Calculus of the probability in the game of Rencontre*

Leonard Euler†

E201
Mémoires de l'académie de Berlin [7] (1751), 1753, p. 255-270

1. The game of rencontre is a game of chance where two persons, each having an entire deck of cards, draw from it at the same time one card after the other, until it happens that they encounter the same card; and then one of the two persons wins. Now, when such an encounter does not happen at all, then it is the other of the two persons who wins. This posed, one asks the probability that each of these two persons will win.

2. In order to fix better our ideas, one is able to suppose that these two persons, of whom the one is named A and the other B, having each a certain and same number of tickets marked with the numbers $1,\ 2,\ 3,\ 4,\ 5$ etc., and that each draws from them one ticket after another, until they encounter the same number at the same time; and that it is the person A who wins then. Now, if it happens that these two persons draw all their tickets without ever encountering the same number, then the person B wins.

3. As it is indifferent by which number each ticket is marked, it is permitted to suppose that A draws his tickets according to the order $1,\ 2,\ 3,\ 4,\ 5$ etc. Now, in order to make application to the cards, one will imagine the cards of both decks so numbered, according to the order as they are drawn successively by A, so that number 1 will be the card that A draws first, number 2 that which he draws second, number 3 third and so on.

4. Thus person A, who is for the encounter, will win, when B draws from his pack of cards: at the first move number 1, or at the second number 2, or at the third number 3, or at the fourth number 4 etc. Now, if it happens that the number of the card drawn by B never corresponds to the number of the card drawn by A at the same move, then it will be B who wins the kitty. By this means, it seems that the research on this game is rendered most easy in order to apply the calculus to it.

5. The question is thus to determine the probability that there will be as much A as B to win the kitty, whatever be the number of cards or numbered tickets. Because first one sees that this determination varies according to the number of tickets and that it becomes more complicated, the greater will be the number of tickets. It will be convenient therefore to begin this research with the smallest numbers of tickets and to proceed from them in order to arrive successively at the greater.

*See also the memoirs 313, 811 and 813.

†Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH

6. Therefore we suppose first that both of the players have only a single card marked with 1, and it is clear that the encounter is not known to fail, so that A will win infallibly. Therefore in this case, the probability of winning by A will be expressed by 1 and that of B by 0, since he has no expectation to win.

7. Let both of the players A and B have now two cards, numbered by 1 and 2, and let A draw his cards according to the numbers 1 and 2. In this supposition there will be two cases, because B will draw his two cards either in the order $1,\ 2$ or in the order $2,\ 1$. The first, giving first at the first move an encounter, will make A win; the other giving no encounter will make B win.

8. Since therefore each of these two cases is equally probable, A as much as B will have each a case to win. And leaving, the probability of each will be expressed by $\frac{1}{2}$, or else each will have the right to take half of the kitty.

9. We posit that the two players have each three cards, marked with the numbers $1,\ 2,\ 3$, and that A draws in first place number 1, in second number 2, in third number 3. Now, B will be able to draw his cards in 6 different ways, of the sort:

A	B					
	1	2	3	4	5	6
1	1	1	2	2	3	3
2	2	3	1	3	2	1
3	3	2	3	1	1	2

and it is equiprobable that each of these 6 cases actually happen.

10. Of these 6 cases, there will be two therefore, the first and the second which will make A win and where the game ends consequently on the first move; of the four other cases, there is only one of them, namely the fifth, which will make A win on the second move and which ends the game. Among the three other cases, there is yet the third, which makes A win on the third move.

11. Thus in all, among all 6 possible cases, there are four which are favorable to A, and the other two, namely the fourth and the sixth, put B in possession of gain. Therefore, A having four cases to win and B two, the expectation of A is $= \frac{4}{6} = \frac{2}{3}$ and that of $B = \frac{2}{6} = \frac{1}{3}$; or the advantage of A is two times greater than that of B.

12. We give now to each of our players 4 cards $1,\ 2,\ 3,\ 4$; and while A draws his cards in the order $1,\ 2,\ 3,\ 4$, the order of the cards of B is able to vary in 24 different ways, of the sort:

A	B																							
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
1	1	1	1	1	1	1	2	2	2	2	2	2	3	3	3	3	3	3	4	4	4	4	4	4
2	2	2	3	3	4	4	3	3	4	4	1	1	4	4	1	1	2	2	1	1	2	2	3	3
3	3	4	4	2	2	3	4	1	1	3	4	3	1	2	2	4	1	4	2	3	3	1	1	2
4	4	3	2	4	3	2	1	4	3	1	3	4	2	1	4	2	1	1	3	2	1	3	2	1

and each of these 24 cases is equiprobable.

13. It is evident that the first six cases make A win first on the first move; and since the game is finished, I have deleted the numbers following of these 6 columns. Of the 18 cases which remain, there are 4, namely the cases 17, 18, 21 and 22, which make A win on the second move, where these columns consequently will be terminated. Fourteen cases will continue the game therefore until the third move; and there are three of them, 10, 12 and 20, which terminate the game in favor of A. Finally of the

eleven cases which remain, there are only two of them which give an encounter for the fourth and last move.

14. Therefore having 6 cases where A wins on the first move, 4 cases where he wins on the second move, three cases where he wins on the third move and two cases where he wins on the fourth move, there will be in all 15 cases favorable to A, and the 9 other cases will make B win. Consequently, the probability of winning by A will be $= \frac{15}{24} = \frac{5}{8}$ and that of $B = \frac{9}{24} = \frac{3}{8}$; or else the expectation of A will be to that of B as 5 to 3.

15. If we put the number of cards $= 5$, one would have in all 120 different cases for the variations which may happen in the order of the cards drawn by B, while A will draw his cards according to the numbers $1, 2, 3, 4, 5$. Now, it would lead too far, if we wished to represent all these cases, in order to see how many of them would be favorable to A and to B; and a still greater number of cards would render this representation completely impractical.

16. Moreover, such an actual denumeration would not serve much to determine in general the expectations of the two players A and B, however great be the number of cards. For this effect, it is necessary to make some general remarks which may lead us to the knowledge of the [probabilities for the] greatest numbers of cards, knowing already the probabilities for the smaller numbers.

17. Therefore I remark first in general that, the number of cards being $= m$, there will be as many different cases as the product of all the numbers $1, 2, 3, 4$ up to m constrained by units; or else this number of cases is

$$= 1 \cdot 2 \cdot 3 \cdot 4 \cdots m.$$

Now, I suppose always that A draws his cards according to the order of numbers $1, 2, 3, 4, \ldots m$, by which they are marked, and the product $1 \cdot 2 \cdot 3 \cdot 4 \cdots m$ will give the number of cases which are able to happen in the order of the cards drawn by B.

18. It is clear by the first principles of combinations, whence one knows that the order of 2 cards is able to vary 2 times, of 3 cards 6 times, of 4 cards 24 times, of 5 cards 120 times, of 6 cards 720 times, of 7 cards 5040 times, of 8 cards 40320 times, and in general of m cards as many times as the product

$$1 \cdot 2 \cdot 3 \cdot 4 \cdots m$$

constrained by units.

19. This number of cases $1 \cdot 2 \cdot 3 \cdot 4 \cdots m$ being put for brevity $= M$, I remark, in second place, that there will be $\frac{M}{m}$ cases where the first card drawn by B is 1, that there will be $\frac{M}{m}$ cases where the first card drawn by B is 2, and there will be as many cases where the first card of B is either 3 or 4 or 5 etc., or finally m.

20. More, if we set aside that the game ends as soon as B will encounter the card of A, and that we suppose that they continue to draw their cards until the end, although there had happened one or many encounters, it is also clear that there will be $\frac{M}{m}$ cases where the second card of B will be 2, and as many cases where the third card will be 3 or the fourth 4 or the fifth 5 or the sixth 6, and thus in order.

21. Therefore, in this supposition that one continues to draw the cards until the end, there will be $\frac{M}{m}$ cases that A wins on the first move, the same $\frac{M}{m}$ cases that he wins on the second move, and always as many cases that he win on the third move or on the fourth or on the fifth etc., or the same on the last move.

22. But in effect, although there are $\frac{M}{m}$ cases which make A win on the first move, there will not be as many cases which makes him win on the second move, since from the $\frac{M}{m}$ cases which would make him win on the second move, in the preceding supposition, it is necessary to subtract those which have already made him win on the first move; because, as soon as he had won on the first move, the game does not continue further.

23. There is the same of the number of cases $\frac{M}{m}$ where B would draw the card number 3; because it is necessary to subtract the cases which were already encountered, either on the first move or on the second. And in order that A win on the fourth move, it is necessary to take off from the number of all the cases where this happens, which is $\frac{M}{m}$, those which will have already been an encounter, either in the first or in the second or in the third move.

24. In general therefore, the number of cases which would make A win on some move being $= \frac{M}{m}$, by the preceding hypothesis, it is necessary to exclude from it all those where there is already found an encounter in any one of the preceding moves; such that the number of cases becomes more and more fewer, the more the move is advanced from the beginning.

25. In order to decide therefore by how much it is necessary to diminish the number of favorable cases $\frac{M}{m}$ at each move, or in order to know the number of those which already obtained an encounter in some preceding move, here is how I myself do it. I imagine that the card which is encountered in a proposed move be removed from both decks, and the order of the cards and the number of the cards will be the same as if the number of the cards were diminished by a unit.

26. In order to render this more intelligible, we consider the case of 4 cards and of the 24 cases which have place there, those where B draws on the third move the card number 3, which are the cases marked 1, 6, 10, 12, 20, 21. We remove from these cases the card marked number 3, and we will have

A	B					
1	1	1	2	2	4	4
2	2	4	4	1	1	2
4	4	2	1	4	2	1

which are precisely the cases where one would have for three cards marked with the numbers 1, 2, 4.

27. Since this is the number of cases where B draws on a third move the card number 3, and since it is necessary to subtract those which already obtained an encounter, either on the first move or on the second, it is clear that this number to subtract is found in the cases of three cards, in adding together the cases where A would win then on the first move or on the second.

28. In general therefore, if the number of cards is $= m$ and if one wishes to know by how much it is necessary to diminish the number of cases $\frac{M}{m}$, which have an encounter in some move, it is necessary to have recourse to the number of cards $= m - 1$ and to

seek in it the cases which would make A win at some one of the preceding moves; and the number of all these cases together will be the one which it is necessary to diminish the number $\frac{M}{m}$, in order to have the number of cases which will make A win actually in a proposed move.

29. We put therefore the number of cards $= m$ and the number of all the cases which correspond to them

$$1 \cdot 2 \cdot 3 \cdot 4 \cdots m = M,$$

and let

a	be the number of cases which make A win	on the first move,
b		on the second move,
c		on the third move,
d		on the fourth move,
e		on the fifth move

etc.,

and we have seen that $a = \frac{M}{m}$; for the other numbers b, c, d, e etc., we look at their progression shortly.

30. Now let the number of cards be greater by one unit, or $= m+1$, and the number of all the cases will be

$$= 1 \cdot 2 \cdot 3 \cdot 4 \cdots (m+1) = M(m+1),$$

which is $= M'$. Let next as before

a'	be the number of cases which make A win	on the first move,
b'		on the second move,
c'		on the third move,
d'		on the fourth move,
e'		on the fifth move

etc.,

31. This posed, we will have

$$a' = \frac{M'}{m+1} = M;$$

and in continuing the game, notwithstanding the encounters already occurred, there will be M cases likewise where there will happen an encounter on the second move; but of these, it is necessary to exclude those which obtained an encounter already on the first move; and, this number being a, as we have seen (§29), we will have

$$b' = M - a$$

for the number of cases which actually make A win on the second move.

32. The number of cases where the encounter happens on the third move being likewise $= M$, and since it is necessary to exclude from them those which obtained an encounter on the first or second move, that is to say those which would make A win on

the first or second move when the number of cards would be less by one, we will have the number of cases which actually will make A win on the third move

$$c' = M - a - b.$$

33. It is the same with the cases which would make A win on some one of the moves following; and therefore, by knowing the numbers a, b, c, d etc. for the number of cards $= m$, we will extract easily the numbers a', b', c', d' etc., when the number of cards is $= m + 1$. Because one will have

$$\begin{aligned} a' &= M, \\ b' &= M - a, \\ c' &= M - a - b, \\ d' &= M - a - b - c, \\ e' &= M - a - b - c - d \\ &\text{etc.} \end{aligned}$$

34. Therefore knowing that, when the number of cards is $m = 1$ and $M = 1$, it is $a = 1$, one will have for two cards

$$a' = 1 \text{ and } b' = 1 - 1 = 0.$$

Now let $m = 2$; and having $M = 2$, $a = 1$, $b = 0$, one will have for three cards

$$a' = 2,\ b' = 2 - 1 = 1,\ c' = 2 - 1 - 0 = 1.$$

We put more $m = 3$; and having $M = 6$, $a = 2$, $b = 1$, $c = 1$, we will have for four cards

$$a' = 6,\ b' = 6 - 2 = 4,\ c' = 6 - 2 - 1 = 3,\ d' = 6 - 2 - 1 - 1 = 2.$$

35. In this fashion, we will be able to continue these numbers to the number of cards as great as one will please, and in order to see better the progression, we represent them in the following manner:

NUMBER OF CARDS

	I	II	III	IV	V	VI	VII	VIII	IX	X
a	1	1	2	6	24	120	720	5040	40320	362880
b		0	1	4	18	96	600	4320	35280	322560
c			1	3	14	78	504	3720	30960	287280
d				2	11	64	426	3216	27240	256320
e					9	53	362	2790	24024	229080
f						44	309	2428	21234	205056
g							265	2119	18806	183822
h								1854	16687	165016
i									14833	148329
k										133496

36. If we divide these numbers by the numbers of all the possible cases which correspond to each number of cards, we will extract from them firstly the expectations of A to win on the first move:

Number of cards	1,	2,	3,	4,	5,	6,	7,	8,	9,	etc.
Expectation of A	1,	$\frac{1}{2}$,	$\frac{1}{3}$,	$\frac{1}{4}$,	$\frac{1}{5}$,	$\frac{1}{6}$,	$\frac{1}{7}$,	$\frac{1}{8}$,	$\frac{1}{9}$,	etc.

Whence we conclude that, if the number of cards is $= n$, the expectation of A to win on the first move will be $= \frac{1}{n}$.

37. If we consider the numbers of the table (§35), we see first that each number is the difference of that which is above and of the one which precedes it. Thus, if for the number of cards m the number of cases which make A win in a certain move, is p, and the number of cases which make him win in the same move, if the number of cards is $= m + 1$, is $= q$, and the number of cases which make him win in the following move, $= r$, the number of cards remaining $= m + 1$, one will always have

$$r = q - p.$$

38. Therefore, for the number of cards $= m$, the number of all the cases being

$$= 1 \cdot 2 \cdot 3 \cdot 4 \cdots m = M,$$

the expectation of A to win on a certain move will be

$$= \frac{p}{M},$$

which I will name $= P$. Now, for the number of cards $= m + 1$, the number of all cases being

$$= M(m + 1),$$

the expectation of A to win at the same move will be

$$= \frac{q}{M(m+1)},$$

which is put $= Q$, and the expectation to win at the following move

$$= \frac{r}{M(m+1)},$$

which is $= R$. This posed, one will have

$$R = \frac{q - p}{M(m+1)},$$

or else

$$R = Q - \frac{P}{m+1}.$$

39. Therefore, putting the number of cards $= n - 1$, since the expectation of A to win on the first move is $= \frac{1}{n-1}$, for the number of cards $= n$ the expectation of A to win on the second move will be

$$= \frac{1}{n} - \frac{1}{(n-1)n} = \frac{n-2}{(n-1)n}.$$

40. Now, the expectation of A to win on the second move, when the number of cards is $= n-1$, being $\frac{n-3}{(n-2)(n-1)}$, we conclude from it that, when the number of cards is n, his expectation to win on the third move will be

$$= \frac{n-2}{(n-1)n} - \frac{n-3}{(n-2)(n-1)n} = \frac{nn-5n+7}{(n-2)(n-1)n} = \frac{(n-2)^2-(n-3)}{n(n-1)(n-2)}.$$

41. Thence we conclude from the same manner that, for the number of cards $= n$, the expectation of A to win on the fourth move will be

$$= \frac{(n-2)^2-(n-3)}{n(n-1)(n-2)} - \frac{(n-3)^2-(n-4)}{n(n-1)(n-2)(n-3)} = \frac{(n-2)^2(n-3)-2(n-3)^2+(n-4)}{n(n-1)(n-2)(n-3)}$$

and his expectation to win on the fifth move

$$= \frac{(n-2)^2(n-3)-2(n-3)^2+(n-4)}{n(n-1)(n-2)(n-3)} - \frac{(n-3)^2(n-4)-2(n-4)^2+(n-5)}{n(n-1)(n-2)(n-3)(n-4)}$$
$$= \frac{(n-2)^2(n-3)(n-4)-3(n-3)^2(n-4)+3(n-4)^2-(n-5)}{n(n-1)(n-2)(n-3)(n-4)}$$

42. For the little while that one reflects on the formation of these formulas, one will find that, the number of cards being $= n$, the expectation of A to win will be on the first move

$$= \frac{1}{n},$$

on the second move

$$= \frac{1}{n} - \frac{1}{n(n-1)},$$

on the third move

$$= \frac{1}{n} - \frac{2}{n(n-1)} + \frac{1}{n(n-1)(n-2)},$$

on the fourth move

$$= \frac{1}{n} - \frac{3}{n(n-1)} + \frac{3}{n(n-1)(n-2)} - \frac{1}{n(n-1)(n-2)(n-3)},$$

on the fifth move

$$= \frac{1}{n} - \frac{4}{n(n-1)} + \frac{6}{n(n-1)(n-2)} - \frac{4}{n(n-1)(n-2)(n-3)} + \frac{1}{n(n-1)\cdots(n-4)},$$

on the sixth move

$$= \frac{1}{n} - \frac{5}{n(n-1)} + \frac{10}{n(n-1)(n-2)} - \frac{10}{n(n-1)(n-2)(n-3)} + \frac{5}{n\cdots(n-4)} - \frac{1}{n\cdots(n-5)},$$

etc.

43. Therefore, the expectation of A to win in general, at any move that this be, will be expressed by the sum of all these formulas taken together. Now, the number of these formulas being equal to the number of cards n, the sum of all the first terms will be

$$= n \cdot \frac{1}{n} = 1.$$

Next, the sum of the numerators of the second terms being

$$= 1 + 2 + 3 + 4 + \cdots + (n-1) = \frac{n(n-1)}{1 \cdot 2},$$

the sum of all the second terms will be

$$= \frac{1}{1 \cdot 2}.$$

More, because

$$1 + 3 + 6 + 10 + \cdots + \frac{(n-1)(n-2)}{1 \cdot 2} = \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3},$$

the sum of the third terms is

$$= \frac{1}{1 \cdot 2 \cdot 3},$$

and the sum of the fourth

$$= \frac{1}{1 \cdot 2 \cdot 3 \cdot 4},$$

of the fifth

$$= \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5},$$

and so forth.

44. Thence, it follows therefore that

the number of cards being	the expectation of A to win will be
1	1
2	$1 - \frac{1}{1\cdot 2}$
3	$1 - \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3}$
4	$1 - \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} - \frac{1}{1\cdot 2\cdot 3\cdot 4}$
5	$1 - \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} - \frac{1}{1\cdot 2\cdot 3\cdot 4} + \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5}$
6	$1 - \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} - \frac{1}{1\cdot 2\cdot 3\cdot 4} + \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}$

always taking in this sequence as many terms as there are cards.

45. The expectation of A is therefore the greatest in the case of one card, and the least in the case of two cards. Next one sees that, when the number of cards is odd, the expectation of A is always greater than that for all even number of cards. Now, if the number of cards is even, then the expectation of A is less than that for all odd number of cards.

46. Having found the expectation of A, one has only to subtract from unity in order to have the expectation of B; because the expectation of each indicates the portion of the kitty to which each is able to claim by virtue of the probability that he has to win the complete whole. Thus, the expectation of A being $= x$, that of B will be $= 1 - x$.

47. The formulas that I have just found for the expectation of A will reduce easily to some decimal fractions, whence one will judge better of their true value. Thus, having made this calculation, I find:

number of cards	expectation of A	expectation of B
1	$1,000000000$	$0,000000000$
2	$0,500000000$	$0,500000000$
3	$0,666666666$	$0,333333333$
4	$0,625000000$	$0,375000000$
5	$0,633333333$	$0,366666666$
6	$0,631944444$	$0,368055555$
7	0.632142857	$0,367857143$
8	$0,632118055$	$0,367881945$
9	$0,632120811$	$0,367879189$
10	$0,632120536$	$0,367879464$
11	$0,632120561$	$0,367879439$
12	$0,632120558$	$0,367879442$
13	$0,632120559$	$0,367879441$
14	$0,632120558$	$0,367879442$
15	$0,632120558$	$0,367879442$
etc.	etc.	etc.

48. Therefore, if we neglect the decimal fractions which follow after the ninth, one is able to say that, as soon as the number of cards is greater than 12, the expectations of A and of B do not vary, however great be the number of cards. Thus, when the number of cards is not below 12, one is able to say that the expectation of A is $= 0,632120558$ and that of $B = 0,367879442$.

49. Therefore provided that the number of cards is not less than 12, the expectation of A will be always to that of B nearly as 12 to 7, or more exactly as 122 to 71, or again more exactly as 1720 to 1001. Or else, among 19 games that one plays, there will be probably 12 which make A win and 7 which make B win.

50. If the number of cards be infinite, the expectation of A will be expressed by this infinite series

$$1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} - \frac{1}{720} + \text{etc.}$$

and the expectation of B by this

$$1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} - \text{etc.}$$

Now, putting e in order to mark the number of which the logarithm is $= 1$, one knows that $\frac{1}{e}$ expresses this last series. Therefore, for the case $n = \infty$, the expectation of A

will be $= 1 - \frac{1}{e}$ and that of [B will be] $= \frac{1}{e}$; but one has

$$e = 2,718281828459045235360.$$

51. Substituting this value for e, one will find that the expectation of A is to that of B as

$$1,718281828459045235360 \text{ to } 1,$$

and this proportion will be correct as soon as the number of cards will be greater than 20. Consequently, it will be very exact for the case of this game as it is ordinarily played, employing for it an entire deck of 52 cards.