ÉLÉMENS DU CALCUL DES PROBABILITÉS*

M. de Condorcet

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ARTICLE V.

On the manner to compare among them some events of different probabilities, and to find a mean value which can represent the different values among them of unequally probable events.

The first applications of the calculus of probabilities had for object games of chance. This application was neither most important nor most useful, but it was most simple; it presented itself first. Pascal and Fermat were the inventors of this calculus; thus one owes it to some French. This remark is not useless; it can serve to refute those who please themselves by repeating that nature has refused the gift of invention, and accords only the one of perfection to men who are born between Perpignan and Dunkirk.

The first questions that these two illustrious geometers proposed, were those to estimate the lot of two players who had an equal probability to win a given sum, and to divide a stake between some players who had an unequal expectation to win it, in the actual state of the game, would agree together to quit it.

They found that the lot of the players, the part of the stake which represents this lot, and that which one must give to each, if they agree to cease to play, must be in ratio of the probability to win the entire stake.

We suppose, for example, that two players A and B play in seven linked games, in a manner that the stake belongs to the one of the two who will have earliest won four games; that they had already played three of them; that A had won two games, and B one only. The probability to win or to lose a game was equal for for the two players, these probabilities were expressed by $\frac{1}{2}$. This posed, we suppose that one had played two more games, there is a probability $\frac{1}{4}$ that A would win them both, $\frac{1}{2}$ that A and B would win each one of them, $\frac{1}{4}$ that B would win both of them. There is therefore a probability $\frac{1}{4}$ that A will win, a probability $\frac{1}{2}$ that he will have three games, and that B will have two of them, and a probability $\frac{1}{4}$ that A will have two of them, and that B will have three of them. We suppose that A has had three games, and B two alone, and that one plays one more of them, one will have $\frac{1}{2}$ in order that A has four games, and $\frac{1}{2}$ that they will have three each; and as the probability of this hypothesis is $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$ or

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. December 14, 2009

$\frac{1}{4}$ will express therefore the probabilities of these events; this which gives us $\frac{1}{4}$ for the probability that A will win, and $\frac{1}{4}$ for that which after the sixth game, they will have each an equal number of them.

We suppose that A has had two games, and that B has had three of them, we will have next, in this case, a probability $\frac{1}{2}$ that A will have a number of games equal to the one of B, and a probability $\frac{1}{2}$ that B will win. But the probability of this hypothesis is $\frac{1}{4}$; therefore these two probabilities will be $\frac{1}{8}$; we will have therefore $\frac{1}{4} + \frac{1}{4}$ or $\frac{1}{2}$ for the probability that A will have won after the sixth game; $\frac{1}{4} + \frac{1}{8} = \frac{3}{8}$ for this that the two players will have been equal, $\frac{1}{8}$ for the one that B will have won. But if A and B have each three games, their probability of winning is $\frac{1}{2}$ for each. That of winning the seventh game will be therefore for each $\frac{3}{8}, \frac{1}{2}$ or $\frac{3}{16}$ and the probabilities, in favor of A and of B, will be $\frac{11}{16}$ and $\frac{5}{16}$.

If therefore the funds of the game is 16 écus, for example, A must then take 11 of them, and B 5 only.

From this first proposition it was easy to conclude that, if two men agreed to play in an unequal game, their stakes must be also in ratio of the probability of winning; since, if they broke the convention before play, they would have right to some parts of the total stake proportional to that probability.

If one supposes that A had a probability $\frac{5}{6}$ to win, and B a probability only $\frac{1}{6}$, the stake of A will be therefore 5, and that of B 1; but if one plays, and if A wins, his profit will be only 1, and if B wins, his profit will be 5. Therefore, one has said, in order to play in an equal game, it is necessary that the gains be in inverse ratio to the probabilities. Likewise A having a probability $\frac{5}{6}$ to win 1, and a probability $\frac{1}{6}$ to lose 5, $\frac{5}{6} - \frac{5}{6}$ or 0 expresses the lot of A; it is likewise of the one of B; and they have therefore neither one nor the other any advantage to play.

We suppose after this that one wishes to evaluate the lot of a man who has a certain number of different probabilities to obtain some events of different values, it will be necessary to multiply the value of each event by its probability; and by taking the sum of these products, if some of these events have an advantage, and if the others produce a disadvantage, the value of the ones will be positive, those of the others will be negative; and then, if the sum is positive, it expresses the mean value of the advantage which results from the hypothesis; if it is negative, it expresses the mean value of the disadvantage which results from the same hypothesis.

This method to take the mean value, was first generally adopted, because, in the different applications which were made of it, it would lead only to some results conformed to common reason. But a celebrated problem known under the name of the problem of Petersburg, gave birth under the generality of this law, to some doubts which, until here, have perhaps not been absolutely dissipated.

One supposes that A play against B *at heads or tails*, under this condition that, if he brings forth tails at the first trial, he will give a coin to B; two, if he brings it forth at the second trial; four, if he brings it forth at the third; eight, if he brings it forth at the fourth, and so forth; and one demands what sum B must give to A before the game, in order that their lot be equal. If x is that sum, it is clear that it must be equal to the mean value of that which B can win, by virtue of the convention. Now, there is a probability $\frac{1}{2}$ to win 1, $\frac{1}{4}$ to win 2, $\frac{1}{8}$ to win 4, $\frac{1}{16}$ to win 8, and so forth. The lot which results from this convention, will be therefore $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} \cdots$, that is to say, a number of coins greater than any given number. Now, this conclusion seems absurd in itself. We suppose, in fact, that the number of possible trials is limited and expressed by n, then it is clear that this sum is expressed by $\frac{n}{2}$, by supposing that if tails is not arrived at the n^{th} trial, B receives nothing, and that if tails arrives at a trial m only B will receive 2^{m-1} coins; it is necessary therefore, in order that B begin to win, that $2^{m-1} > \frac{n}{2}$. We suppose in consequence that $\frac{n}{2}$ be 2^{10} one would have had $m = 11$, and for B a probability only $\frac{1}{2^{11}}$ or $\frac{1}{2^n}$ of not losing; if $\frac{n}{2} = 2^{100}$ the probability of not losing will be for B $\frac{1}{2^{100}}$, and thus in sequence, so that if one supposes n greater than any given number, there results from it that B would have given a sum greater than any given quantity, in order to acquire a probability to win smaller than any given quantity.

Finally m is greater also, under the same hypothesis, than any given quantity, since, whatever determined value that one supposes to m, as soon as $\frac{n}{2} = 2^{m-1}$, it would follow that $\frac{n}{2}$ would not be then greater than any given quantity, this which is against the hypothesis, and there results from it that B would be able to win only after a number of trials greater than any given number, at the end of a time greater than any given time.

One would be deceived, if one believed that this idea of a quantity greater than any given quantity is the sole cause of the difficulty which the application of the rule to his problem presents, and that thus one can regard this difficulty as one of these paradoxical conclusions which seem to appear in metaphysics, and of which many parts of geometry give some examples.

We take therefore a case which can be real. We suppose that the coin taken here for unity is one denier; that the number of trials is limited to 36, in a manner that B gives eighteen deniers or six liards. In order to be able to fill all the conditions of the game, it will be necessary that A is able to pay, when tails itself will arrive only at the thirty-sixth trial; then he must have 2^{35} deniers which are a sum of more than 14,316,000 *l*. Now, it is evident that any man playing from this fortune, who would have his reason, would not consent to play in a game where he is able to win only 18 deniers, and where he exposes himself to a danger, very small, in truth, but to a real danger to lose his fortune or a considerable part of his fortune.

We are going therefore to seek to examine if one must substitute another law to that which has been adopted until here, or to conserve it, by distinguishing the cases to which one must apply it.

I will observe first that this law is one consequence of the principle according to which one takes a mean value. Indeed, let there be two events of which the probability is as a to b, and the value A and B; the rule gives $aA + bB$ for the mean value. We suppose that a is expressed by $\frac{n}{n+m}$, and b by $\frac{m}{m+n}$, n and m being whole numbers, we will have therefore n equally possible events of which the value is A, m equally possible events of which the value is B, and consequently $aB + bB = \frac{nA+mB}{n+m}$ is the same thing as the sum of the values of all the events, divided by their number.

The ordinary rule gives us therefore the true mean value of events of different and equally probable values.

One can distinguish two kinds of mean value; the first destined to represent a determined value which is unknown, and of which there is a question to have a value also as near as it is possible, in order to employ it in the place of the true value which one does not know, and that one is counted to not know always; and, in this case, the true value is not necessarily one of the values given by observation, and among which one chooses a mean value. We suppose, for example, that I employ the method of the corresponding altitudes in order to determine the hour that a given pendulum marks at true noon. One of the observations will give me, for example, noon plus two minutes and twenty seconds; another, noon plus two minutes and thirty seconds; a third, noon plus two minutes and fifteen seconds; a fourth, noon plus two minutes and 27 seconds. I take the sum of these values, I divide it by 4, and I have, for mean value, noon two minutes plus twenty-three seconds.

One can suppose also that there are many possible determined and different values among them, and that the mean value sought is not regarded as an approximate value, but only as an equivalent value in certain circumstances, however one would be sure that it can not be equal to the true value, nor even approach it; and in this case, the true value is one of those among which one takes the mean value.

For example, if one takes the mean value of the duration of life of a man of a given age, by taking the sum of the real duration of the lives of a great number of men of that age, and dividing this sum by their number, one has only a very small probability that this mean value expresses the real duration of a given person.

If I have, in a any game, a probability $\frac{99}{100}$ to win 1, and a probability $\frac{1}{100}$ to lose 99, the mean value of the loss and of the gain is 0, and however the event which would give a real value equal to 0, is impossible.

There is no question then to prove that the method to find these mean values, by taking the sums of the equally possible values, and dividing it by their number, or, that which returns to the same, by taking the sum of the values multiplied by their probability, gives either certainly or very probably the true value, either rigorously, or in a manner very nearly, but only one must indicate that the mean value thus found, is the only one which can be substituted to the true unknown value, all the time that, by the nature of the questions proposed, this substitution must take place.

The mean value that one deduces by the method that we come to expose, possessed exclusively the following properties:

1. That the sum of the positive and negative differences between it and the values given by observation, or else the sum of the differences between this value and the equally possible true values, is equal to zero.

2. That the difference between the unknown true value, or any possible true value, and the mean value, is equal to the sum of the differences between each of these true values and the other observed values, or the possible values, divided by their number.

3. That by taking, under similar circumstances, this mean value for a true value, the most probable event will be the one where the differences more or less between reality and hypothesis, will be compensated; that one will have an always increasing probability that their sum will not exceed a part as small as one will wish of the greatest possible sum of these differences, while at the same time there exists no law which is able to give a probability also always increasing to differ only by a quantity always constant. Finally, one will have a probability always approaching to $\frac{1}{2}$, or always tending to equality that this sum will be positive rather than negative, or reciprocally.

One sees therefore that, in all the cases where one is able to substitute a mean value for the true value, that which the ordinary method gives, is the only one which one must choose. Otherwise, one will be exposed to deceive oneself by a greater quantity more than less, to fall into some errors so much greater as one will use a greater number of times the mean values, by having a probability always increasing to be deceived more or less; finally, one will not choose the value for which it is most probable that the errors will be compensated.

It is necessary then to use this mean value, but only to examine before, for each case, if it is one of those in which the substitution of a mean value to the true value, is able to take place.

Thus when one proposes to substitute, either in the true but uncertain and unknown value of the event of the game, from one enterprise of commerce, the mean value of this event, or the mean value of the duration of human life, or of annual advantages depending on the duration of life, for their real and unknown value, or a mean value of the values given by different observations, for the true value of which one supposes that it gives an approximate value, it is necessary to examine to what point this substitution is legitimate.

We examine first the usage of this mean value in the different games. We will speak, in a separate article, of the way to find an approximate mean value, according to the observations.

We will distinguish first two cases; the one where the game is already undertaken, and the one where it is not begun; the one where the players would have to divide a given sum, and the one where one must determine their stake.

In the first case, it is evident that one must divide the sum by giving to each player the mean value of his expectation. Indeed, one is not able, if the probability to win or to lose are unequal, to place him into a state where he has an equal probability to win or to lose equal sums; and one sees moreover that, if one would establish another law, whatever it may be, and if one employed it a great number of times, the mass of players who would have the same lot as this first, would have an always increasing probability, either to have more, or to have lass than they would have had by the lot; instead that by following the ordinary law, the probabilities to have more or to have less, always approach to equality; that besides they have a probability always increasing that this which gives the law to them, will not deviate beyond certain limits of that which the lot would have given them.

If the concern is to determine the stake, one finds likewise that every other law would lead, by playing a very great number of trials, to give to one of the players a very great probability to win or a very great probability to lose; instead that by making the stake equal to the mean value of the events that the game is able to bring forth, the probability to win or to lose will always approach equality for each player, and that each of them will have a probability always increasing to not lose beyond a certain part of the total stake.

But at the same time one sees that it is to that which the greatest equality possible is limited between the lot of the two players, who do not have absolutely equal conditions, between the lot of a player who has decided to play, and the one which he had before the game.

Therefore before being engaged in a game, each must examine if this equality, the greatest possible which is able to exist between the state of a player, before engaging in a game, and the one which will result from the game, suffices in order to determine it to change the state. Now, it is easy to see that, in every case where the probabilities to win or to lose are very different among themselves, it is necessary to suppose a great number of trials in order that this equality had a sensible place. One will determine therefore to play only as much this hypothesis will be judged admissible.

Thus, in the example drawn from the game of *heads and tails*, which we have exposed above, one sees that the one who gives a stake $\frac{n}{2}$ and which has probability $\frac{1}{2^n}$ to win 2^{n-1} at the n^{th} trial, must determine to play, only so much as he will be able to repeat the game often enough to have a near equal probability to win or to lose. Likewise the one who, on the contrary, has to give 2^{n-1} after having received only $\frac{n}{2}$, must play, in spite of the great probability that he has to win, only when he can regard $\frac{n}{2}$, or rather the least sums that he has an expectation founded on winning, as a compensation of the very small risk to lose the much greater sum 2^{n-1}, which which obliges to make n very small. Then the players would be able to determine themselves to play the game; and the paradox would vanish.

We are going to give some other proper examples to show the different applications of this rule.

There exist many kinds of games where one of the players plays all at once against all the others, who besides have a more or less great expectation to win many times their stake. This player is called the banker, and the others punters.

There results from this convention that the banker is exposed to the risk of loss of sums much more considerable than each of the players in particular; in a manner that if one supposes the game equal according to the principles exposed above, he remains exposed to the danger of very considerable loss, and that, in order that this loss not exceed a very small part of the total stake, it would be necessary to continue the game a long time, and to suppose this stake very large. It would be imprudent therefore to make a trade of playing an equal game; it is therefore necessary that the punters, who would find pleasure to play this game, or who were tempted by the expectation of a gross gain, consent to make a sacrifice in favor of the banker; and thus there results from it, 1° that he has a probability always increasing of winning; 2° that a given probability of losing only a given part of the stake, takes place at the end of a much smaller time.

We suppose now that one proposes a lottery of 100000 tickets at twenty sols per ticket; in order to play at an equal game, it is necessary, according to the general rule, that the sum of the lots equal 100000 *l.* Now one will find in placing the tickets, however much one would give the lots for a much less sum. In this case, it is clear first that those who set into this lottery, are not able to have in view this equality that it would be able to have between the probability to win or that to lose, after a great number of drawings, nor this other probability always increasing to lose, after a great number of drawings, only a given part of the total stake. Indeed, in the lotteries which have the most success, in those where the first lots are considerable, it would be necessary to consider a sequence of drawings continued during many centuries in order to approach this equality, and in order that the probability to not lose or to not win beyond a very small part of the total stake, was very great.

The sole motive which determines them to play, is therefore the little importance of the stake which they risk, and the great advantage that they expect, in the case where the lot would favor them. Whence there results that the more the sum that one risks is small, the more the disproportion is able to be great.

If on the contrary one proposed a lottery of 100000 tickets, each of one hundred louis, with the expectation of a single lot of 2400000 *l.* according to the ordinary rule, the game would be equal; however it would be doubtful that a similar lottery was filled it would not be likewise, when the lot would be 3000000 *l.*, although he had then advantage. Indeed, it would be necessary to suppose the game repeated too many times, and consequently the stake too considerable, in order to attain, despite a real advantage, to this probability nearly equal to win or to lose; to this very great probability to lose only a small part of the stake which constitutes the equality; and in the case that one examines here, the loss to which one is exposed in playing this game, is too considerable in order to be regarded as of little importance.

In the circumstances where a trader makes a speculation accompanied by a sensible risk, one sees that he would not permit at all that his profit was such that the mean value of his expectations was equal to his stake, augmented by the interest that a commerce without risk would procure to him. It is necessary yet that he has, by continuing this kind of commerce, a very great probability that he will endure at length no loss.

It would be necessary therefore, in order to submit to the calculus the speculations of this kind, to determine, according to the funds that each merchant is able to have in order to use successively in this perilous commerce, what excess of profit it is necessary that he finds in order to have a sufficient probability of not losing at all the totality of his funds, to lose them only in part, to conserve them only, to conserve them, plus a certain interest; probabilities that it would be necessary to determine also by some particular principles that we will expose in the following.

There remains to us in order to end this article, only to examine one question. We have supposed here that the only kind of equality which is able to exist among some unequal advantages and some unequal probabilities, by making, according to the general rule, these advantages in inverse ratio of the probability, it would be established by the hypothesis of a continuation more or less repeated of the same game, of the same hazard in general. This hypothesis carries no difficulty, if one supposes the different trials linked among them; for example, if one supposes to wager that one will bring forth a rafle with three dice; that stake of the one who wagers, let it be 1, the one of the adversary 35; that finally one agrees to play one hundred thousand trials, one will see that this equality will be established between the players. But if one does not suppose the matches linked, then it presents an objection. We suppose, in fact, that, in the example proposed, the one who wagers that he will bring forth a rafle, has brought it forth two times, he will have won 70. If his adversary wishes to continue the game, as the preceding events do not influence the probabilities of the others, it is necessary to recommence the calculation, and these are no longer the probabilities to lose and to win, these are the probabilities to lose or to not lose above 70, which approaches more and more to be equals. Likewise the player who would have to lose the half of the total stake which he is able to risk, has no longer, at this moment of the game, the probability to lose only a certain part of his total stake, such as he would have at the beginning.

There would result from it therefore that an equal game at the beginning, according to the established rule, to which one would be able to be delivered, even with prudence, will be able to cease to be at a certain period of the game.

In order to remedy this inconvenience, it would be necessary: 1°: to determine, in each circumstance, the total stake which is able to be employed in the game successively.

2°: To divide this game into a certain number of parts, into ten, for example.

3°: To combine the game in a manner to have a certain probability that, in a great number of trials, there will be no moment where the loss exceeds this tenth part.

4°: In the case where it exceeds, then to continue the game only by diminishing proportionally the stake, in order to be able thus, in all cases, to be found in a state, not the same; but similar to the one in which one has commenced the game.

We propose here this division, which is arbitrary, because, in rigor, it would be necessary at each loss, to lower the value of the stake at each trial proportionally to this total stake; but one senses that this method will be incommodious in practice.

In many games, there arrives on the contrary that a player increases continually his stake, in a manner to this that a favorable trial compensates him in all that which he would have been able to lose in the preceding trials; this which is called *to make the martingale*. If one follows the game in this manner, even in playing against a banker who has some advantage, and if the stake or the number of trials is not limited, one arrives to a result of the same kind as the one of the question of Petersburg. But if the game is contained within a certain limit, one finds only that by supposing the parts linked, the player who increases thus his stake, changes the nature of the game, that is to say, that instead of a game where with a mediocre stake he would have a rather small probability to win much, he has on the contrary a very great probability to win little by exposing a large stake.

If one does not play in linked games, there will arrive on the contrary that, each time that one recommences with this condition, the two players are found in the same case as at the beginning of the game, by supposing only that the one who has won, continues to play a small game, and the one who has lost, a greater game than at the beginning; but that one has never a very great probability to win.

It results from these reflections that every time that there is concern of the too great stake in order to regard the loss of it nearly null, it is prudent to play only with a very great probability of not losing at all beyond a limit where one is no longer able to continue the game a rather long time in order to have a probability equal to the first. Whence one sees that, in each case, neither the large game, nor of the considerable enterprises exposed to some risks, are able to be followed with prudence, not only if the lot is equal, according to the general rule, but even when he would have some advantage.

Article VI.

Application of the calculus of probabilities to the questions where the probability is determined.

We have seen above that one would be able to apply the calculus, either to the events of which the probability is fixed and determined, as when these events depend on the throw of dice that one supposes similar, of the drawing of cards or of tickets; or to the events which are susceptible only of a mean probability, as when the real probability being unknown, one seeks to find from it a mean value after the observation of past events of the same kind, that one supposes to have constantly a like probability (*See article III. p. 76*).

We are going to give successively some examples of these two kinds of applications, and we will choose, for that of the first kind, Biribi and the royal Lottery of France.

I. One plays Biribi[1] with 70 numbers, each punter puts a certain sum on the number that he chooses; the banker draws next a number, pays to those who have chosen the drawn number, 63 times their stake, and all those of the punters belong to him.

We suppose therefore that a punter has staked a piece on a given number; since there are 70 numbers, the probability that this number will arrive is $\frac{1}{70}$; that it will not arrive is $\frac{69}{70}$. In the first case, the punter will receive 63 pieces, and having given one of them, will win 62: in the second, he will lose the piece which he has given. His expectation will be therefore (*article V.*) $\frac{1}{70} \times 62 - \frac{69}{70} \times 1$, or $-\frac{7}{70}$, this is $-\frac{1}{10}$. The expectation of the banker will be on the contrary $\frac{69}{70}1 - \frac{1}{70}62 = \frac{1}{10}$. If one supposes that the punters were in any number, they are able, at each trial, to put indifferently on such numbers as they have wished, any sum, but contained in some certain limits, the expectation of the banker will be equal to $\frac{1}{10}$ of the mean value of that which the punters will have been able to risk, and consequently $\frac{1}{620}$ of all the sum that the banker is able to lose.

But it follows equally from the established principles in the preceding article, that the probability for the banker to have a gain which approaches the value of that mean expectation, is able to begin to be great only by supposing that he plays a rather great number of trials. It is necessary therefore, in order to calculate with exactitude the lot of the banker, when he is determined to play this game, to evaluate first the number of trials that he must naturally play, to see next what sum he must possess in order to have a very great probability that he will not be forced to quit the game, that he will not be bankrupted; to regard then that sum as a fund that he exposes in a commerce accompanied with risks, and to seek the probability that he will not lose at all a considerable part of this funds, that he will conserve it complete, that he will win a sum which has one such ratio with that total sum.

If one is able to suppose that the banker has more considerable funds, and if he is proposed to make this business a nearly known number of times, during one year, it is necessary to know his advantage, to see (the funds that he must have made being given also) what probability he will have to not encroach upon his funds, and to draw from them on the contrary a certain rate of interest. One will see by this means what are the true advantages and the risks of this commerce, compared with those of the other kinds of commerce; and one will judge if this advantage of $\frac{1}{10}$ at each trial is above or below the one which it was reasonable to accord to him.

To this first observation it is necessary to join another. In the games of this kind the conduct of the punters has on the advantages of the banker and on their proper risks, an influence which is necessary to evaluate.

We suppose, in fact, that each punter is able to put at will on a number from one piece to thirty-two, and that he chooses the resolution to increase his stake successively, in a manner that, as soon as a number exits to him, there is a profit for him, or at least that he experiences no loss. This combination changes much the state of the game.

It is clear, in fact, 1° that it embraces 254 trials; 2° that the punter, instead of a greater probability of loss, will have on the contrary a probability to win greater than

[1] *Translator's note*: Biribi, also known as cavagnole, is a game of chance similar to Lotto and played for small stakes. It was played on a board on which the numbers 1 to 70 are marked. Players placed their stakes on the numbers on which they wished to wager. The banker drew a case containing a ticket from a bag. Each ticket corresponded to a number on the board.

$\frac{37}{38}$; 3° that the banker, instead of a great probability to win and a small risk to lose much, will have a very small probability to win much, and a great probability to lose little; 4° that the mean value of the gain of the banker or of the loss of the punter will be, not the tenth part of that which the punter has been able to risk, but a much more feeble sum, while on the contrary, instead of being the $\frac{1}{620}$ part of that which the banker has risked, it will be much higher; but also the stake of the punter will be much greater, and the sum risked by the banker much smaller.

One sees here that it is necessary to fix a limit to the stakes. In fact, in the example that we choose here, and where it is from 1 to 32, the probability to win that the banker has, is only $\frac{1}{38}$, for each of these combinations of trials, that one is able to regard as one alone. If one had supposed a latitude greater in the stakes, it would have been much smaller. For example, for a latitude from 1 to 127, the banker would have more than a probability to win of around $\frac{1}{133}$; it is true that he would not lose beyond 62, and that the punter, at the end of 340 trials, which contains this combination, would have been able to lose to 7989 times his stake. On another side the gain of the banker is able to arrive only after 340 trials.

But all the punters do not follow this method; on the contrary, the greater part play at random, nearly always after some superstitious ideas which they attach to certain numbers. Some diminish their game, when they lose, because they believe in bad luck; others increase it in order to recoup earlier.

It is necessary therefore, in calculating the advantage of the banker, to have regard to the conduct of the players, and to see if, even in supposing that they follow the particular combination that we just exposed, the banker will have sufficient advantage. Now, it is easy to sense that that is not able to happen, at least that the limits of the stakes are such only the banker has, in the space of some sessions, a rather great probability to win, even against those who follow this manner of playing.

This example, taken from the simplest game among the games of this kind, where the banker has an advantage, proves that it suffices not at all, in order to calculate it, to take the mean value of the expectation, such as an abstract consideration of the game would be able to give.

We terminate this first example with the discussion of two questions.

1° Does a punter find a real advantage to choose the combination which we have exposed and which procures to him a great probability to win?

I believe that one must respond negatively. In fact, he acquires this probability only for the gain of a small sum, and he risks to lose a very considerable sum in a game disadvantageous to himself. Also the punters must consider the game only as a diversion which they pay; and the manner to calculate their state would consist in examining if, without being exposed to make a ruinous loss, they have in the games of this kind, a sufficient probability to not pay too dear the pleasure that the game procures to them. The only prudent conduct for them is to fix, according to their fortune, a sum above which they themselves would determine to never lose in a session and to reserve themselves, when chance favors them, a part of the gain which they have made.

By this means, they will have a mean value of their expectation, always negative, I admit, but much below the tenth part of their total stake, and would be able to conserve, for a long space of time, a probability to win greater than that to lose, as they have in biribi, from the 48^{th} trial to the 63^{rd}.

One is able to demand 2° if, in the case where one makes the combination which we have exposed, that which is called *make the martingale*, must one continue to put on the same number, until it exits, or to put at random at each trial, on any number whatsoever, even on the one which just exited? The calculus, contrary to the former in common opinion and even to natural sentiment, responds that nothing is more indifferent. Whatever be in fact the preceding events, the calculus will give always here $\frac{1}{70}$ for the probability of each number. The common opinion appears based only on natural sentiment, and this sentiment has only a single cause, it is that one is more struck to see in biribi, for example, the number 3 to exit twice in sequence, than to see the numbers 2 and 10 to exit one after the other. One of these events strikes us as remarkable; the other is not perceived. Now, these events as one remarks, are less numerous than the others; one is therefore carried to regard them as extraordinary. One does not make a reflection that, if, in $70 \cdot 70$ or 4900 possible combinations of two trials of biribi, there are only 70 combinations which give the same number repeated twice and 4830 which give two different numbers, the combination 3 and 3 is unique among the 70 combinations of the first class, and the combination 3 and 10 among the 4830 of the second, so that, if the probabilities to have a combination of the second class, or one of the first, are $\frac{4830}{4900}$ $\frac{70}{4900}$ or $\frac{69}{70}$ and $\frac{1}{70}$; that to have next a determined event of the second, being $\frac{1}{4830}$ and a determined event of the first $\frac{1}{70}$, the real probability of a determined event will be therefore, for one of the classes, $\frac{4830}{4900}$ $\frac{1}{4830}$ or $\frac{1}{4900}$ and for the other, $\frac{1}{70} \cdot \frac{1}{70} = \frac{1}{4900}$: these two probabilities are therefore equals.

If this manner to play gives a probability to win at the end of a certain number of trials, it is not that after having lost 253 times, for example, one has a probability greater than $\frac{1}{70}$ to win the 254; it remains always the same at each trial, and the probability $\frac{37}{38}$ to not lose, has place only for the one who is determined to follow this combination at the beginning of the game, and after having knowledge of the events which precede this last trial; because from the moment where the events are known, for example, if he has lost 253 trials in sequence, he has for the 254^{th} a probability $\frac{1}{70}$ to win 18 times the first stake, and a probability $\frac{69}{70}$ to lose 1998 times this same stake.

But it will be useless to give more expanse to this example. We have said enough on it for those who, without having the rage of the game, would be able to be tempted to be delivered on the faith of some false combination; and that which we would add would not cure those which are animated with this passion. The only way to play which is not able to be imprudent, consists in never exposing oneself to some considerable risks, and consequently one would be wrong to hope that they were able to adopt it.

The royal Lottery of France is formed from 90 numbers; there exit five of them at each drawing. One is able to set there in 7 different ways.

1° On a single number by extract; if it exits, one has five times his stake.

2° On a single number by extract determined, that is to say, by determining that it will exit first, the second of the drawing; and if it exits at the indicated rank, one has seventy times his stake.

3° By ambe, respecting two numbers; if the two chosen numbers are among the number of five exited, one receives two hundred seventy times his stake.

4° By ambe determined; it is necessary that the two numbers exit from the drawing at the indicated rank, one the first, and the other the third, but without reciprocity: one obtains 5100 times his stake.

5° By terne, respecting three numbers; it is necessary that all three exit in the drawing: one obtains 5500 times his stake.

6° By quaterne, respecting four numbers; all four must exit in the drawing, and one obtains 75000 times his stake.

Finally, by quine, respecting five numbers, and one obtains 1000000 times his stake.

This put, it is necessary, for each manner of setting in the lottery, to evaluate the expectation of the player or that of the banker.

1° By extrait. Since there are 90 numbers, and since there exit 5 of them, the probability that a given number will exit, is $\frac{5}{90}$, since 90 is here the total number of numbers, and 5 the one of the numbers which are able to exit. This probability is therefore $\frac{1}{18}$; that which it will not exit is consequently $\frac{17}{18}$: but if the ticket exits, the player wins 14 times his stake, since one gives to him 15; if the ticket does not exit, he loses his stake. His expectation is therefore $\frac{1}{18}\,14 - \frac{17}{18}\,1$ or $-\frac{3}{18} = -\frac{1}{6}$. The lot of the player is therefore equal to the loss of the sixth of his stake, the one of the banker to the gain of a sixth of the stake of the player.

2° By determined extrait. Since here the rank where a ticket must exit is determined, there is only one combination to exit; the probability in his favor is therefore $\frac{1}{90}$, the probability against $\frac{89}{90}$; and as he obtains 69 times his stake, when the ticket exits, his chance will be expressed by $\frac{1}{90}\,69 - \frac{89}{90}\,1 = -\frac{20}{90}$ or $-\frac{2}{9}$ of his stake.

3° By ambe. The number of combinations two-by-two of 90 numbers is expressed by $\frac{90.89}{1.2} = 4005$. The one of the combinations two-by-two of the the five exiting numbers is $\frac{5.4}{1.2} = 10$. The probability of the exit of an ambe will be therefore $\frac{10}{4005}$, that which it will not exit at all $\frac{3995}{4005}$. The player will win, in the first case, 269 times his stake; his lot will be therefore expressed by $\frac{2690}{4005} - \frac{3995}{4005}$ or $-\frac{1305}{4005} = -\frac{29}{89}$ of this stake.

4° By determined ambe. Here the total number of combinations two-by-two is double, since each of the two, placed the first or the second, form a particular combination. The one of the determined ambes of 90 numbers will be $90.89 = 8010$; and since the rank of the drawing where each number must exit is determined, there is only one of these 8010 combinations which give the chosen ambe, we will have therefore the probability that it will exit equal to $\frac{1}{8010}$, that it will not exit equal to $\frac{8009}{8010}$; and as one will receive then 5100 times his stake, the lot of the player will be expressed by $\frac{1}{8010}\,5099 - \frac{8009}{8010} = -\frac{291}{801}$.

5° By tern. The number of combinations three-by-three of 90 numbers is expressed by $\frac{90.89.88}{1.2.3} = 117480$, the one of the combinations three-by-three of 5 numerals is $\frac{5.4.3}{1.2.3} = 10$. The probability that a given tern will exit, is therefore $\frac{1}{11748}$, and that it will not exit $\frac{11747}{11748}$; one gives for a tern 5500 times his stake. The lot of the player will be therefore:

$$\frac{5499}{11748} - \frac{11747}{11748} = -\frac{6248}{11748} = -\frac{142}{267}$$

6° For the quaternes. The number of the combinations four-by-four of 90 numbers is $\frac{90.98.88.87}{1.2.3.4} = 2555190$. The one of the combinations four-by-four of five numbers $\frac{5.4.3.2}{1.2.3.4} = 5$; the probability that the quaterne will exit, will be therefore $\frac{1}{511038}$; that it will not exit $\frac{511037}{511038}$; and as one gives 75000 times the stake, the lot of the player will be:

$$\frac{74999}{511038} - \frac{511037}{511038} = -\frac{436038}{511038} = -\frac{218019}{255519}.$$

7° Finally for quine. The number of combinations five-by-five of 90 numbers is $\frac{90.89.88.87.86}{1.2.3.4.5} = 43949268$. The number of combinations of five numbers five-by-five is unity. The probability that the quine will exit is therefore $\frac{1}{43949268}$, that it will not exit $\frac{43949267}{43949268}$. The lot of the player will be therefore (since the quine is payed one million times the stake) equal to

$$\frac{999999}{43949268} - \frac{43949267}{43949268} = -\frac{42949268}{43949268},$$

a quantity greater than $\frac{42}{43}$.

In order to give an idea of these last probabilities to win to those who are not accustomed to this kind of calculation, we will observe that the expectation to win a quaterne is less than the risk that a man of fifty years has to die of apoplexy in an hour.

That to win a quine, is less than the risk to see two persons of fifty years; out of eight, to be struck by apoplexy during the day; or, if one wishes, the risk that two persons of that age, are both struck by apoplexy in three days.

The one who would take a quaterne at each drawing, would have an equal probability for the exit or the non-exit of his ticket after only nearly 376288 drawings or more than 15678 years.

The one who would take a quine would have an equal expectation of exit or non-exit only after 30103000 or 1254292 or more than one million two hundred fifty thousand years.

If one supposes the lottery created there are six thousand years at the same time as the world, there would be yet odds more than 99668 against 362, or 300 against 1 that a quine would not yet exit.

If one observes finally that the more the probability of a fortunate event diminishes, the more the profit that one is able to withdraw from an increased stake, but much less as the probability diminishes, and so much less as that probability is more feeble; it is easy to see that there is no possible combination which gives to a player an expectation based on winning; that thus one must risk only the sums on which one is able to regard the loss as nearly null, and of which, by making another use of it, one is able to expect neither much utility nor much pleasure.

For example, a man who, playing with an honest fortune, finds some pleasure to be entertained by chimerical expectations, and who has the sagacity to not surrender too much, would very well be able to sacrifice an écu to quaternes or to quines at each drawing. It would be a way to have always at least the possibility to make a considerable gain, to be amused by projects in the air, without these projects derange his fortune.

If now one seeks the lot of the banker, one will find that his mean expectation is to win:

1°. $\frac{1}{6}$ or $\frac{166666}{1000000}$ of all that which is staked by extrait.

2°. $\frac{2}{9}$ or $\frac{222222}{1000000}$ of all that which is staked by extrait determined.

3°. $\frac{29}{89}$ or $\frac{325842}{1000000}$ of all that which is staked by ambe.

4°. $\frac{291}{801}$ or $\frac{363259}{1000000}$ of all that which is staked by ambe determined.
5°. $\frac{142}{267}$ or $\frac{422416}{1000000}$ of all that which is staked by tern.
6°. $\frac{218019}{255519}$ or $\frac{853240}{1000000}$ of all that which is staked by quaterne.
7°. $\frac{42949268}{43949268}$ or $\frac{977248}{1000000}$ of all that which is staked by quine.

In view of the great number of players, one must have a very great probability that after a small number of drawings the profit will not deviate much from this proportion, especially for the first five ways to set into this lottery, which are also those where the stakes are much the highest.

One has need in the lotteries, 1° of being reserved a greater advantage on the less probable combinations, in order to not expose the banker to be ruined. If one would be content of an advantage of $\frac{1}{2}$, the quine of 2 liv. 10 s. would be paid more than fifty millions.

2° To fix to the stakes on each number or on each combination, a value contained between certain limits, by that same reason, and next to have a probability of the greatest to obtain at length a profit which approaches the mean value of the advantage. In fact, one would be certain of it, if the sums were exactly distributed on all the combinations; and the more one will tighten the stakes between some narrow limits the more one will be sure to approach this equal distribution.

3° In order to be assured yet of this more equal distribution, one finds in many lotteries, and particularly in that of France, combinations all imprinted, and that the players purchase, instead leaving to them the trouble to form these combinations themselves. Then one takes care to arrange them in a manner that nearly all the numbers are equally distributed.

4° As the limits of the stakes out of the different kinds of combinations are able to be fixed, since one repeats them in many offices, there arrives many times to form certain numbers on which one has place to believe that one would carry the higher sums. There results from this violation of the first convention no real harm for the players, but only a greater probability of a less unequal repartition of the chances on the numbers.

One would be able to apply here, in some regards, that which we have said on biribi, and to find, instead of a means to have a great expectation to win a small sum, or rather to not lose, by exposing oneself to a small risk, to lose much, to have all at once, but with the same condition, a very great expectation to not lose, and a very weak expectation to make a considerable gain.

We suppose, in fact, that 10 sols represent here unity, that 3 liv. or six times ten sols, consequently, represent a fixed sum that one places at each drawing on some quaternes and some quines; in order to have this feeble expectation of the possibility of a great gain and, and that one disposes his stakes on five numbers at each drawing, in a manner that the exit of a single number makes the totality of the stake return to us.

If a expresses the number of units or times 10 sols that it is necessary to put onto a number at a drawing n, r the profit of the player, if there were exited a number at this drawing, and b the sum that it is necessary to put onto each number at the $\overline{n+1}$ drawing, in order to withdraw the totality of the stake, one will have $b = \frac{5a+6-r}{10}$. If, in order to avoid the fractions, instead of taking b exactly, one takes b' or the entire number greater than b which differs from it the least, one will have for remainder, after the $\overline{n+1}$ drawing, or for the profit of the player, if there exits a number at this drawing, $r' = 10b' - 15a - 6 + r$, and the total stake, after the $\overline{n+1}^{\text{th}}$ trial, will be $5b' + 15a + 6 - r$.

This put, if one makes $a = 1$ at the first trial, one will have for the drawings, the stakes and the remainders:

Trials	...	Stakes	...	Rest
1	......	1	......	4
2	......	2	......	3
3	......	4	......	7
4	......	6	......	1
5	......	10	......	5
6	......	16	......	9
7	......	24	......	3
8	......	37	......	7
9	......	56	......	6
10	......	84	......	0
11	......	127	......	4
12	......	191	......	3
13	......	287	......	2
14	......	431	......	1
15	......	648	......	0
16	......	973	......	4

The column of the remainders mark in multiples of 10 sols that which the player would win, if there would exit a single ticket at this drawing; that of the stakes, that which it is necessary to put for each drawing on a ticket.

The total stake would be here 14581, or 7290 *l.* 10 sols.

Now since $\frac{90.89.88.87.86}{1.2.3.4.5}$ expresses all the combinations five-by-five of the 90 numbers, and since, by the same reason, $\frac{85.84.83.82.81}{1.2.3.4.5}$ expresses all the combinations of the 85 numbers that the player has not taken, the second of these numbers divided by the first, will express the probability that none of these five numbers will exit, and it will be $\frac{85.84.83.82.81}{90.89.88.87.86}$ or $\frac{746347}{1000000}$. This same fraction raised to the sixteenth power or $\frac{129}{10000}$ will express the probability that there will exit none of the five tickets in 16 drawings. There will be therefore odds of 9871 against 129, or odds of more than 76 against 1, that there will exit at least one; and the player will have this probability to not lose his stake and to win on the contrary at least one of the small sums which express the rest.

We have supposed that the player placed at each drawing a sum of 6 or of 3 liv. on quines or quaternes, this which offered to him the very small probability or, to say better, the possibility to win a considerable sum.

But there is another advantage, the one of the exit of two, three, four, five numbers in a single drawing, a case in which he wins moreover one, two , three or four times his total stake, plus the rest. We examine only here the probability that he wins it at least one time, that is, that there exits at least two of his five tickets in a drawing. The probability of the other events must be set in the number of extraordinary chances.

In order to determine this probability, we will consider the following formula, of which the successive terms give the probability that the total number of numbers being

p, the one of the numbers which exit on a drawing being q, and s the one of the numbers which a player has retained, there will exit none of these tickets, there will exit only one, there will exit only two, three, etc. We place here this formula, because it is practical for them who wish to amuse themselves by calculating the probabilities of this kind.

In that formula a term $\left(\frac{a}{b}\right)$ expresses in general the coefficient of the $\overline{b+1}^{\text{st}}$ term of the formula of the binomial for the power a. That formula will be therefore:

$$\frac{\left(\frac{p-s}{q}\right)}{\left(\frac{p}{q}\right)} + \frac{\left(\frac{s}{1}\right)\left(\frac{p-s}{q-1}\right)}{\left(\frac{p}{q}\right)} + \frac{\left(\frac{s}{2}\right)\left(\frac{p-s}{q-2}\right)}{\left(\frac{p}{q}\right)}$$

$$+ \frac{\left(\frac{s}{s-1}\right)\left(\frac{p-s}{q-s+1}\right)}{\left(\frac{p}{q}\right)} + \frac{\left(\frac{p-s}{q-s}\right)}{\left(\frac{p}{q}\right)}$$

In the case that we consider here $p = 90$, $q = 5$, $s = 5$, and the first two terms which represent the probability that there will exit less than two chosen numbers, will be:

$$\frac{\frac{85.84.83.82.81}{1.2.3.4.5} + 5 \cdot \frac{85.84.83.82}{1.2.3.4}}{\frac{90.89.88.87.86}{1.2.3.4.5}}$$

or $\frac{976701}{1000000}$, and for 16 drawings, the sixteenth power of this number, or $\frac{670873}{1000000}$. There will be therefore odds of 670173 against 329827, or nearly 67 against 33, or a little more that 2 against 1, that in the 16 drawings linked together, one will not obtain the advantage by doubling or less his total stake, and at each drawing one will have a probability $\frac{253653}{1000000}$ to withdraw his stake, and a probability $\frac{23299}{1000000}$ only by doubling it; one being a little more than a fourth, and the other less than a fortieth.

Thus, in each linked game of 16 drawings, the probability to lose the total stake, that of withdrawing the advanced stake, and that of the double would be able to be expressed by:

$$\frac{129}{10000} \qquad \frac{6573}{10000} \qquad \frac{3298}{10000} \qquad \frac{9871}{10000}$$

a sum of the two last; but this probability is only that of double the advanced stake, and not the value of the total stake that one risks in this combination. Thus there would arrive, if one was proposed to continue this manner of play, that one exposed oneself to lose this total stake after a certain number of trials, and that he would have only a very small probability to have been compensated of this loss by the preceding.

The player would have a probability $\frac{1}{2}$ of loss in 51 successive combinations of the same nature. The mean value of the number of drawings which correspond, is around 105, this which demands four years four months and one-half. Thus the one who would play this game, would have an expectation $\frac{1}{2}$ to be able to continue this game during a mean duration of four years four months and one-half without losing his stake.

But we are able to make here the same reflection as on the game of biribi; it is that if one is limited to that which is not an extraordinary event, to that which does not require thousands of years in order to arrive to a probable expectation, one would have, in playing this game, the inconvenience to expose high stakes with the expectation of a very small profit: one would be able therefore yet to regard it only as a manner to amuse oneself, where the loss is less probable, but where the sums exposed are also much higher than if one was limited to risk only a very small stake at each drawing.

It would agree therefore yet less to a reasonable man. That which we have said, in speaking of the game of biribi, on the illusion which makes to prefer some numbers to others, those which are not at all to exit for a long time to those which come to exit, is able to applied equally to lotteries. There does not exist, there is not able to exist any real way to play with advantage in any of those games which are disadvantageous by themselves; and, if one excepts from them, for the ones, the pleasure to play, for the others the one of entertaining in the illusory expectation of a very slightly probable fortune, there exists no motive to be delivered; and consequently a wise man is able to risk only some not only very small sums for his fortune, but also too small in order that another use of these sums is able to procure to him a real advantage.

These are uniquely the avidity to win carried to a point that leaves only the quiet usage of reason, the common prejudice on the more probable exit of the numbers which are not exited since a long time, or finally of the superstitious ideas much more common than they must be in an enlightened century; they are these motives alone which engage to deliver themselves, with a kind of furor, to those curious games. The most certain secret in order to feel aversion would be to spread as much as it is possible the knowledge of the calculus of probabilities. There is no point at all moreover proper to destroy the speculative or practical errors which arrest the progress and are opposed to the happiness of human kind, and one must neglect in no way to render finally these popular notions.

ARTICLE VI

Concerning the way to establish terms of comparison between the different risks to which one is able to be delivered with prudence, in the hope to obtain advantages from a given value.

The results of the calculus of probabilities being able to serve to regulate on our conduct in the speculations which interest our life and our fortune, it is of greatest importance to compare them to some objects which make an appreciable impression on us, and on which we are able to set our judgment. In fact, after having determined the mathematical probability that an event which interests us will arrive or not arrive, it is necessary to have some means to understand if it is sufficient to determine for us to incur the risks to which we will be exposed in these enterprises. One is able to arrive to his end only by comparing these risks with those that we experience every day, and to which we attach an importance more or less great, but determined relatively to our conduct.

If we ourselves stop first at some speculations of commerce, one will see that the most natural manner to evaluate the risks to which a prudent man is able to be exposed in order to have the expectation of a certain profit, is to compare them to those that one neglects every day or to which one attaches little importance. This method supposes that one has observed with care the rare and unexpected accidents which are the object of these small risks. It is necessary next to take care that in speaking of the risk to incur in some speculations of commerce, we do not understand the absolute risk which is able sometimes to be rather great in an isolated circumstance, but the one which results really from the set of enterprises of the same kind, formed by the same merchant. This consideration is so much more necessary as it is easy to see that when the probability of an event is below $\frac{1}{2}$, the risk which results from it diminishes in measure as one embraces more events, and one is able to arrive in this case to a very great probability to not lose beyond a given sum.

It is therefore in order to appreciate this last that one must seek a term of comparison capable to make a known impression on us.

But in the absense of these rare and unexpected events, of which the observation has not been followed to the present, to the point to be able to construct some tables, those of mortality are able to fulfill rather commodiously the object that we ourselves propose. The diverse risks of which they give the value are rather rare in order to be susceptible of a general application, and striking besides nearly in the same manner the common of men. The risks to die in an interval of time given relatively at each age, and the differences of these risks among them, furnish some degrees of probability that one will be able to employ in some determinations more or less important.

We come to an application; let be taken in any of some tables of mortality, those of Deparcieux, for example, the risk of dying during the year, for a young man of twenty years, one will find that out of 814 persons of that age, there die of them 8 in the year, and consequently that the sought risk is $\frac{8}{814}$, or around $\frac{1}{102}$; for the age of 50 years, this same risk is $\frac{10}{581}$, or $\frac{1}{58}$ nearly. The difference of these two fractions, $\frac{44}{5616}$ or $\frac{1}{134}$, will be the increase of risk, relative to the increase of age. By dividing it by 52, one will have that of the danger to die in a month, under the same circumstances, $\frac{1}{6968}$. Now, a man of 50 years, in good health, and of a chosen class, as was that of members of a tontine of France, on which Déparcieux has calculated his tables, fear scarcely more to die within the month than a young man of 20 years.

By taking for second term of comparison an individual of age of 65 years, the danger to die within the year is then $\frac{15}{395}$ or $\frac{3}{79}$. The difference between this value and that which corresponds to the age of 20 years, is $\frac{227}{8058}$, nearly $\frac{1}{35}$; that of the risks to die within the month in each of these two ages, will be $\frac{1}{1820}$. This quantity is rather small in order to serve as base to some speculations of commerce, especially when one is able to continue it a rather long time in order to obtain a much greater probability, not only to not encroach upon his funds, but still to be found in gain by the rest.

If one has had regard, in the construction of the tables of mortality, to the physical constitution of the individuals, to their sex, and in general to all the distinctions which are able to carry with them some differences in the order of mortality, one would see likewise to chose some points of comparison more exact and more varied. Thus, if one had a table apart from the individuals of each age, in good health, and of those who out of this number are dead of instantaneous maladies, it would be easy to deduce from it the risk of this kind of death for an age and an interval of given time; a risk which is nearly always regarded as null, or of little importance, and thence very proper to serve as term of comparison for those in which one is exposed voluntarily in the hope of an gain.

It appears that the deaths caused by some acute maladies and of short duration, compose nearly the tenth part of the total number of deaths; the risk to die of a malady of this kind in the course of the year, will be for a man of age of 20 years $\frac{1}{1020}$, and

$\frac{1}{53040}$ only for a month.

It is from the cases, those where the life of a citizen is compromised, which demands that one know the *maximum* of the risk that one is able to neglect, and that determination is indispensable in order to regulate the formation of the tribunals which must judge definitely to death. The probability for the accused to not be condemned to the death unjustly must be such that a man who would have this same probability to not perish, would be believed exposed to no danger.

It would be necessary to employ to that determination only rare accidents, of very slight risks and to which one is exposed voluntarily for the smallest interest, such that the one to perish with the packet boat in the course a voyage from Calais to Dover, and reciprocally, and departing in a time regarded as good and sure, the one to perish in going from Europe to America on a vessel well equipt, which has put to sail in a favorable season, or in others of this kind. Unhappily one has none given on these objects; but one is able to replace them by some combinations of the elements, drawn from tables of mortality, as one has seen previously. It is necessary then to bring together the epochs of the life that one wishes to compare among them, so that it does not have near sensible difference in the impression that the danger of death makes in an interval of very short time, in one or in the other of these epochs. We will have occasion to return to these objects, in speaking of the application of the calculus of the probabilities to the decisions rendered in the plurality of votes.

ARTICLE VII

On the application of the calculus of probabilities to games of chance.

That which has been said in the preceding articles contains the general principals of the calculus of probabilities, and the application of diverse questions which will be able to present themselves will have no other difficulties than those which will be able to be born of the research of the number of equally possible combinations in the proposed cases.

There does not enter into our plan to give some details on the diverse games of chance, and we must be expected to find here only some examples proper to show the application of the principles proposed previously.

1°. *Someone having a number m of tokens in one of his hands, and set a part into the other, and proposes to divine if this last contains in it an even number, or odd; one demands the probability of each of these events.*

It is evident that the even numbers will be the results of all the combinations 2 by 2, 4 by 4, 6 by 6, etc. of tokens, and that the odd numbers will be able to come only from the combinations 1 by 1, 3 by 3, 5 by 5, etc. All these numbers are expressed by each of the coefficients of the power m, of the binomial $(a+b)$, the first of those of a odd rank, departing from the first term, and the others of those of even rank: there results thence that one will have for the number of events of the first kind

$$\frac{m.m-1}{1.2}+\frac{m.m-1.m-2.m-3}{1.2.3.4}$$
$$+\frac{m.m-1.m-2.m-3.m-4.m-5}{1.2.3.4.5.6}\text{ etc.}$$

and for those of the second

$$\frac{m}{1}+\frac{m.m-1.m-2}{1.2.3}+\frac{m.m-1.m-2.m-3.m-4}{1.2.3.4.5}$$
$$+\text{ etc.}$$

Let A be the first sum, and B the second; and, adding them together, one will have

$$A+B=\frac{m}{1}+\frac{m.m-1}{1.2}+\frac{m.m-1.m-2}{1.2.3}$$
$$+\frac{m.m-1.m-2.m-3}{1.2.3.4}+\text{ etc.}$$

an expression which is nothing other than the development of $(1+1)^m-1$, and consequently equal to 2^m-1. Now, if one subtracts B from A, one will have

$$A-B=-\frac{m}{1}+\frac{m.m-1}{1.2}-\frac{m.m-1.m-2}{1.2.3}$$
$$+\frac{m.m-1.m-2.m-3}{1.2.3.4}-\text{ etc.}$$

this which is the development of $(1-1)^m-1$, and consequently equal to -1: one has therefore, in order to resolve the proposed question, the two equations that are here

$$A+B=2^m-1$$
$$A-B=-1$$

whence one draws $A=2^{(m-1)}-1$ and $B=2^{m-1}$. The total number of combinations, or $A+B$, being 2^m-1, the probability of the even number will be $\frac{2^{m-1}}{2^m-1}$, and that of the odd $\frac{2^{m-1}}{2^m-1}$, greater than the first. There is therefore greater odds for the second event than for the first; and, in the case of a similar game, the stake of the one who would hold for even, must be, to that of his adversary, in the ratio of 2^{m-1} to $2^{m-1}-1$. If $m=4$, one will have, for the respective probabilities of even and of odd, $\frac{7}{15}$ and $\frac{8}{15}$.[2]

2°. *One demands the lot of two players of whom one wagers to bring forth a given point,* $p+1$*, with a number* n *of dice.*

This problem has already been treated by Mr. Montmort; but, as he has given of it only some arithmetic solutions, we will place here a formula of Mr. Moivre, which merits to be known.

If one supposes that each of the dice have one face marked with limit, a number k marked with 2 units, a number k^2 marked with 3 units, etc., and thus consecutively to K^{f-1}, f designating the number of points marked on the faces which have the most of them, the sum of all the possible chances, with a single die, will be expressed by the following geometric progression:

$$1+k+k^2+k^3+k^4+k^5+k^{f-1};$$

[2]The solution of this question is drawn from a Memoir presented to the Académie by Mr. Bertrand, professor at Geneva.

Translator's note: According to Lacroix, *Traité élémentaire du calcul des probabilités* the memoir referenced was presented to the Academie of Sciences of Paris in 1786, but was not printed.

and if one raises it to the power n, each of these terms, from unity, will mark the number of ways in which one is able to obtain the points n, $n+1$, $n+2$, $n+3 \ldots n+h$, with a number n of dice. The concern is then no longer, in order to arrive to the solution of the problem, but to know the term in which one has $n+h=p+1$; now it is easy to see that this term must have before it $p+1-n$ terms, because the power the least raised ok k, which will be found in the preceding formula, will be the power n. This put, the geometric progression $1+k+k^2+k^3+$ etc. $\ldots K^f$, has for sum

$$\frac{1-K^f}{1-K} = (1-K)^{-1} \times (1-K)^f;$$

therefore

$$\begin{aligned} &\{1+K+K^2+K^3+ \text{ etc.} \ldots K^f\}^n \\ =&(1-K)^{-n} \times (1-K^f)^n; \end{aligned}$$

but

$$\begin{aligned} (1-K)^{-n} = 1 + nK + \frac{n.n+1}{1.2}K^2 \\ + \frac{n.n+1.n+2}{1.2.3}K^3 + \text{ etc.} \end{aligned}$$

and

$$\begin{aligned} (1-K^f)^n = 1 - nK^f + \frac{n.n-1}{1.2}K^{2f} \\ - \frac{n.n-1.n-2}{1.2.3}K^{3f} + \text{etc.} \end{aligned}$$

If one multiplies these two series by one another, the terms of the product which will have for exponent $p+1-n$ will belong to the sought quantity. We make, for brevity, $p+1-n=l$; let EK^f be the term of the first series, where the exponent is l; and we represent by DK^{l-f}, CK^{l-2f}, and thus consecutively, those which precede it in the progression $-f$, $-2f$, $-3f$, $-4f$, etc., it is clear that, if one writes below this quantity $EK^l + DK^{l-f} + CK^{l-2f}$, etc. the quantity $1 - nK^f + \frac{n.n-1}{1.2}K^{2f}$, etc., and that one multiplies the terms which correspond, one will have all the terms of the product of the two original series, in which the exponent is l, and they will be...
$EK^l - nDK^l + \frac{n.n-1}{1.2}CK^l$, etc.

But one sees, by the formula of the binomial, that the coefficient E is

$$\frac{n.n+1.n+2.n+3\ldots n+l-1}{1.2.3.4.5\ldots -l},$$

and is able to change into this other

$$\frac{n+l-1.n+l-2\ldots l+1}{1.2.3.4\ldots n-1}.$$

By writing the factors of the numerator in an inverse order, and by erasing, at the same time in the denominator, all those which are contained between n and l inclusively, as being common to the two terms of the fraction, one will have thus

$$EK^l = \frac{n+l-1.n+l-2\ldots l+1K^l}{1.2.3.4\ldots n-1};$$

but, because of $p+1-n=l$, one has $p=n+l-1$, and consequently

$$EK^l = \left\{\frac{p.p-1.p-2.p-3.\text{ etc.}\ldots p+2-n}{1.2.3.4\ldots n-1}\right\}K^l.$$

One will find likewise that

$$-nDK^l = -\left\{\frac{p-f}{1}\cdot\frac{p-f-1}{2}\cdot\frac{p-f-2}{3}\text{ etc.}\ldots\frac{p-f+2}{n-1}\right\}nK^l$$

$$\frac{n.n-1}{1.2}cK^l = \left\{\frac{p-2f}{1}\cdot\frac{p-f-1}{2}\cdot\frac{p-f-2}{3}\text{ etc.}\ldots\frac{p-2f+2-n}{n-1}\right\}\cdot\frac{n.n-1}{1.2}l$$

and thus consecutively. If therefore one makes $K=1, p-f=q, q-f=r, r-f=s$, one will have for the number of ways in which one is able to bring forth the point $p+1$, with n of dice,

$$\begin{aligned}
&+\frac{p}{1}\cdot\frac{p-1}{2}\cdot\frac{p-2}{3}\cdot\text{ etc.}\ldots\frac{p+2-n}{n-1}\\
&-\left\{\frac{q}{1}\cdot\frac{q-1}{2}\cdot\frac{q-2}{3}\cdot\text{ etc.}\ldots\frac{q+2-n}{n-1}\right\}\frac{n}{1}\\
&-\left\{\frac{r}{1}\cdot\frac{r-1}{2}\cdot\frac{r-2}{3}\cdot\text{ etc.}\ldots\frac{r+2-n}{n-1}\right\}\frac{n}{1}\cdot\frac{n-1}{2}\\
&-\left\{\frac{S}{1}\cdot\frac{S-1}{2}\cdot\frac{S-2}{3}\cdot\text{ etc.}\ldots\frac{S+2-n}{n-1}\right\}\frac{n}{1}\cdot\frac{n-1}{2}\frac{n-2}{3}
\end{aligned}$$

We take for example the number of ways to bring forth the point 8 with 4 dice. The formula here joined becomes, in this case, $7\cdot\frac{6}{2}\cdot\frac{5}{3}=35$. There is no need to avert that it is necessary to stop when one encounters some null or negative factors.

In calculating for three dice the number of ways to bring forth the points 11, 12, 13, 14, 15, 16, 17, 18, one will find their sum 108, that is to say, the half of the total number of possible combinations with three dice, which is 216; whence one sees that to pass 10 with this number of dice, or to not make it, are two events of equal probability.

The evaluation of the number of possible trials in the greater part of card games, demands that one is able to know the number of ways to partition into many equal or unequal parts the quantity of cards which compose the game. By taking for example piquet, one is able to propose fo find in how many ways the 32 cards which compose this game are able to be partitioned between two players and the talon.

Likewise as the coefficients of the terms of the power m of the binomial make known the number of different combinations that one is able to make of m, things taken in given number, or the number of ways in which one is able to partition m, things into two parts, one of them containing n and the other $m-n$ of them; likewise also the coefficients of the trinomial, of the quadrinomial, or in general of the polynomial, raised to a power m, are the expressions of the number of ways in which one is able to distribute m, things in three, four, or a greater number of parts, composed of n, p, q, r, of these things, but of which the sum is equal to the total number of quantities, or to the exponent of the power of the polynomial.

In fact, let there be the trinomial $(x+x'+x'')^m$, we make $x+x'=p$, and one will have the binomial $(p+x'')^m$, which has for development

$$p^m + \frac{m}{1}p^{m-1}x'' + \frac{m.m-1}{1.2}x''^2 \text{ etc.}\dots + \frac{m.m-1.m-2\dots m-n+1}{1.2.3.4\dots n}p^{m-1}x''^n$$

But if one develops the factor p^{m-n} of the general term of this formula, which is itself a binomial $(x+x')^{m-n}$, one will have

$$p^{m-n} = x^{m-n} + \frac{m-n-1}{1}x^{m-n-1}x'\dots + \frac{(m-n)(m-1)}{1.2}x^{m-n-2}x''^2\dots + \frac{(m-n)(m-n-1)\dots m-n-q+1}{1.2.3\dots q}x^{m-n-q}x''^q$$

One sees therefore that each term of the first series will furnish, by the substitution of the second, a series of terms, and the result will have for general term

$$\frac{m.m-1.m-2\dots m-n+1}{1.2.3\dots n} \cdot \frac{(m-n)(m-n-1)\dots(m-n-q+1)}{1.2.3\dots q}x^{m-n}x'^q x''^{n-q}$$

The coefficient of this term expresses in how many ways one is able to partition a number of things into three parts, of which one contains n of them, the other q, and the last $m-n-q$; this is evident by the theory of multiplication of similar factors; but one is able still to render the reason for it in this way: m quantities combined n by n give, as one knows,

$$\frac{m.m-1.m-2\dots m-n+1}{1.2.3\dots n}$$

combinations, to each of those here, is able to correspond some one of the

$$\frac{(m-n)(m-n+1)\dots(m-n-q+1)}{1.2.3\dots q}$$

combinations of the $m-n$ remaining things taken q by q, of which the total number of partitions of m quantities into two parts, the one of m, and the other of q, will be expressed by the product of the quantities which we just found, and the last part which contains a number $m-n-q$ of things, is a necessary consequence of the two others.

If one wished to take account of the diverse arrangements or orders which are able to present each combination, it would be necessary to eliminate the divisors.

The analogie leads us to the general term of the quadrinomial, that is to say, to the one of the development of $(x + x' + x'' + x''')^m$, and which is

$$\frac{m.m-1\ldots m-n+1}{1.2.3\ldots n}\cdot\frac{(m.m-n).(m-n-1)\ldots(m-n-q+1)}{1.2.3\ldots q}$$
$$\cdot\frac{(m-n-q).(m-n-q-1)\ldots(m-n-q+1)}{1.2.3\ldots r}x^{m-n-q-r}x'^{n}x''^{q}x''''$$

This formula is able to be set under a simpler form, by making $m - n - q - r = p$, and by multiplying the coefficient by the quantity

$$\frac{(m-n-q-r).(m-n-q-r-1)\ldots-1}{(m-n-q-r).(m-n-q-r-1)\ldots-1},$$

which is evidently equal to unity, one will have then

$$\frac{1.2.3\ldots m}{(1.2.3\ldots p).(1.2.3\ldots n).(1.2.3\ldots q).(1.2.3\ldots r)}x^p x'^n x''^q x'''^r$$

by observing besides that $p + n + q + r = m$.

The coefficient of $x^p x'^n x''^q x'''^r$ marks in how many ways one is able to partition a number m of things into four parts: the first of n, the second of q, the third of r, and the fourth of p.

By applying these formulas to the cases of piquet proposed above, it is clear that the general coefficient of the binomial will give us the sought solution, and that then $m = 32$, $n = 12$, $q = 8$, one will have for the sought number.

$$\frac{32.31\ldots21}{1.2.3\ldots12}\cdot\frac{20.21\ldots13}{1.2.3\ldots8};$$

and, if one adds to the numerator and to the denominator of the last product the factors 9, 10, 11 and 12, one will have

$$\frac{32.31\ldots9}{1.2.3\ldots12};$$

and by developing these products, one will find 28,443,124,054,800 for the number which marks in how many ways the game is able to be partitioned between the players and the talon.

One has seen (*page 26*) that in the case of many trials of two contradictory events, the term

$$\frac{p.(p-1)\ldots(p-q+1)}{1.2.3\ldots q}\cdot\frac{n^{p-q}m^q}{(n+m)^p}$$

will express the probability to bring forth q times the event of which the probability is $\frac{m}{m+n}$, and $p - q$ the contradictory event of which the probability is $\frac{n}{m+n}$ in a number p of trials.

If one has three events of which the probabilities are $\frac{x}{x+x'+x''}$, $\frac{x'}{x+x'+x''}$, $\frac{x''}{x+x'+x''}$, the general term of the formula of the binomial found above, being divided by $(x +$

$x' + x'')^m$, will express the probability to bring forth in a number m of trials $m-n$ times the event of which the probability is $\frac{x}{x+x'+x''}$, at the same time q times the event of which the probability is $\frac{x'}{x+x'+x''}$, and at the same time $n-q$ times the one of the probability is $\frac{x''}{x+x'+x''}$, one will extend easily these remarks, to the case of four or a greater number of events, by aid of the formula of the quadrinomial or of that of the polynomial.

In order to give a very simple application of these formulas, we ourselves will propose this question: *In a lottery composed of i numbers, of which there exit from it at each drawing, one demands the probability that all will be exited after a number m of drawings.* It is clear that, if one designates for a moment by x, x', x'', x''' the number of ways to bring forth each number, one will have a polynomial of which the number of terms will be i, by raising to the power m; and, after that which one just saw, the terms where all the letters x, x', x'', x''' will enter at once, will be those to which it is necessary to have regard, since these are the only ones which designate that all the events have successively or alternately taken place, a number of times more or less great. The sum of all the terms divided by $(x + x' + x'' + x'''$ etc.$)^m$ will be the sought probability; but one will observe that it will be necessary to make in these terms x, x', x'', x''' etc., each equal to unity, and that the divisor will become thus i^m.

We suppose that one demands the probability that after seven casts one will bring forth the different points of a die, this question is, at base, the same as the preceding: the number of numbers is here 6; and the one of the trials or drawings 7; it will be necessary therefore to raise $(x+x'+x''+x'''+x^{iv}+x^{v})^7$; but one sees that the terms of this power, which form all the letters, are able to be only the following:

$$\left.\begin{array}{cccccc}
x^2 & .x' & .x'' & .x''' & .x^{iv} & .x^{v} \\
x & .x'^2 & .x'' & .x''' & .x^{iv} & .x^{v} \\
x & .x' & .x''^2 & .x''' & .x^{iv} & .x^{v} \\
x & .x' & .x'' & .x'''^2 & .x^{iv} & .x^{v} \\
x & .x' & .x'' & .x''' & .x^{iv^2} & .x^{v} \\
x & .x' & .x'' & .x''' & .x^{iv} & .x^{v^2}
\end{array}\right\}$$

since their exponents are formed of the sole ways in which one is able to make 7 with six numbers.

Their coefficients will be found, by the following formula, drawn by induction from those which we have just given above:

$$\frac{1.2.3\ldots m}{1.2.3\ldots p \times 1.2.3\ldots n \times 1.2.3\ldots q.1.2.3\ldots r \times 1.2.3\ldots s \times 1.2.3\ldots f}$$

One sees that they must all be the same, since each of them, one makes only by changing successively p into n, into q, into r, into s, into t: one has therefore, for one of them,

$$\frac{1.2.3\ldots 7}{1.2} = 3.4.5.6.7,$$

and for their sum,

$$6.3.4.5.6.7;$$

finally, for the probability of the sought event,

$$\frac{6.3.4.5.6.7}{6^7} = \frac{3.4.5.7}{6^5} = \frac{420}{7776} \text{ or around } \frac{1}{18}.$$

One has said above (*page 83*) that the greatest term of the quantity $(n+m)^{pq(n+m)}$ was the one where was found the factor $n^{pqn}.m^{pqm}$; we are going to demonstrate that *à priori*, and to give at the same time the mean having the greatest term of the binomial, of the quadrinomial, etc.

The general term of the power of the binomial $m+n$, raised to the power r, is, as one knows,

$$\frac{r.r-1\dots r-h+1}{1.2.3\dots h} m^{r-h} n^h.$$

The one which precedes, and the one which follows it, are:

$$\frac{r.r-1\dots r-h+2}{1.2.3\dots h-1} m^{r-h+1} n^{h-1}$$

and

$$\frac{r.r-1\dots r-h}{1.2.3\dots h+1} m^{r-h-1} n^{h+1}.$$

If the number h is supposed to correspond to the greatest term of the series, these two here must be smaller; and, by designating the greatest by M, they become

$$\left.\begin{array}{c} \frac{hM}{r-h+1} \cdot \frac{m}{n} \\ \frac{(r-h)M}{h+1} \cdot \frac{n}{m} \end{array}\right\}$$

and one must have

$$\left.\begin{array}{c} M > \frac{h}{r-h+1} M \frac{m}{n} \\ M > \frac{(r-h)}{h+1} M \frac{n}{m} \end{array}\right\}$$

or, that which reverts to the same,

$$\left.\begin{array}{c} 1 > \frac{h}{r-h+1} \cdot \frac{m}{n} \\ 1 > \frac{(r-h)}{h+1} \cdot \frac{n}{m} \end{array}\right\}$$

whence it follows,

$$\left.\begin{array}{c} nr - nh + n > mh \\ m + mh > nr - nh \end{array}\right\}$$

and finally

$$\left.\begin{array}{c} h < \frac{nr+n}{m+n} \\ h > \frac{nr-m}{m+n} \end{array}\right\}$$

The whole number which will fall between these two limits will be the sought number; or else, if these two quantities are whole, there will be two consecutive equal terms, and the greatest of all the formula. This case holds when $m = n$, and when r is an odd number; because the limits remain then $\frac{r+1}{2}$ and $\frac{r-1}{2}$, which are whole.

If one makes $r = pq(m+n)$, one will have

$$\left.\begin{array}{l} h < \frac{npq(m+n)+n}{m+n} = npq + \frac{n}{m+n} \\ h > \frac{npq(m+n)+m}{m+n} = npq - \frac{m}{m+n} \end{array}\right\}$$

whence it follows $h = npq$, as we have said in the place cited.

If n was a binomial $u+s$, so that one demanded the greatest term of the trinomial $(m+u+s)^r$, after having found the greatest term of the series

$$n^r + rm^{r-1}(u+s) + \frac{r.r-1}{2}m^{r-2}(u+s)^2$$

that we will suppose $= 2M$. As it contains implicitly the series which results from the development n^h, or from $(u+s)^h$, one sought, as previously, the greatest term of that series; and, naming h' the exponent of s in this term, one would have

$$\left.\begin{array}{l} h' < \frac{sh+s}{u+s} \\ h' > \frac{sh-s}{u+s} \end{array}\right\}$$

and consequently

$$\frac{1.2.3\ldots r}{1.2.3\ldots r-h-h' \times 1.2.3\ldots h-h' \times 1.2.3\ldots h'} \cdot m^{r-h}u^{h-h'}x^{h'}$$

for the greatest term of the trinomial. Let $r = p(m+u+s)$, one will have first $h = p(u+s)$ and $h' = ps$, of which the greatest term of the trinomial $(m+u+s)^{p(m+u+s)}$ is

$$\frac{1.2.3\ldots p(m+u+s)}{1.2.3\ldots pm.1.2.3\ldots pu\ldots 1.2.3\ldots ps}m^{pm}u^{pn}s^{ps}.$$

There results from this formula that, in the case of $p(m+u+s)$ trials of three possible events, the most probable event is the one which will bring forth each of them a number of times proportional to their respective probability, in the same way as one shows it for the case of two contradictory events. We will not push this object further; one sees rather that the march and the results will be the same for all the other cases.

We ourselves will propose, in order to end this article, the following problem, an extract *from the Analytica Miscellanea* of *Moivre*, in the same way as the greatest part of that which precedes:

Three players, designated by A, B, C, *must draw, in the order of their letter, each a ball, from an urn that one supposes to contain a number* a *of white balls, and* b *of black balls. The one who first will draw a white ball, will win. One demands, by supposing that the gain is represented by unity, what must be the stake of each player.*

1°. *A* has for the probability $\frac{a}{a+b}$ to bring forth a white ball, and against him $\frac{b}{a+b}$ to bring forth a black ball.

2°. After the drawing of *A*, there remains, in the case that he had not won, a number $b-1$ of black balls. Thus, *B* has for it, under this hypothesis, the probability $\frac{a}{a+b-1}$; but this hypothesis itself has for probability proper $\frac{b}{a+b}$: one will have therefore the absolute probability in favor of *B* $\frac{a.b}{(a+b)(a+b-1)}$.

3°. *C*, if he comes to draw, will have $\frac{a}{a+b-2}$ in order to bring forth a white ball; but he is able to incur this chance only in the case where *A* and *B* would not have won. The particular probability of each of these events is $\frac{a}{a+b}$ and $\frac{b-1}{a+b-2}$; the probability of their concurrence will be $\frac{b.(b-1)}{(a+b)(a+b-1)}$; and consequently that of the gain of *C* $\frac{b.b-a}{(a+b)(a+b-1)(a+b-2)}$.

If *A* recommences to draw before the lot is decided, he will have $\frac{a}{a+b-3}$ for the probability to have a white ball; but, as this drawing supposes the concurrence of the three events contrary to *A*, *B* and *C*, of which the probabilities are $\frac{b}{a+b}$, $\frac{b-1}{a+b-1}$, $\frac{b-2}{a+b-1}$, one will have

$$\frac{ab.(b-1)(b-2)}{(a+b)(a+b-1)(a+b-2)(a+b-3)},\text{ etc.}$$

in a way that by making, for brevity $a+b=n$, if one writes the series

$$\frac{a}{n}+\frac{ab}{n.n-1}+\frac{ab\times b-1}{n.n-1.n-2}+\frac{a.b.(b-1)(b-2)}{n.n-1.n-2.n-3},$$

of which the law is easy to know, the sum of all the terms taken in the ranks 1.4.7.10, or taken 3 by 3, to count from the first, will be the expectation of the first player; that of the terms of which the rank is 2.5.8.11. etc., or taken 3 by 3, to count from the second, will be the expectation of the second; finally, the terms of the ranks 3.6.9.12, taken 3 by 3, to depart from the third, will form the expectation of *C*. If one make

$$\left.\begin{array}{r}n=12\\a=4\\b=3\end{array}\right\}$$

the preceding sequence will become

$$\frac{4}{12}+\frac{4.8.7}{12.11}+\frac{4.8.7}{12.11.10}+\frac{4.8.7.6}{12.10.11.9}+\frac{4.8.7.6.5}{12.11.10.9.8}$$
$$+\frac{4.8.7.6.5.4}{12.11.10.9.8.7}+\frac{4.8.7.6.5.4.3}{12.11.10.9.8.7.6}+\frac{4.8.7.6.5.4.3.2}{12.11.10.9.8.7.6.5}$$
$$+\frac{4.8.7.6.5.4.3.2.1}{12.11.10.9.8.7.6.5.4}$$

By taking the terms that we have designated, and by reducing the fractions to their simpler expressions and to the same denominator, one will find the expectations of the players *A.B.C* proportionals to the numbers 77.53.35 below, which must give evidence before playing the sums

$$\frac{77}{77+53+35}M,\quad\frac{53M}{77+53+35},\quad\frac{35M}{77+53+35}$$

in order to have right to the sum M.[3]

[3]The development of all the possible combinations, in the greater part of the questions relative to the calculus of probabilities, give place to some formulas which contain the successive products of a very great number of factors, and often even the sum of a considerable number of expressions of this kind: but, as the concern is never butt to assign the ratios between the quantities, the geometers have found some methods of approximation, by the aid of which one will arrive very promptly to the value sufficiently exact of a formula that its length would have rendered impossible to calculate rigorously. One is able to consult, on this object, the differential calculus of Euler, *pag. 467*, and the *Mémoires de l'Académie*, year 1782 and 1783.