On the Advantage to the Banker in the game of Pharaon*

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E313

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In order to render the research more general, I will mark the number of cards that the Banker has in hand by the letter n. Among these cards, there will be either one or two or three or four similar to that on which the Punter will have wagered. I will name these cards *significants*, in order to distinguish them from the others of which the kind does not enter into consideration.

Although the number of significant cards does not surpass four, in order to render the calculation more complete, I will pass to greater number; a like case would have place if, for example, the Punter would wager without distinction on a king or a queen at the time, such that he wins or loses, whether a king or a queen come out first; in this case, the number of significant cards would climb as far as eight. I will have therefore as many cases to examine as there are significant cards, and I will determine for each the advantage to the Banker; for this purpose, I will mark by unity the wager of the Punter, and I will seek the claim that the Banker would make on it consistent to the rules of probability. Because, if the game were fair, the Banker would not know how to make any claim; and this is only because of his advantage that he is able to pretend to a certain share.

PROBLEM 1

1. *The number of all the cards being* $= n$*, if there is only a single significant card, to find the advantage of the Banker.*

SOLUTION

Let the Banker draw one card after another; and the probability that the first card be significant is $\frac{1}{n}$; and that it is not it, the probability will be $\frac{n-1}{n}$. In the first case, he wins the wager$= 1$, this which is worth $\frac{1}{n} \cdot 1 = \frac{1}{n}$; in the second case, it is necessary to pass beyond.

*See also the memoirs 201, 811 and 813.

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We suppose therefore that the advantage of the Banker, when he has drawn the second card, is worth a to him; and since the probability that this case occur is $\frac{n-1}{n}$, his advantage at the beginning is

$$x = \frac{1}{n} + \frac{n-1}{n}a,$$

putting x for the advantage of the Banker at the beginning of the game.

Now, in order to find a, I consider the second card; and because the number of cards$= n-1$, the probability that the second card is the significant is $\frac{1}{n-1}$, and that it is not, $= \frac{n-2}{n-1}$; there, he loses 1, and here, he passes to the expectation that he will have at the third card, which is put $= b$; thence, we will have

$$a = -\frac{1}{n-1} + \frac{n-2}{n-1}b$$

and therefore

$$x = \frac{n-2}{n}b.$$

Let him draw at present the third card; and in order that it be the significant, the probability is $\frac{1}{n-2}$, and that it not be the one, $= \frac{n-3}{n-2}$; the first case wins 1 for him, and the other sets to him the expectation that he will have at the fourth card, which let be $= c$; whence we will have

$$b = \frac{1}{n-2} + \frac{n-3}{n-2}c$$

and therefore

$$x = \frac{1}{n} + \frac{n-3}{n}c.$$

Let him draw the fourth card; and the probability that it is the significant being $= \frac{1}{n-3}$, and that it not be, $= \frac{n-4}{n-3}$; he loses 1 in the first case, and in the other, he obtains the expectation that he is able to have at the fifth, which let be $= d$; thence, we will have

$$c = -\frac{1}{n-3} + \frac{n-4}{n-3}d$$

and

$$x = \frac{n-4}{n}d.$$

In the same manner, if we put the expectation of the Banker

at commencement	$=x$,
at the second card	$= a$,
at the third card	$= b$,
at the fourth card	$=c$,
at the fifth card	$= d$,
at the sixth card	$= e$,
etc.,	

we will have the following values:

$$x = \tfrac{1}{n} + \tfrac{n-1}{n}a,$$
$$x = \tfrac{n-2}{n}b,$$
$$x = \tfrac{1}{n} + \tfrac{n-3}{n}c,$$
$$x = \tfrac{n-4}{n}d,$$
$$x = \tfrac{1}{n} + \tfrac{n-5}{n}e,$$
$$x = \tfrac{n-6}{n}f,$$
etc.

But it is necessary to remark that the number n of all the cards is even and that the Banker loses nothing, when the significant card is the last, although it be even in order. Thus, if there should be $n = 2$, there would be $a = 0$; if $n = 4$, there would be $c = 0$; if $n = 6$, there would be $e = 0$ and so forth; whence one sees that there will be in general $x = \frac{1}{n}$, whatever even number that n be.

REMARK 1

2. It is evident that this advantage of the Banker, $x = \frac{1}{n}$, arises uniquely from the law that the last card does not make the Punter win; because, if without exception all the even cards were favorable to him, as all the odds are to the Banker, the share would be perfectly equal, and the advantage to him would be $x = 0$.

REMARK 2

3. This sole consideration would have been able first to lead me to the solution of the problem. Because, since the probability that the significant card be the last is $= \frac{1}{n}$, and that in this case the Banker does not lose 1, as he would if the advantage were equal, it is as much as if this case returned 1 to him, beyond the equality; whence his advantage is estimated

$$x = \frac{1}{n} \cdot 1 = \frac{1}{n}.$$

REMARK 3

4. Since one entire deck of cards, which contains 52 of them, contains four significant cards, this case could take place only when the stock is already diminished to below 49 cards. But, since one draws the cards always two by two, the number of cards in this case, n, will be 48 at most, or whatever other smaller even number. Therefore, the smallest advantage to the Banker in this case, which he will have when $n = 48$, will be $\frac{1}{48}$, or will be worth a little more than two per 100; and if the Banker would have not more than 10 cards in hand, his advantage would be $\frac{1}{10}$ or 10 per cent.

PROBLEM 2

5. *The number of all cards being $= n$, if there are two significant cards, to find the advantage of the Banker.*

SOLUTION

Let x be the advantage of the Banker at the beginning of the game; and since he draws the cards two by two, let a be his advantage after he will have drawn two cards (supposing that each of the significant cards did not come out), and b the one after he will have drawn 4 cards, c the one after he will have drawn 6, d the one after he will have drawn 8, and so on up to the end.

Now, for the first pair of cards, it is necessary to consider 4 cases:

1. when the first and the second are significant; where the game ends and the Banker wins the half of the wager, or $\frac{1}{2}$, according to the rules of the game;

2. when the first is significant and the other not; in this case, the Banker wins all the wager 1;

3. when the first is not significant, but that the second is significant; in this case, the Banker loses 1;

4. when neither the first nor the second is significant; in this case, one continues the game and the Banker arrives to the advantage that I have marked by the letter a.

Now, since there are two significant cards among all, of which the number is n, the probability that the first be significant is $= \frac{2}{n}$, and that it not be, $= \frac{n-2}{n}$.

Let the first be significant; and since in the rest of the cards, of which the number is $n-1$, there is no more than one significant, the probability that it be the second is $= \frac{1}{n-1}$, and that it is not, $= \frac{n-2}{n-1}$; therefore, in order that the first case occur, the probability is

$$= \frac{2}{n} \cdot \frac{1}{n-1},$$

and that the second occur, the probability is

$$= \frac{2}{n} \cdot \frac{n-2}{n-1}.$$

Let the first not be significant; and since there are yet 2 significant cards among the others, of which the number is $n-1$, the probability that the second be significant is $= \frac{2}{n-1}$, and that it not be, $= \frac{n-3}{n-1}$. Therefore, that the third case occur, the probability is

$$= \frac{2(n-2)}{n(n-1)},$$

the fourth case, the probability is

$$= \frac{(n-2)(n-3)}{n(n-1)}.$$

The probability of the occurrence of this fourth case, with the profit or the loss that each brings to the Banker, is thus

$$x = \frac{2}{n(n-1)} \cdot \frac{1}{2} + \frac{2(n-2)}{n(n-1)} \cdot 1 - \frac{2(n-2)}{n(n-1)} \cdot 1 + \frac{(n-2)(n-3)}{n(n-1)}a,$$

or else

$$x = \frac{1}{n(n-1)} + \frac{(n-2)(n-3)}{n(n-1)}a.$$

In the same manner, we will find the advantage of the Banker, a, that he will have after having already drawn 2 cards; because, having now yet $n-2$ cards, among which are found two significants, setting $n-2$ in place of n and b in place of a, we will have

$$a = \frac{1}{(n-2)(n-3)} + \frac{(n-4)(n-5)}{(n-2)(n-3)}b,$$

and continuing the same reasoning, we will find

$$\begin{gathered} b = \tfrac{1}{(n-4)(n-5)} + \tfrac{(n-6)(n-7)}{(n-4)(n-5)}c, \\ c = \tfrac{1}{(n-6)(n-7)} + \tfrac{(n-8)(n-9)}{(n-6)(n-7)}d, \\ d = \tfrac{1}{(n-8)(n-9)} + \tfrac{(n-10)(n-11)}{(n-8)(n-9)}e, \\ \text{etc.} \end{gathered}$$

But it is necessary here to have regard to one exception that the rules of the game contain, which is that, when the two significant cards are the last, the Banker wins the entire wager, and not only the half, as happens when the two significant cards are encountered in one other arbitrary pair. Now, the probability that the two significant cards are the last being $= \frac{2}{n(n-1)}$, and this case returns to him a gain of $\frac{1}{2}$ above the ordinary, we will have only to add again

$$\frac{2}{n(n-1)} \cdot \frac{1}{2} = \frac{1}{n(n-1)}$$

to the advantage that the preceding determinations provide to us.

Therefore we substitute successively the values found for a, b, c, d etc. and we will find

$$\begin{gathered} x = \tfrac{1+1+(n-4)(n-5)b}{n(n-1)}, \\ x = \tfrac{1+1+1+(n-6)(n-7)c}{n(n-1)}, \\ x = \tfrac{1+1+1+1+(n-8)(n-9)d}{n(n-1)}, \\ \text{etc.} \end{gathered}$$

Therefore, if there were $n = 4$, we would have

$$x = \frac{2}{n(n-1)};$$

if there were $n = 6$, we would have

$$x = \frac{3}{n(n-1)};$$

if there were $n = 8$, we would have

$$x = \tfrac{4}{n(n-1)};$$
etc.,

and hence in general, some number pair that may be n, we will have

$$x = \frac{n:2}{n(n-1)} = \frac{1}{2(n-1)}.$$

We have therefore only to add the advantage which results from the case when the two significant cards are the last, and we will obtain the entire advantage of the Banker

$$x = \frac{1}{2(n-1)} + \frac{1}{n(n-1)} = \frac{n+2}{2n(n-1)}.$$

REMARK 1

6. When, among the n cards, there is only a single significant card, the advantage of the Banker is worth

$$\frac{1}{n} = \frac{2n-2}{2n(n-1)};$$

therefore, unless n not be four or two, the advantage to the Banker is greater when there is only one significant card than when there are two.

REMARK 2

7. It is therefore, on the contrary, more profitable for the Punter when there are yet two significant cards in the stock, provided that the stock contains yet more than 4 cards.

REMARK 3

8. This case is able to take place when $n = 50$, and then the advantage of the Banker will be $\frac{52}{100 \cdot 49} = \frac{13}{1225}$, or will be worth a little more than one per cent. But, if the number of cards, n, were only 10, his advantage would be $\frac{12}{20 \cdot 9} = \frac{1}{15}$, or would be worth almost 7 per cent.

PROBLEM 3

9. *The number of all cards being* $= n$, *if there are three significant cards, to find the advantage of the Banker.*

SOLUTION

Put the sought advantage of the Banker$= x$, and let a, b, c, d etc. mark his advantage, that he will have after having drawn $2, 4, 6, 8$ etc. cards; we will have to consider the same four cases which were exposed in the solution of the second problem.

Therefore, since there are 3 significant cards among n cards, the probability that the first drawn be significant will be $= \frac{3}{n}$, and that it not be, $= \frac{n-3}{n}$.

Let the first be significant; and since there are yet 2 among $n-1$ cards, the probability that the second be significant will be $\frac{2}{n-1}$, and that it not be, $= \frac{n-3}{n-1}$; therefore, that the first case occur, the probability is

$$= \frac{3 \cdot 2}{n(n-1)},$$

the second case, the probability is

$$= \frac{3(n-3)}{n(n-1)}.$$

Now, if the first is not significant, since among the $n-1$ remaining cards there are 3 significant cards, the probability that the second be significant is $= \frac{3}{n-1}$, and that it not be, $= \frac{n-4}{n-1}$.

Therefore, in order that the third case occur, the probability is

$$= \frac{3(n-3)}{n(n-1)},$$

the fourth case, the probability is

$$= \frac{(n-3)(n-4)}{n(n-1)}.$$

Thence, we conclude the advantage to the Banker

$$x = \frac{3 \cdot 2}{n(n-1)} \cdot \frac{1}{2} + \frac{3(n-3)}{n(n-1)} \cdot 1 - \frac{3(n-3)}{n(n-1)} \cdot 1 + \frac{(n-3)(n-4)}{n(n-1)} a$$

or else

$$x = \frac{3 + (n-3)(n-4)a}{n(n-1)}.$$

By a reasoning entirely similar, we will determine the value of a in putting the number of cards $= n-2$,

$$a = \frac{3 + (n-5)(n-6)b}{(n-2)(n-3)},$$

and following

$$b = \tfrac{3+(n-7)(n-8)c}{(n-4)(n-5)},$$
$$c = \tfrac{3+(n-9)(n-10)d}{(n-6)(n-7)},$$
etc.

We substitute these found values, and we will have from them

$$x = \tfrac{3}{n(n-1)} + \tfrac{3(n-4)}{n(n-1)(n-2)} + \tfrac{(n-4)(n-5)(n-6)}{n(n-1)(n-2)} b,$$
$$x = \tfrac{3}{n(n-1)} + \tfrac{3(n-4)}{n(n-1)(n-2)} + \tfrac{3(n-6)}{n(n-1)(n-2)} + \tfrac{(n-6)(n-7)(n-8)}{n(n-1)(n-2)} c,$$

and pursuing these substitutions to the end

$$x = \frac{3}{n(n-1)(n-2)} \left((n-2) + (n-4) + (n-6) + (n-8) + \cdots + 0\right);$$

we have therefore an arithmetic progression to sum, of which the number of terms is $\frac{n}{2}$ and the sum of the first and last $= n-2$; whence the sum of the progression is

$$= \frac{n(n-2)}{4},$$

and therefore

$$x = \frac{3}{n(n-1)}.$$

This is also the true value of the advantage of the Banker, since the irregular case of the two last cards cannot take place here. Therefore, the sought advantage of the Banker is in this case in general

$$x = \frac{3}{4(n-1)}.$$

REMARK 1

10. If, among the n cards, there is only one significant, the advantage of the Banker is

$$= \frac{1}{n};$$

and if there are only two of them, his advantage is

$$= \frac{n+2}{2n(n-1)}.$$

Therefore, according as there are one or two or three significant cards, the advantage of the Banker follows the ratio of these three numbers

$$4n-4,\ 2n+4,\ 3n.$$

Therefore, provided that there be $n > 4$, the advantage of the Banker is the least when the stock contains yet two significant cards.

REMARK 2

11. Now, if $n > 4$, the advantage of the Banker is greater if the stock contains one significant card than if it contains three of them. Therefore, the number of the cards in the stock remaining the same, the Punter will act most advantageously when he wagers on a card which is found yet twice in the stock.

REMARK 3

12. When the stock contains three significant cards, the advantage of the Banker is as much smaller as the number of cards is great. The least advantage will be therefore when $n = 50$, which is worth $\frac{3}{4\cdot 49} = \frac{3}{196}$, or $1\frac{1}{2}$ per cent nearly.

When the stock contains no more than 10 cards, the advantage of the Banker will be $= \frac{3}{4\cdot 9} = \frac{1}{12}$, or $8\frac{1}{3}$ per cent.

PROBLEM 4

13. *The number of cards being* $= n$*, if there are four significant cards, to find the advantage of the Banker.*

SOLUTION

In operating in the preceding manner; since there are 4 significant cards, the probability that the first drawn be one of them will be $\frac{4}{n}$, and that it not be, $= \frac{n-4}{n}$.

Let therefore the first be significant; and since there are yet 3 of them among the $n-1$ remaining cards, the probability that the second be significant is $\frac{3}{n-1}$, and that it not be, $= \frac{n-4}{n-1}$.

Therefore, in order that the first case occur, the probability is

$$= \frac{4 \cdot 3}{n(n-1)},$$

the second case, the probability is

$$= \frac{4(n-4)}{n(n-1)}.$$

Now, if the first is not significant, since there are 4 of them among the $n-1$ remaining, the probability that the second be significant is $= \frac{4}{n-1}$, and that it not be, $= \frac{n-5}{n-1}$.

Therefore, in order that the third case occur, the probability is

$$= \frac{4(n-4)}{n(n-1)},$$

the fourth case, the probability is

$$= \frac{(n-4)(n-5)}{n(n-1)}.$$

Thence, we conclude

$$x = \frac{12}{n(n-1)} \cdot \frac{1}{2} + \frac{4(n-4)}{n(n-1)} \cdot 1 - \frac{4(n-4)}{n(n-1)} \cdot 1 + \frac{(n-4)(n-5)}{n(n-1)} a$$

or else

$$x = \frac{6}{n(n-1)} + \frac{(n-4)(n-5)}{n(n-1)} a;$$

and in the same manner

$$a = \frac{6+(n-6)(n-7)b}{(n-2)(n-3)},$$
$$b = \frac{6+(n-8)(n-9)c}{(n-4)(n-5)},$$
$$c = \frac{6+(n-10)(n-11)d}{(n-6)(n-7)},$$
etc.

Thence, we will draw the value of x sought:

$$x = \frac{6}{n(n-1)(n-2)(n-3)} \left((n-2)(n-3) + (n-4)(n-5) + (n-6)(n-7) + \cdots + 0\right).$$

This progression being algebraic, if we seek the sum by the known rules, we find

$$= \frac{n(n-2)(2n-5)}{12},$$

and therefore, the advantage of the Banker will be

$$x = \frac{2n-5}{2(n-1)(n-3)}.$$

THE ADVANTAGE OF THE BANKER
is worth as many per cent as the following table indicates

Number of all cards	For one card significant	For two cards significant	For three cards significant	For four cards significant
2	50,000	100,000		
4	25,000	25,000	25,000	50,000
6	16,667	13,333	15,000	23,333
8	12,500	8,929	10,714	15,714
10	10,000	6,667	8,333	11,905
12	8,333	5,303	6,818	9,596
14	7,143	4,395	5,769	8,042
16	6,250	3,750	5,000	6,923
18	5,556	3,268	4,412	6,078
20	5,000	2,895	3,947	5,418
22	4,545	2,597	3,571	4,887
24	4,167	2,355	3,261	4,451
26	3,846	2,154	3,000	4,087
28	3,571	1,984	2,778	3,777
30	3,333	1,839	2,586	3,512
32	3,125	1,714	2,419	3,281[1]
34	2,941	1,604	2,273	3,079[2]
36	2,778	1,508	2,143	2,900[3]
38	2,632	1,422	2,027	2,741
40	2,500	1,346	1,923	2,599
42	2,381	1,277	1,829	2,470
44	2,273	1,216	1,744	2,354
46	2,174	1,160	1,667	2,248
48	2,083[5]	1,109[6]	1,596	2,151
50		1,061	1,531	2,062
52				1,981

[1] Original edition: 3,272.
[2] Original edition: 3,078.
[3] Original edition: 2,901.
[4] Original edition: 2,084.
[5] Original edition: 1,108.

REMARK 1

14. In considering this table, one is able to give to the Punters this rule, so that they risk the least:

That they wait until two cards of a kind have come out, and as soon as this has occured, then they choose this card for their wager.

REMARK 2

15. The most advantageous case for the Punter is therefore, when the Banker draws on the first move two similar cards, such that there remain yet 50 cards in the hand. Because then, when the Punter wagers on this card, the advantage of the Banker will be the least possible.

REMARK 3

16. However, if it did not happen that two similar cards come forth before the Banker has drawn 16 cards, it would be worth as much as the Punter wagers first, on the second move, on a card which would come forth in the first move; but, as there are only 13 cards of each kind, this case cannot occur.

REMARK 4

17. When the Banker has no more than 28 cards in the hand, or even less, it is no longer apropos to wager on one card, although it is found only twice in the stock. It will be worth more to wait for the Banker to begin again, and to wager then on any one card; but the most certain way is always to wait yet then the second move, and to wager on a card which will have come forth on the first.

REMARK 5

18. There is yet a rule very essential for the Punters, that they never wager on a card, whatever it be, when the stock is already very diminished. It is also never good to wager on a card which is found no more than one time in the stock. Because, even when it would occur already in the third move, the Banker would have more than 2 per cent advantage on it. Now, a prudent Punter is able always to wager in a way that the advantage of the Banker surpasses hardly a per cent.

PROBLEM 5

19. *The number of cards being* $= n$*, if there are five significant cards, to find the advantage of the Banker.*

SOLUTION

Putting the sought advantage of the Banker $= x$ and making the same reasoning as before, one will attain finally to this equation[1]

$$x = \frac{10\left((n-2)(n-3)(n-4) + (n-4)(n-5)(n-6) + (n-6)(n-7)(n-8) + \cdots + 0\right)}{n(n-1)(n-2)(n-3)(n-4)}.$$

Now, the sum of the progression which makes here part of the numerator is found

$$= \frac{n(n-2)^2(n-4)}{8},$$

whence we draw the advantage of the Banker

$$x = \frac{5(n-2)}{4(n-1)(n-3)}.$$

PROBLEM 6

20. *The number of cards being* $= n$*, if there are six significant cards, to find the advantage of the Banker.*

SOLUTION

To find this advantage, that I name $= x$, one reaches this progression

$$s = (n-2)(n-3)(n-4)(n-5) + (n-4)(n-5)(n-6)(n-7) + \cdots + 0,$$

of which one finds the value

$$s = \frac{n(n-2)(n-4)(2nn - 13n + 16)}{20},$$

and that of x will be

$$x = \frac{15s}{n(n-1)(n-2)(n-3)(n-4)(n-5)}.$$

Consequently, the advantage of the Banker will be

$$x = \frac{3(2nn - 13n + 16)}{4(n-1)(n-3)(n-5)}.$$

[1] The original edition has the error, in the numerator,

$$10\left((n-2)(n-3)(n-4) + (n-4)(n-5)(n-6) + (n-5)(n-6)(n-7) + \cdots + 0\right).$$

PROBLEM 7

21. *The number of cards being* $= n$*, if there are seven significant cards, to find the advantage of the Banker.*

SOLUTION

One will reach this equation

$$x = \frac{21s}{n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)}$$

and

$$s = (n-2)(n-3)(n-4)(n-5)(n-6) + (n-4)\cdots(n-8) + \cdots + 0.$$

Now, the sum is

$$s = \frac{n(n-2)(n-4)(n-6)(2nn - 12n + 13)}{24}.$$

Therefore

$$x = \frac{7(2nn - 12n + 13)}{8(n-1)(n-3)(n-5)},$$

this which is the advantage of the Banker.

PROBLEM 8

22. *The number of cards being* $= n$*, if there are eight significant cards, to find the advantage of the Banker.*

SOLUTION

Now, there is concern to find the sum of this progression

$$s = (n-2)\cdots(n-7) + (n-4)\cdots(n-9) + (n-6)\cdots(n-11) + \cdots + 0.$$

Now, the known rules furnish us this sum[2]

$$s = \frac{1}{56}n(n-2)(n-4)(n-6)(4n^3 - 50nn + 176n - 151),$$

and then one will have

$$x = \frac{28s}{n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)(n-7)}.$$

Consequently, the advantage of the Banker will be

$$x = \frac{4n^3 - 50nn + 176n - 151}{2(n-1)(n-3)(n-5)(n-7)}.$$

[2] In the original edition, the first factor of s is $\frac{1}{12}$, instead of $\frac{1}{56}$.

REMARK 1

23. If one wishes to go further and to suppose the number of significant cards greater, all returns to the summation of similar progressions; which being algebraic, one has only to make application of the known rules in order to find the sum of them.

REMARK 2

24. In order to put before the eyes all that we come to find, in marking the number of all cards by n, one has

	the advantage of the Banker
for 1 significant card	$\frac{1}{n}$,
for 2 significant cards	$\frac{n+2}{2n(n-1)}$,
for 3 significant cards	$\frac{3}{4(n-1)}$.
for 4 significant cards	$\frac{2n-5}{2(n-1)(n-3)}$,
for 5 significant cards	$\frac{5n-10}{4(n-1)(n-3)}$,
for 6 significant cards	$\frac{3(2nn-13n+16)}{4(n-1)(n-3)(n-5)}$,
for 7 significant cards	$\frac{7(2nn-12n+13)}{8(n-1)(n-3)(n-5)}$,
for 8 significant cards	$\frac{4n^3-50nn+176n-151}{2(n-1)(n-3)(n-5)(n-7)}$

etc.

REMARK 3

25. It is difficult to discover a law in these expressions; also it is not necessary to seek them among all, since the first and the second are subject to some irregularities which are not present in the following. Now, if we neglect these anomalies of the case of one and two significant cards, the advantage is found in the first$= 0$ and in the second $= \frac{1}{2(n-1)}$; and these are the formulas which, with the following, must observe a certain law of progression.

REMARK 4

26. Any tangledness that this law should show, will appear sufficiently clear, if we decompose the found fractions into some simple fractions, according to the factors of the denominator of each. By this manner, one will change these expressions into the

following:

Number of significant cards	**Advantage of the Banker**
	0
	$\frac{1}{2}\cdot\frac{1}{n-1}$
3	$\frac{1}{4}\cdot\frac{3}{n-1}$
4	$\frac{1}{8}\left(\frac{6}{n-1}+\frac{2}{n-3}\right)$
5	$\frac{1}{16}\left(\frac{10}{n-1}+\frac{10}{n-3}\right)$
6	$\frac{1}{32}\left(\frac{15}{n-1}+\frac{30}{n-3}+\frac{3}{n-5}\right)$
7	$\frac{1}{64}\left(\frac{21}{n-1}+\frac{70}{n-3}+\frac{21}{n-5}\right)$
8	$\frac{1}{128}\left(\frac{28}{n-1}+\frac{140}{n-3}+\frac{84}{n-5}+\frac{4}{n-7}\right)$

In considering these formulas, one will discover easily the law of progression; and putting in general the number of significant cards $=\nu$, the number of all cards being $=n$, the advantage of the Banker will be

$$\frac{1}{2^{\nu-1}}\left\{\begin{array}{l}\frac{\nu(\nu-1)}{1\cdot 2}\cdot\frac{1}{n-1}+\frac{2\nu(\nu-1)(\nu-2)(\nu-3)}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{1}{n-3}\\ +\frac{3\nu(\nu-1)(\nu-2)(\nu-3)(\nu-4)(\nu-5)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}\cdot\frac{1}{n-5}+\text{etc.}\end{array}\right\},$$

which is changed into this

$$\frac{\nu}{2^\nu}\left(\frac{\nu-1}{1(n-1)}+\frac{(\nu-1)(\nu-2)(\nu-3)}{1\cdot 2\cdot 3(n-3)}+\frac{(\nu-1)(\nu-2)(\nu-3)(\nu-4)(\nu-5)}{1\cdot 2\cdot 3\cdot 4\cdot 5(n-5)}+\text{etc.}\right).$$

GENERAL PROBLEM

27. *The number of all cards being $=n$, if the number of significant cards is $=\nu$, to find the advantage of the Banker.*

SOLUTION

We come to see that the advantage of the Banker will be

$$\frac{\nu}{2^\nu}\left(\frac{\nu-1}{1(n-1)}+\frac{(\nu-1)(\nu-2)(\nu-3)}{1\cdot 2\cdot 3(n-3)}+\frac{(\nu-1)(\nu-2)(\nu-3)(\nu-4)(\nu-5)}{1\cdot 2\cdot 3\cdot 4\cdot 5(n-5)}+\text{etc.}\right),$$

except the case where one has $\nu=1$ or $\nu=2$. Now, if we consider this progression, we see easily that it is able to be reformulated in a finite integral expression

$$\frac{\nu}{2^{\nu+1}}\left(\int z^{n-1}dz\left(1+\frac{1}{z}\right)^{\nu-1}-\int z^{n-1}dz\left(1-\frac{1}{z}\right)^{\nu-1}\right).$$

Because, having taken these integrals in such way that they vanish putting $z=0$, one has only to put in $z=1$.

In observing this rule in the integrations, the advantage of the Banker would also be expressed in a way

$$\frac{\nu}{2^{\nu+1}}\left(\int z^{n-\nu}dz\,(z+1)^{\nu-1}-\int z^{n-\nu}dz\,(z-1)^{\nu-1}\right),$$

putting after the integration $z=1$.

One sees first that this formula cannot take place, unless there be $n>\nu$, since one would not render the integral $=0$, in the case $z=0$, this which is conformed to the nature of the question.

REMARK 1

28. Following the direct method, we would have had to sum this progression

$$s=(n-2)(n-3)\cdots(n-\nu+1)+(n-4)(n-5)\cdots(n-\nu-1)+\cdots+0,$$

and the advantage of the Banker would have been[3]

$$x=\frac{\nu(\nu-1)s}{2n(n-1)(n-2)\cdots(n-\nu+1)}.$$

Therefore, reciprocally, one will obtain the sum of the progression

$$\frac{n(n-1)(n-2)\cdots(n-\nu+1)}{(\nu-1)2^{\nu}}\left(\int z^{n-\nu}dz\,(z+1)^{\nu-1}-\int z^{n-\nu}dz\,(z-1)^{\nu-1}\right).$$

REMARK 2

29. Now, putting $n-2=2t$ or $n=2t+2$, the quantity s marks the sum of one algebraic progression of which the general term, or the one which corresponds to the exponent t, is

$$T=2t(2t-1)(2t-2)(2t-3)\cdots(2t-\nu+3).$$

And therefore, we will be able to assign the sum $S.T$ which agrees to this general term by the following integral expression

$$S.T=\frac{(t+1)(2t+1)}{(\nu-1)2^{\nu-1}}T\left(\int z^{2t+2-\nu}dz\left((z+1)^{\nu-1}-(z-1)^{\nu-1}\right)\right).$$

REMARK 3

30. But, in developing this integral formula, we will have

$$S.T=\frac{(2t+2)(2t+1)}{2^{\nu-1}}T\left(\frac{1}{2t+1}+\frac{(\nu-2)(\nu-3)}{2\cdot3(2t-1)}+\frac{(\nu-2)(\nu-3)(\nu-4)(\nu-5)}{2\cdot3\cdot4\cdot5(2t-3)}+\text{etc.}\right),$$

and this summation is correct, whatever whole numbers that one puts for the letters t and ν in a way that $\nu<2t+2$, or rather that ν not be greater than $2t+2$. This sum corresponds therefore to the general term

$$T=2t(2t-1)(2t-2)(2t-3)\cdots(2t-\nu+3),$$

the exponent of the last term of the progression being $=t$.

[3] In the original edition, the factor 2 is wanting in the denominator of the fraction which gives x, and this error has repercussions in the following formulas. We have worked the necessary modifications. L.G.D.