the just value of x. Now, in order to render the calculation more general, instead of $\frac{105}{100}$ or $\frac{21}{20}$ we write the letter λ, & the sum of all the annuities which the entrepreneur must pay successively will be

worth at present

$$\frac{(m+1)r}{\lambda} + \frac{(m+2)r}{\lambda^2} + \frac{(m+3)r}{\lambda^3} + \frac{(m+4)r}{\lambda^4} + \text{\&c.}$$

which being equal to $(m)x$ will give

$$x = \frac{r}{(m)}\left(\frac{(m+1)}{\lambda} + \frac{(m+2)}{\lambda^2} + \frac{(m+3)}{\lambda^3} + \frac{(m+4)}{\lambda^4} + \text{\&c.}\right)$$

8. Here is therefore the just price that a man aged m years must pay in order to be set into the possession of an annual annuity r during all his life, & which being first invested at 5 percent sets the entrepreneur precisely in a state to pay in the future the annuities, provided that the number of annuitants is considerable enough. One understands well, that having thus invested first all the capital that the entrepreneur will have received, the following year the interests will not be sufficient to pay the annuities, but that it will be necessary to use a part of the capital, whence the capital will suffer each year a diminution: however it will be entirely extinguished only when all the annuitants will be dead. By this reason, the entrepreneur will be well obliged to raise the price of the annuities that I just found, according to the circumstances & the particular expenses that one such establishment requires.

9. One sees well that the determination of this price named x demands a calculation as long as annoying, especially for the low ages, where the number of terms to add together is quite considerable. But it is not difficult to perceive, that having already made this calculation for a certain age, one can easily draw from it the one which corresponds to one year more or less. In order to explicate more clearly this artifice, I will serve myself of this character[1] $\overbrace{m}\, r$ in order to mark the price that a man aged m years must pay for the life annuity r: so that

$$\overbrace{m} = \frac{1}{(m)}\left(\frac{(m+1)}{\lambda} + \frac{(m+2)}{\lambda^2} + \frac{(m+3)}{\lambda^3} + \frac{(m+4)}{\lambda^4} + \text{\&c.}\right)$$

thence, for the men aged $m+1$ years, we will have

$$\overbrace{m+1} = \frac{1}{(m+1)}\left(\frac{(m+2)}{\lambda} + \frac{(m+3)}{\lambda^2} + \frac{(m+4)}{\lambda^3} + \frac{(m+5)}{\lambda^4} + \text{\&c.}\right)$$

whence we conclude:

$$\lambda(m)\overbrace{m} = (m+1) + (m+1)\overbrace{m+1},$$

& hence

$$\overbrace{m} = \frac{1}{\lambda}\cdot\frac{(m+1)}{(m)}(1 + \overbrace{m+1}),$$

so that having found the value of $\overbrace{m+1}$, one will calculate from it rather easily that of $\overbrace{m}$.

[1] *Translator's note:* It is impossible to replicate the character which is similar to the bracket [set horizontally. As a near approximation, I have used the horizontal brace.

10. By aid of this artifice, after having commenced through the age of 90 years, I have calculated the price of the annuity r successively for all the inferior ages until the newly born infants; whence I have obtained the following table, by fixing the annuity r at 100 écus, & the interest at 5 percent.

TABLE

which mark the price of one life annuity of 100 écus for all the ages.

age years	number of living	price of the annuity	age years	number of living	price of the annuity
0	1000	1155.50	25	552	1403.60
1	804	1409.04	26	544	1395.45
2	768	1448.84	27	535	1389.87
3	736	1487.43	28	525	1387.16
4	709	1521.27	29	516	1382.54
5	690	1541.32	30	507	1376.82
6	676	1551.90	31	499	1368.84
7	664	1558.94	32	490	1363.68
8	654	1561.92	33	482	1355.63
9	646	1560.33	34	475	1344.38
10	639	1556.29	35	468	1332.71
11	633	1549.59	36	461	1320.60
12	627	1542.64	37	454	1308.01
13	621	1535.42	38	446	1298.04
14	616	1525.28	39	439	1284.67
15	611	1514.65	40	432	1270.76
16	606	1503.50	41	426	1253.09
17	601	1491.81	42	420	1234.54
18	596	1479.54	43	413	1218.24
19	590	1469.31	44	406	1201.21
20	584	1458.63	45	400	1180.19
21	577	1450.18	46	393	1161.27
22	571	1438.68	47	386	1141.44
23	565	1426.66	48	378	1123.88
24	559	1414.07	49	370	1105.59
25	552	1403.60	50	362	1086.52
m	(m)	$\overbrace{m}$	m	(m)	$\overbrace{m}$

age years	number of living	price of the annuity	age years	number of living	price of the annuity
50	362	1086.52	70	175	638.30
51	354	1066.62	71	165	610.83
52	345	1049.17	72	155	582.75
53	336	1031.14	73	145	554.09
54	327	1012.49	74	135	524.89
55	319	989.78	75	125	495.22
56	310	969.44	76	114	470.16
57	301	948.35	77	104	441.13
58	291	929.98	78	93	417.98
59	282	907.64	79	82	397.75
60	273	884.44	80	72	375.64
61	264	860.32	81	63	350.77
62	254	838.90	82	54	329.69
63	245	813.21	83	46	306.38
64	235	790.20	84	39	279.44
65	225	766.59	85	32	257.60
66	215	742.30	86	26	232.90
67	205	717.43	87	20	217.91
68	195	691.93	88	15	205.07
69	185	665.14	89	11	193.62
70	175	638.30	90	8	179.54
m	(m)	$\overbrace{m}$	m	(m)	$\overbrace{m}$

11. Mr. Kerseboom has continued his Table on mortality only to 95 years, & for this reason I have not judged appropriate to continue this one to beyond 90 years, since a person at this age will have probably no longer a view toward the life annuity. At least, since in all the plans, these old men find themselves ranked in the same class as those of 60 or of 70 years: notwithstanding that it would be quite unjust, if one would wish to require of a nonagenarian more than the third of the price that a septuagenarian must pay, & more than the fourth of a sexagenarian. However, if one is curious to see the continuation of my table, here it is:

m	90	91	92	93	94
(m)	8	6	4	3	2
$\overbrace{m}$	179.54	151.35	138.38	93.73	47.62

But I would not wish to advise to an entrepreneur to trouble oneself with such old men, at least when their number was not considerable enough; that which is a general rule for all the establishments founded on probabilities.

12. Thence one will conclude easily how much the entrepreneur must pay in interest at each age, for any sum whatever, that one would have taken first within his hands. It is not necessary to enter here into the same detail, & it will suffice to mark from 5 by

5 years the *procents*,[2] that the annuitants could require.

age	Procents	age	Procents	age	Procents
0	$8\frac{2}{3}$	30	$7\frac{1}{4}$	60	$11\frac{1}{3}$
5	$6\frac{1}{2}$	35	$7\frac{1}{2}$	65	13
10	$6\frac{1}{3}$	40	8	70	$15\frac{2}{3}$
15	$6\frac{1}{2}$	45	$8\frac{1}{2}$	75	20
20	$6\frac{3}{4}$	50	9	80	$25\frac{2}{3}$
25	7	55	10	85	$38\frac{1}{2}$
30	$7\frac{1}{4}$	60	$11\frac{1}{3}$	90	$55\frac{1}{2}$

On this footing the entrepreneur would have no profit, at least he was not in a state to turn to account his money at more than 5 percent.

13. Therefore, if a state had need of money, & if it was able to find at 5 percent interest as much as it is necessary for it from it, it would be assuredly quite bad, if it would wish to establish such life annuities, as I just determined on this same footing of 5 percent, because in regard to the impediment that one such establishment requires necessarily, it would be always better to borrow the sum of which there is need at 5 percent, that it would be able next to pay off according to the circumstances, instead that the life annuities would remain to him at expense during very long times. Or else, it would be necessary to raise the price of the annuities beyond that which I have fixed them, in order to procure for him any benefit; but then it would be much to be feared that he found himself annuitants no longer, at least were there not some old men beyond 60 years, that the interests of 10 & more percent would be able to dazzle.

14. But to wish to establish more advantageous life annuities for the annuitants, it would be a project little proper to comfort a State; since that would return to the same, as if one wished to undertake debt at six & more percent: while one would wish to make some loans at 5 percent without submitting to the impediment that the life annuities demand. Indeed, if a State would wish to establish the annuities exposed here, & calculated on the basis of 5 percent, it would know how to regard this expense only as an expense taken at 6 for 100, because of so many arrangements which would be required. Thus I see nearly no more cases, where the establishment of the life annuities could be advantageous to the State, as long as one can borrow money at 5 percent, & perhaps less. But one can imagine another kind of annuity, which would be perhaps more appreciated, although it was equally based on the footing of 5 percent. I wish to speak of annuities, which must commence to incur only after 10 or even 20 years; & one understands easily, that the price of such annuities will be quite mediocre, & hence capable to attract the public.

15. We imagine therefore this question also in general, & seek how much a man aged m years must pay at present, in order to acquire an annual annuity r, which will commence to him to be paid only after n years, so that from that time he is able to enjoy it regularly to his death. Let x be the present price of this annuity, & we will find

[2] *Translator's note*: The rates presented here are those required for a perpetuity. For example, in the case of the payment of a 90 year old man, we note that 179.54 invested at 55.7% annually would increase to 279.54 in one year. Thus, an annual annuity of 100 is generated forever.

as above:

$$x = \frac{r}{(m)} \left(\frac{(m+n)}{\lambda^n} + \frac{(m+n+1)}{\lambda^{n+1}} + \frac{(m+n+2)}{\lambda^{n+2}} + \text{\&c.} \right)$$

Now, by this calculation of the ordinary annuities explicated before, we will have:

$$\overbrace{m+n-1} = \frac{1}{(m+n-1)} \left(\frac{(m+n)}{\lambda} + \frac{(m+n+1)}{\lambda^2} + \frac{(m+n+2)}{\lambda^3} + \text{\&c.} \right)$$

whence we conclude:

$$x = \frac{r}{(m)} \frac{(m+n-1)}{\lambda^{n-1}} \cdot \overbrace{m+n-1} = \frac{r}{\lambda^{n-1}} \frac{(m+n+1)}{(m)} \cdot \overbrace{m+n-1},$$

where $\overbrace{m+n-1}\, r$ expresses the present price of the ordinary annuity for a man aged $m+n-1$ years.

16. Therefore, if one demands the present price of an annual annuity of 100 écus, which will commence to be paid only at the end of 10 years for a man aged m years, one will take from the table developed in § 10 the price of the ordinary annuity which matches to the age of $m+9$ years, & one will multiply it by $\left(\frac{20}{21}\right)^9 \frac{(m+9)}{(m)}$ in order to have the value sought of x. Thence I have calculated the following tables from 5 by 5 years:

TABLE

of the price of a life annuity of 100 écus which must begin to run only at the end of 10 years

age years	price of the annuity	age years	price of the annuity	age years	price of the annuity
0	649.75	30	717.05	60	290.55
5	877.77	35	671.73	65	203.11
10	874.50	40	610.40	70	120.14
15	833.95	45	533.55	75	56.20
20	787.43	50	455.78	80	19.07
25	745.72	55	375.25		
30	717.05	60	290.55		

TABLE

of the price of a life annuity of 100 écus which must begin to run only at the end of 20 years

age years	price of the annuity	age years	price of the annuity	age years	price of the annuity
0	343.06	30	319.30	60	47.28
5	453.36	35	272.96	65	19.17
10	441.81	40	234.47	70	4.82
15	413.60	45	183.72		
20	382.17	50	134.52		
25	349.63	55	87.91		
30	319.30	60	47.28		

17. Perhaps one such project of life annuities would succeed better, notwithstanding that they are fixed on the footing of 5 percent. It seems that it would always be advantageous for a newborn infant to be able to assure himself, for the price of 343, or else of 350 écus, a fixed annuity of 100 écus per year, although it begins to be paid only when the infant will have attained the age of 20 years: & if one would wish to employ the sum of 3500 écus, this would be always a good establishment, as to enjoy from the age of 20 years a fixed pension of 1000 écus. However it is still doubtful, if there would be found many parents who would much wish to make such sacrifice for the good of their infant. Perhaps there would be found more men of 60 years, who would not at all hold in equilibrium to pay first 3000 écus in order to be assured to enjoy a fixed pension of 1000 écus per year as soon as they would have passed their 70^{th} year.