PROBABILITÉ*

Condorcet

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PROBABILITÉ. We will limit ourselves to give here the general principles of the calculus of *probabilities*, of which one finds some applications in diverse articles.

I.

1. The fundamental principle of this calculus can be expressed thus.

Let *A* be an event, & *N* another event contradictory to the first (that is to say, which, under the hypothesis, cannot exist at the same time); let n express the total number of the equally possible combinations, m the one of the combinations which give the event *A*, m' that of the combinations which give the event *N*, $\frac{m}{n}$ will express the *probability* of the event *A*, & $\frac{m'}{n}$ that of the event *N*. $n = m + m'$.

2. If one has three events *A*, *N*, *N'*, if n is always the total number of combinations, m the one of the combinations which give *A*, m' the one of the combinations which give *N*, m'' the one of the combinations which give *N'*, the *probability* of *A* will be $\frac{m}{n}$, that of *N* will be $\frac{m'}{n}$, that of *N'* $\frac{m''}{n}$, & one will have $n = m + m' + m''$.

It is easy to see that this second definition is a sequel to the first; in effect, here the *probability* of *A* is, by this first definition, $\frac{m}{n}$, & that of not having *A* $\frac{m_{\prime}}{n}$, $m_{\prime} = m' + m''$; but, if one has not *A*, the *probability* of *N* is $\frac{m'}{m_{\prime}}$, & that of *N'* $\frac{m''}{m_{\prime}}$: therefore that of *N* will be, in general, $\frac{m_{\prime}}{n} \times \frac{m'}{m_{\prime}} = \frac{m'}{n}$, & that of *N'* $\frac{m_{\prime}}{n} \times \frac{m''}{m_{\prime}} = \frac{m''}{n}$. Thus, in the following of our reflections on this first principle, we will consider only two events.

3. If $m' = 0$, $n = m$, & the *probability* of *A* is $\frac{m}{n} = 1$; but, if no possible combination gives the event *N*, this event is therefore impossible, the event *A* will arrive therefore necessarily; thus, 1 will express the probability of a necessary event, or certitude; likewise one of the two events *A* or *N* arrive necessarily, & the sum of their *probabilities* is $\frac{m}{n} + \frac{m'}{n} = 1$. This which leads still to the same conclusion.

4. It follows from the same definition that, if one takes any number t whatever of successive combinations of the events *A* & *N*, the *probability* of each will be expressed by the sequence of terms of the binomial $\frac{\overline{m+m'}^{\,t}}{n^t}$; so that the *probability* to have t times, the event *A* will be $\frac{m^t}{n^t}$, that to have $t-1$ times the event *A*, & one time the event *N* will be $\frac{t.m^{t-1}m'}{n^t}$, that to have the event *A* $t - t'$ times, & the event *N*, t' times will be $\frac{t.t-1\ldots t-t'+1}{1.2\ldots t'} m^{t-t'}\, m'^{\,t'}$.

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5. We have called *probability* of an event, the number of the equally possible combinations which give it, divided by the total number of combinations which give this event or the contradictory event. Until now this is a pure definition of name, but one intends more:

1. $^\circ$ That, if the number m of the combinations, which give *A*, surpasses the number of the combinations m' which give *N*, so that the probability $\frac{m}{n} > \frac{1}{2}$, there is place to believe that the event *A* will arrive rather than to believe that it will not arrive.

2. $^\circ$ That this motive to believe is so much greater as $\frac{m}{n}$ is also greater, & approaches unity.

3. $^\circ$ That it increases proportionally in this same ratio.

6. We are going to demonstrate first that these three propositions depend on the 1st, and that it suffices even that it be true for the case where m is much greater than m', & we will expose what is, in this last case, the nature of the motive to believe that *A* will arrive rather than *N*.

First the third proposition depends on the truth of the first two; we suppose in effect that the events *A* or *N* can arrive a number t times; & that $t = mp + m'p$, it follows from the definition of the 1st article, that the *probabilities* that the event *A* will arrive t times, & the event 0 times, the event *A*, $t-1$ times, & the event *N* one time, & thus in sequence will be expressed by the terms of the binomial,

$$\left(\frac{m}{n}\right)^t, \quad t\left(\frac{m}{n}\right)^{t-1}\cdot\frac{m'}{n}, \quad \frac{t.t-1}{2}\left(\frac{m}{n}\right)^{t-2}\left(\frac{m'}{n}\right)^2$$

so that a term

$$\frac{t.t-1\ldots t-q+1}{1.2\ldots q}\left(\frac{m}{n}\right)^{t-q}\left(\frac{m'}{n}\right)^q$$

will express the *probability* that the event *A* will arrive $t-q$ times, & the event *N* q times.

This put, it is clear, by considering the formula of the binomial, 1. $^\circ$ that the greatest term will be,

$$\frac{t.t-1\ldots t-m'p+1}{1.2\ldots m'p}\left(\frac{m}{n}\right)^{t-m'p}\left(\frac{m'}{n}\right)^{m'p} = M.$$

The most probable combination will be therefore that where the ratio of the number of events *A* to the one of the events *N*, will be the one of mp to $m'p$, that is to say, the same as the ratio of the *probability* of the events.

2. $^\circ$ That the greater will be p, the more it will be probable that the ratio of the number of the events *A* to the one of the events *N*, will deviate little from $\frac{m}{m'}$.

Indeed, let a fraction $\frac{t}{r}$ of t, r being any whatsoever, one will prove, by considering the formula of the binomial, that, if one takes the number $\frac{t}{r}$, so much above as below from this mean term M, & if one divides successively the rest of the terms in the number of $\frac{r-t}{r}t$ into $\frac{t}{r}$ parts of $r-1$ terms each, one will be able to take t great enough, in order that each of these $\frac{t}{r}$ terms the nearest to the term M surpasses, in such ratio as one will wish, the $r-1$ which correspond to it (*See* the 4th part of the *Ars Conjectandi* of Jacques Bernoulli. Besides this demonstration has no difficulty by employing the theorem of Stirling & of Mr. Euler on the indefinite products of a very great number of terms.)

If therefore, having $m > m'$, I judge that the event m will arrive rather than m', I will have a *probability* as great as one will wish, from an indefinite number of judgments, the ratio of the number of true judgments, a total number will deviate very little from $\frac{m}{n}$; therefore, if a greater *probability* gives me a motive to judge conformably to this *probability*, the motive must be proportional to it.

The truth of the 2$^{\text{nd}}$ proposition that this motive must be stronger, if the *probability* is greater, depends yet on the truth of the first. In effect, calling u the probability of *A* & e that of *N*, the probability of being mistaken only p times out of t judgments, in pronouncing that one will have *A* will be expressed by

$$u^t + t.u^{t-1}e + \frac{t.t-1}{2}u^{t-2}e^2 \cdots + \frac{t.t-1\ldots t-p+1}{1.2\ldots p}u^{t-p}e^p = V.$$

Now, if one supposes to V & to t a constant value, it is easy to see that the greater u will be, the smaller will be p; therefore, by pronouncing that the event *A* will take place rather than *N*, one will have an equal *probability* to be mistaken so much less often as u will be greater, & consequently, the greater u will be, the more the motive to judge that the event *A* will take place, will have to be strong. It remains therefore to prove only that a greater probability in favor of an event *A*, is a motive to believe that *A* will take place rather than *N*, & for that it suffices to prove that this motive exists when this *probability* is very great; in effect, preserving the preceding demonstrations, one will have the *probability* that *A* will take place more often than *N* in $2t+1$ judgments, expressed by

$$u + (u-e)(ue + 3u^2e^2 + \frac{5.4}{2}u^3e^3 \cdots + \frac{2t-1.2t-2\ldots t+1}{1.2\ldots t-1}u^t te^t);$$

which, when $u > e$, approaches continuously from unity in measure as t increases: therefore one will have a probability as great a one will wish, at least to be mistaken in believing that *A* must arrive more often than *N*, in a very great number of judgments; therefore, if this very great *probability* is a motive to believe; however little that u exceeds e, one will have still a motive to judge that *A* will take place rather than *N*.

7. There remains to us therefore now only to examine the nature of the motive which carries us to judge conformably to a very great *probability*.

I suppose that I have in an urn a billion white tickets, & one black, &, 1.° that one has drawn no ticket; 2.° that one has drawn a white ticket, & that it is not exposed; 3.° that one has drawn a black ticket which is not exposed either.

It is clear that, in the three hypotheses, I have the same *probability* that the ticket is white; however the fact is uncertain in the first case, it is certainly true in the second, & certainly false in the third. Another who will have seen the tickets would be able to be sure respecting the falsity of an event of which the truth is for me very probable; there is therefore no real liaison between the *probability* of a future or unknown event, & reality.

Suppose therefore now an urn which contains the black tickets & the white tickets in an unknown ratio, that I have drawn a number t of white tickets without a single black,

& that I seek the *probability* to bring forth in a following trial a white ticket rather than a black. Let n be the number of tickets, & x that of the white tickets, the number of combinations which give $t+1$ white tickets will be $\frac{x^{t+1}}{n^{t+1}}$, the number of combinations which give t white tickets & one black, will be $\frac{x^t\overline{n-x}}{n^{t+1}}$, & the number of all the possible combinations under the hypothesis, will be consequently $\frac{x^{t+1}+x^t\cdot\overline{n-x}}{n^{t+1}} = \frac{x^t}{n^t}$;[1] but x is susceptible of all the values from $x=0$, to $x=n$; therefore $\frac{\sum x^{t+1}}{n\sum x^t}$, the integrals being taken from $x=n$ to $x=0$, will express the probability to have a white ticket.

$$\sum x^{t+1} = \frac{n^{t+2}}{t+2}an^{t+1} + b.\frac{t+1}{2.3}n^t - c.\frac{t+1.t.t-1}{2.3.4.5}n^{t-2}\cdots$$

$$\sum x^t = \frac{n^{t+1}}{t+1}an^t + b\frac{t}{2.3}n^{t-1} - c.\frac{t.t-1.t-2}{2.3.4.5}n^{t-3}$$

Whence

$$\begin{aligned}\frac{\sum x^{t+1}}{n\sum x^t} &= \frac{t+1-a.\frac{t+2.t+1}{n}+b\,\frac{t+2.\overline{t+1}^2}{2.3.n^2}-c\frac{t+2.\overline{t+1}^2.t.t-1}{2.3.4.5n^4}}{t+2-a.\frac{t+2.t+1}{n}+b.\frac{t+2.t+1.t}{2.3.n^2}-c\frac{t+2.t+1.t.t-1.t-2}{2.3.4.5.n^4}}\\ &= \frac{t+1}{t+2}\cdot\frac{1-a.\frac{t+2}{n}+b.\frac{t+2.t+1}{2.3.n^2}\cdots}{1-a\frac{t+1}{n}+b.\frac{t+2.t}{2.3n^2}\cdots}\end{aligned}$$

Now it is easy to see that, if we suppose n incomparably greater than t,

$$\frac{t+1}{t+2} \quad \& \quad \frac{t+1}{t+2}\cdot\left(1-\frac{a}{n}\right) = \frac{t+1}{t+2}\cdot\left(1-\frac{1}{2n}\right),$$

will express the approximate values, the one greater, the other smaller of this *probability*. We suppose finally t very great, $\frac{t+1}{t+2}$ or $\frac{t+1}{t+2}\cdot\left(1-\frac{1}{2n}\right)$ will give us a very great *probability*. If we examine at present what motive we have to believe according to this *probability*, we will find that it is the same which carries us to believe that a fact arrived steadily will continue to arrive again.

But this motive is the one which makes us admit this general principle to us, that the natural events are subject to some constant laws, since we can base this opinion only respecting the observation of the order of the past events, & on the assumption that it will continue to be the same for future events. This motive is still the same which makes us believe in the persistence of objects which strike our sense, & consequently in the existence of bodies. It is again by the same motive that we believe the truth of the demonstrations of which we cannot embrace the chain by a single glimpse; indeed, we cannot be sure that a proposition of which we ourselves remember to have followed the demonstration is true, that by long experience that our memory does not deceive us in this case; & that as often as we have wished to follow anew the same reasonings, they have led us to the same results.

If therefore one excepts the intuitive knowledge of the propositions which our mind can embrace immediately, all our knowledge respecting nature, all the propositions according to which we guide our behavior & all our movements, & even as far as the

[1] *Translator's note*: The right-hand side has been corrected from $\frac{x^n}{t^n}$.

best demonstrated mathematical truths, can not have for us veritable certitude, & we have no other motive to believe them, than this tendency to regard as constant that which has been constantly observed, that is to say, as a very great *probability*.

One sees only to be born here different orders of truths; indeed, in mathematical demonstrations, for example, this motive acts only in order to make us suppose that the truths which have been demonstrated to us once, will appear always to us, instead that, in some other kinds of knowledge, we have need of the same motive in order to suppose the reality of the same facts on which we reason, then on the observed order in these facts, &c. &c.

This motive is for us a natural tendency, which is confounded even sometimes with sensation. It is by virtue of this motive that two men, seen at some distances doubles the one of the other, appear sensibly equal, although being seen under some different angles, the more extended must appear one time smaller; it is in effect experience which alone has been able to mix in our sensation a secret judgment which is confounded with it.

It is by virtue of this motive that, if I roll a ball between two crossed fingers, I sense really two balls, while my reflection, supported out of some more constant experiences, forces me to believe that there is only one of them. One sees, in this last example, how this tendency, which seems in itself purely natural, & proportional perhaps to the force of the impression of the objects, can however cede to reason, but without being destroyed, & even without having no loss of force.

We will not carry further the consequences of this observation, which we believe very important, & which can serve to explicate many phenomena relative to the force of prejudices, to the power of reason, to liberty, &c.

It suffices us here to have shown, 1.° that the consequences which one draws from the calculus of *probabilities*, relative to the reality of the objects, are some truths of the same kind as those which are born of observations & of the reasonings which one makes according to them. 2.° That they differ from them only in this point, that one knows then, by the calculus, the value of the motive which carries us to believe, & which one has the veritable measure of it, instead to cede uniquely to a natural tendency, which, in many circumstances, can deceive us.

II.

1. The second principle of the calculus of *probabilities* is this one.

Suppose that we have many events A, A', A'', ... of which the *probabilities* are p, p', p'', ... & that e, e', e'', ... represent the values or the effects of these events, effects or values which one supposes of the same kind; the mean value of the event A as probable will be expressed by $\frac{pe}{p+p'+p''\cdots}$; that of the event A' it will be by $\frac{p'e'}{p+p'+p''\cdots}$, & that of the event A'' by $\frac{p''e''}{p+p'+p''}\cdots$, & the mean value of the events or of any event whatever which will necessarily arrive, will be $\frac{pe+p'e'+p''e''\cdots}{p+p'+p''\cdots}$, the values of e, e', e'' can be of different signs. Thus, for example, if I have the *probability* $\frac{1}{2}$ to win an écu, the *probability* $\frac{1}{4}$ to win a half-écu, & the probability $\frac{1}{4}$ to lose two écus; the value of the expectation to win by the event A, will be consequently $\frac{1}{2}$ écu, & my total expectation to win will be $\frac{1}{2}+\frac{1}{8}-\frac{1}{2}$ écu or $\frac{1}{8}$ écu.

In order to demonstrate this rule, it suffices to observe that one can consider the event A having p possible combinations which produce it as p events of the same value each having one combination; that it will be the same of the other events, & that thus the rule is reduced to taking for the mean value of a certain number of equally possible events, the sum of their values, & to divide it by the number of these events.

2. Thus it is not at all against this rule in itself, but against the usage which one can make of it that there is raised some objections of which some have been insoluble until now.

We suppose two men *A* & *B* play together, with the condition such that *A* has a *probability* p to win a sum s, & *B* a probability p' to win the same sum, & that $p+p'=1$; the mean value of the expectation of *A* will be, by the preceding rule, equal to ps, & that of *B* equal to $p's$. If now this sum has ought to be furnished by *A* & *B* at the beginning of the game, & if one demands in what proportion they must furnish it in order to play at an equal game, either with respect to the other player, or in an arbitrary manner; one will respond that each must give a sum equal to the value of their expectations, that *A* consequently must give ps & *B*, $p's'$.

What does one intend now by an equal game? $1.^\circ$ this is not that the lot of the players is the same as before the game, since, even by supposing $p=p'$ & the lot of the players absolutely similar, each before the game, is $\frac{1}{2}s$, & that according to the game the one will certainly have s, & the other 0. $2.^\circ$ It is not that the state of the two players is similar; because player *A*, after the game, will have a sum $p's$ of gain, or a sum ps of loss, & player *B* a sum ps of gain, or $p's$ of loss, so that their state is essentially different, except when one has $p=p'$.

3. One understands therefore that the players having agreed to change the state by putting into the game, & before, after the game, to have a state different from the first, & different also for each player, one has proportioned their stakes in a way that it is either between their state before playing, & their state after the game, or between the state of them after the game, the greatest equality that the nature of the things can permit.

Now here this equality can be established only by supposing the game repeated a great number of times; & then one can require, $1.^\circ$ that the most probable case is precisely the one which changes nothing in the state of the two players; $2.^\circ$ that the *probabilities* to win or to lose for *A* as for *B* approaches more and more to be equal to $\frac{1}{2}$, in measure as the number of trials is multiplied.

3. Finally, let one have, under the same hypothesis, a *probability* always increasing, that the loss of *A*, or that of *B*, will not exceed, either a fixed sum, or a proportional part of the total stake, if the preceding condition can not be executed.

4. Let therefore, $1.^\circ$ $t\overline{p+p'}$ be the number of trials, the most probable event or the greatest term of the series $\overline{p+p'}^{\,t\overline{p+p'}}$ will be

$$\frac{t.\overline{p+p'}.t\overline{p+p'}-1\ldots tp'+1}{1.2\ldots tp}p^{tp}p'^{tp'}.$$

Now let x be the wager of *A*, & $s-x$ that of *B*, $tps-tx$ will express that *A* will have won; therefore, making $tps-tx=0$, one will have $x=ps$ for the value of the stake of A.

2.° In supposing the same law established, it is clear that, in $\overline{p+p'}^{t\overline{p+p'}}$ all the combinations where the exponent of p is greater than tp, will give a positive sum for the gain of *A*, & that all the terms where this exponent is below, will give a positive sum for the gain of B; now, the more t will be great, the more the sum of these terms which are favorable to *A*, & that of the terms which are favorable to *B*, will approach the one & the other to the quantity $\frac{1}{2}$; in a manner that, supposing t always increasing, if $p > p'$, the sum of the terms favorable to *A* will be first greater than $\frac{1}{2}$, & will diminish in bringing them together, instead that, if $p < p'$, the sum of the same terms will be first smaller than $\frac{1}{2}$, & will increase in bring them together.

3.° If, in the sequence of terms of the formula $\overline{p+p'}^{t\overline{p+p'}}$, one takes all those where the exponents of p are above $t.\overline{p-a}$, whatever be a, in measure as t will become great, the sum of these terms will approach to unity to which it will become equal, if t is infinity. Now let this exponent be $t.\overline{p-a}$, the loss of *A* will be $tps - ts.\overline{p-a} = ats$; therefore the more one will multiply the trials, the more *A* will have a continually increasing *probability* of not losing beyond tas, that is to say, beyond a fraction $\frac{a}{p}$ of his total stake tps, a being a quantity as small as one will wish, provided that it is finite: &, since the term where the power of p is $p^{t\overline{p-a}}$, is the limit to where it is necessary to go in order that the total sum of the preceding terms can approach indefinitely to unity, in measure as t increases (a being always a quantity as small as one will wish, but finite), one sees that this same condition to have a *probability* always increasing to not lose beyond a certain sum, can take place only for a sum proportional to ts.

One sees therefore here not only that the established law put between the state of the players before or after the game, & between their respective states, the greatest equality possible, or the sole one which can be compatible with the difference of these states, but one sees at the same time that with any other, one cannot fulfill the same conditions.

5. That which we have said on the money destined to form the sum s, that *A* & *B* have an unequal expectation to win, will be applied equally to the case where it will be necessary to partition the sum, the game being supposed stopped, & their expectations being p & p' if it had been continued; &, in general, to the case where one buys, for a given sum, the expectation of another sum, or else, where one divides among many persons a sum to which they have some more or less probable claims.

6. In the free agreements, as the game, one sees that this law must have been established finally that there is no advantage to play or to not play, to choose rather the lot of *A* than the one of *B*, & that one is not absolutely determined by some particular social conventions.

This is very nearly that which arrives in commerce, where a common price of commodities is established in a manner as such day, for example, a septier de bled,[2] an ell of such material equivalent to four ounces of silver, & that consequently the preference given by certain persons to one of these things over the other, will keep to their needs or to their particular wishes, without that one can say, in general, that one is preferable to the other.

If the convention is forced, then one must adopt the same law, since it is that in

[2]*Translator's note*: This is a unit of poor land.

which it is most probable that there will result a smaller sum of injustices from a great number of distributions made by virtue of this law. *See* ABSENS.

But there results from this that we just said, a remarkable difference between these two cases. In effect, in the second where the agreement is forced, the law must always be followed; but in the first, if the kind of equality that this law established does not appear sufficient, there must result from it that, as little as one acts with prudence, one will not wish at all to form the agreement. In the first case, one decides according to the law because one can only consider the total mass of similar agreements, & to seek to do so that there results from it the least possible inequality. In the second, if one wishes to act with prudence, if the object is important, one must lend oneself to the agreement only as much as one can envisage the possibility to establish between the two parties a sufficient equality.

7. This put, consider two players, of whom one *A* has an expectation e to win, & a risk $1 - e$ to lose; & the other *B* an expectation $1 - e$ to win, & a risk e to lose, & let the stake of *A* be to the stake of *B* as e is to $1 - e$; so that in winning *A* will win $1 - e$ times, & that in losing he will lose e times a certain sum regarded as unity.

If $e < 1 - e$ in measure as the number of trials will be multiplied, the probability that *A* will have to win will approach $\frac{1}{2}$, but it will always remain below, & e can be small enough in order that, even for a great number of trials, this probability is still quite inferior to $\frac{1}{2}$, while the probability to lose that the same player would have, would always be quite above $\frac{1}{2}$.

In the same case, the probability to not lose above an n^{th} part of the total stake, will increase in favor of *A* whatever be n; but if n is very small, it will be necessary to suppose the game continued a very great number of times in order that this probability becomes great enough.

It is necessary to observe next that, for the same player *A*, the probability to not win beyond a certain portion of his total stake, increases at the same time as that to not lose beyond the same proportion.

It is likewise of it in all the cases which one could choose, so that in general the one of the two players who have the least probability, wins in the combination of the greatest number of trials on the side of the expectation of not losing, & loses as much to the expectation to win much while to the contrary the one who has a great probability loses from the expectation to win, & at the same time is exposed to a smaller risk to lose much.

8. This manner to consider the law which we examine, & which consists in regarding the value which results from it for the expectations & for the risks, as a proper mean value to restore the greatest possibility equality between those who exchange between them a certain value & an uncertain expectation, or two uncertain expectations, &c. has seemed to us to be able to make vanish most of the difficulties which this rule has appeared to present in its application. We are going to examine here some & we will begin with those which the famous *problem* of Petersburg presents.

In this problem, one supposes that a player *A* must give to a player *B* a coin if he brings forth tails on the first toss, two if he brings it forth on the second, four if he brings it forth on the third, & thus in sequence; & one demands what is the value of the expectation of *B*, or what sum he must give to *A* in order to play an equal game. The rule of the calculus gives this sum equal to

$$\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\cdots=\frac{1}{2}(1+1+1\cdots)=\frac{1}{2}\times\frac{1}{0}.$$

A conclusion which appears so much more absurd, as this stake of *B* being supposed greater than any given quantity; one can have a probability as great as one will wish, that *B* will lose in this agreement.

But one can observe 1.$^\circ$ that the case which becomes the most probable, by supposing that one continues the game, the one where there is neither loss nor gain, can not take place here, at least if one does not suppose the game repeated an infinite number of times.

2.$^\circ$ That the probability of *B* to win, will no longer approach to be equal to $\frac{1}{2}$, & consequently to be equal to the probability of loss, but by supposing also the game repeated an infinite number of times; it begins even to be finite only at this term.

3.$^\circ$ That the probability to lose only a certain part of the total stake as we have seen should increase with the number of trials, is finite for *B* only by supposing infinite the number of times that the game is repeated, & that in this case this part of the stake is necessarily still an infinite quantity.

One sees therefore that the principle on which we have said that the general rule must be founded, the one to put the greatest possible equality between two essentially different states, can have no place here, since this equality would require that one embraces the combination of an infinite number of games, so that the limit which, in the ordinary problems is an infinite number of games is necessarily here an infinity of the second order.

It is therefore not the rule which is in default, but the application of the rule to a case which one presents as real, & which however can not be, since it supposes the reality of an infinite sum, of an infinite number of trials in each game, & of an infinite number of games. Thus, the problem must be considered not as a real case, but as the limit of the real questions of the same kind as one can have in view.

This explication however is not yet satisfactory. Indeed, one has remarked, with reason, that the rule would appear to be in default even when one would limit the number of possible trials, because the sum that *B* must give to *A* under this hypothesis in order to play at an equal game, is yet such, if the number of trials is in the least great, that any reasonable man would risk to give it. Nonetheless in most of the solutions given to this question, one is confined to say that it was necessary to limit the number of trials, either because beyond a certain number, it was necessary to regard the probability as too small, or because it was necessary that this number was such that the wealth of *A*, or the sum that he reserves to this game, suffices to pay that which he must to *B*, if tails does not arrive at the last trial.

Such is therefore now the case which remains to us to examine, the one where the number of trials is fixed, & where the sum which *B* must give, & the probability which he has to win being finite, the problem becomes a real problem.

We suppose that each game is limited to n trials, & that one pays 1 if tails arrives the first trial, 2 if it arrives the second, 4 if it arrives the third, 8 if it arrives the fourth...

2^{n-1} if it arrives the n^{th}, & 2^n if it does not arrive at all. The probabilities will be

$$\frac{1}{2}, \quad \frac{1}{4}, \quad \frac{1}{8}, \quad \cdots \quad \frac{1}{2^{n-1}}, \quad \frac{1}{2^n},$$

& the stake of *B* must be

$$\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\cdots+\frac{1}{2}+1=\frac{n}{2}+1,$$

& we will find first that *B* will begin to win when tails will arrive at a trial p, such that $2^{p-1} > \frac{n}{2}+1$, or $n < 2^p - 2$. If $n = 2^p - 2$, then there is neither loss nor gain; but in the same case $\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^{p-1}}$, or $1-\frac{1}{2^{p-1}}$, expresses the probability that *B* has to lose.

Suppose, for example, $p = 4$ & $n = 2^p - 2 = 14$; the stake of *B* will be 8. We will have $\frac{7}{8}$ for the probability that *A* will win, $\frac{1}{16}$ for the probability that he will have neither loss nor gain, & $\frac{1}{16}$ for this that *B* will win. At the same time because $n = 14$, it will be possible that he wins 16376, in truth the probability of this gain will be only $\frac{1}{16384}$. On his side *A* will have a probability $\frac{15}{16}$ of no loss, but he would win only 7 in the most favorable case, & could lose up to 16376.

One sees therefore that there is a very great inequality between the positions of *A* & of *B*, by considering only a single trial, & that not only there are some circumstances where neither one nor the other must wish to consent to change the state where they are before the game against the one which results from this agreement, but that this must take place nearly generally. If one considers next a sequence of games, then one will seek to determine, either the sum regarded as unity, or the number n of trials, in a way 1.$^\circ$ that the probability to win for *A* & for *B* approaches to equality, 2.$^\circ$ that one has a great enough probability that neither *A* nor *B* in a number m of games will lose beyond a value which is a given proportion with m.

The number of games must then be determined by the condition to be such that it has a probability nearly equal to unity, or to certitude that the loss that *A* can have will not exceed at all his wealth, or the sum that one believes that he will wish or is able to place into the game. The same assumption of certitude is the only rigorous, it is the sole way that *B* is not here at disadvantage. We take, for example, a simpler case, the one where of 100 tickets, *B* chooses 1 of them, & gives 1 to *A*, on the condition that if this ticket arrives, *A* will give to him 100, & we suppose that one plays 200 trials, the probability that *A* will win will be expressed by $\frac{40464}{100{,}000}$, that he will neither lose, nor gain by $\frac{27203}{100{,}000}$, and that *B* will win by $\frac{32333}{100{,}000}$, the probabilities to win for *A* & for *B* will be therefore here very nearly as 5 to 4, & consequently already neighboring to equality. In the same case, the probability for *A* to not lose beyond 500 will be $\frac{235}{100{,}000}$, a risk already very small.[3]

One sees therefore that provided that *B* has the expectation to be able to play 200 trials, there is established in the game a sort of equality. It is true that the established

[3] *Translator's note*: Condorcet has made an error here. Let X have a binomial distribution with parameters $n = 200$ and $p = .01$. $\Pr(\text{B wins } 100(k-2)) = \Pr(X = k)$ for $k = 0, 1, 2 \ldots 200$. For A to lose more than 500 requires B to win more than 500, namely, $\Pr(X \geq 8) = .001013$. Furthermore, each of the remaining probabilities in this and the following paragraph are slightly in error.

law can take place only by supposing that if *A* would lose 200 times, he would pay 200×100, or 20,000; but even though he would not have them, as the probability that *A* will not lose above 10,000, for example, is then nearly equal to unity, & that in the other very rare cases, *B* would win always 10,000; it is easy to see that even though *A* would pay only this sum, *B* would still consent to play this game, where he can expect to win 10,000 by risking only 200. *B* under this hypothesis would keep besides a probability $\frac{32{,}333}{100{,}000}$ to win against a probability $\frac{40{,}464}{100{,}000}$ to lose, a probability $\frac{18{,}145}{100{,}000}$ to win 100 against a probability $\frac{27{,}066}{100{,}000}$ to lose 100, & a probability $\frac{14{,}188}{100{,}000}$ to win 200 or more against a probability $\frac{13{,}398}{100{,}000}$ to lose 200. Thus, even though *B* would not have the absolute certitude that *A* would pay all the possible loss, his state in regard to *A* would conserve still a sufficient equality.

9. It is necessary however here to consider two quite distinct cases, the one where, for example, the two hundred trials above form a linked game, so that if *A* & *B* agree one time to play it, they are engaged to continue the number of trials; & under this hypothesis, the state of each player, & the kind of equality which subsists between them, & which can be regarded as sufficient, is expressed as we just said it.

But if *A* & *B* conserve the liberty to make at each trial the same convention, there is moreover an observation to make: since it is by considering at each time the system of future trials that *A* & *B* are determined to play, there results from it that they must regulate the stake regarded as unity, in a manner that at each trial they can envision as possible the number of trials necessary in order to establish a sufficient equality, that is to say that it is necessary that the wealth of each of the two, or the sum that one has cause to believe that he would wish to risk, can suffice to this number of trials; thus in order to conserve the necessary equality, the stake must change after a certain number of trials. In some of the possible combinations, that is to say, in those where the wealth of one of the two players is arrived to a value which obliges to this change, if one makes to enter this diminution of the stake in the calculus, one will see that there must result from it necessarily the possibility to play a much greater number of trials; whence there must result also between the players a greater equality; because this kind of equality consists in this that if one considers the sequence of future trials, one has a probability nearly equal for each of the players, to lose or to win, & a very great probability that the loss or the gain of any of the two, will not exceed a very small part of the total stake: now in this case, the first condition holds as in the preceding, & the part of the total stake can even be, under this last hypothesis, regarded as a given quantity.

10. The manner in which we have considered the established rule, can explicate also two contradictory phenomena which are themselves presented in the applications of this rule to some real cases.

It happens equally, & let a reasonable man *A* refuse to give a sum b for the probability n to win a sum $a > \frac{b}{n}$, & also let a reasonable man *B* consent to give a sum b' for the probability n' to win a sum $a' < \frac{b'}{n'}$.

The first case takes place when b is a considerable sum with respect to the state of the fortune of *A*, either in itself, & when n is very small.

The second takes place to the contrary particularly when b' is a very small sum, & when n' is an extremely small quantity.

In the first case, although, if the game were supposed to be repeated a very great number of times, it was favorable to A, however he will refuse to play it; 1.° because he can not continue it a great enough number of times; 2.° because for a single trial he has a very great probability to lose his stake, & by hypothesis, to make a loss which inconveniences him, or which deprives him of agreeable enjoyments.

In the second, B agrees to play, because the small sum b' is a very moderate sum of which he does not regret the loss, & of which the expectation to win the considerable sum $\frac{b'}{n'}$, engages him to expose himself, even with disadvantage, to this loss regarded as light: this is here the case of the lotteries.

There are some games where the strength of the players is not equal, & where one gives advantage to a banker; as the banker is obliged to play a very considerable game, which requires some advances, & exposes to the possibility of enormous losses; which besides he is subject for the stakes, to be submitted, with certain limits, at the will of the punters; & that finally if he would have no advantage, he would have, especially when the number of punters is great, & when they play very nearly the same game, a very great probability to make only very little loss & gain: it has appeared necessary to accord to him an advantage which gave to him an assurance to win at length; & the punters have consented to buy at this price the pleasure to play, & to conduct their game at their whim up to a certain point.

11. One has observed that among the games which depend altogether on chance & on good play, the ones had only a very short duration, while others conserved their vogue a very longtime: one of the causes of this difference, is the way to combine in these games, the influence of chance & of good play, so that the difference in strength of the players, when it is small, alters not at all sensibly in the two or three games which one wishes to play in a day, the equality of the probability to win, as they could have among them of some equal players. If one gives too much to chance, one takes off to these games a great part of their pleasure; if the chance influences too little, the difference of strength becomes too sensible, it humiliates the self-respect.

We will note finally, that in the enterprises where men expose themselves to a loss in view of a profit, it is necessary that the profit be greater than the one which follows the general rule, it establishes equality: indeed, as in general one is delivered not at all as in the game, by the appeal of the pleasure to play, or as in the lotteries, by the expectation to win much with a small stake, one can have motive to risk, only an advantage which, by envisioning a series of similar risks, produces an assurance great enough to win, & a probability nearly equal to certitude of no loss at all beyond a certain part of the stake.

These reflections have appeared to us proper to accommodate the rule established in the calculus of probabilities, with the sentiment & with the behavior of reasonable & prudent men, in most of the cases where this rule would appear at first glance to be contrary to it.

III.

1. Until here we have regarded the number of combinations which give each event as determined & known. We are going now to suppose this number unknown & variable, so that it has no longer a determined *probability* of the events, but only a mean *probability* according to which one can determine that of their production.

2. Suppose, for example, that one has an urn containing some black balls & some white balls, that one has drawn n white & m black balls, & that one demands what is the *probability* to draw p white balls & q black balls.

Suppose moreover that the number of these balls is infinite, so that the ratio of the white balls, to the total number, can have all the values from 1 to 0. Let x be this unknown ratio, the *probability* to draw first n white balls, & m blacks, & next p whites & q blacks, will be

$$\frac{m+n\ldots n+1}{1.2\ldots m}\,\frac{p+q\ldots p+1}{1.2\ldots q}\,x^{m+p}\,\overline{1-x}^{p+q};$$

& that to draw first n white balls, & m blacks, & next $p+q$ white or black balls, will be

$$\frac{m+n\ldots n+1}{1.2\ldots m}\,x^{n}\overline{1-x}^{m}\cdot(x+1-x)^{p+q}=\frac{m+n\ldots n+1}{1.2\ldots m}x^{n}.\overline{1-x}^{m},$$

where x can have all the values from 1 to 0; therefore

$$\frac{m+n\ldots n+1}{1.2\ldots m}\,\frac{p+q\ldots p+1}{1.2\ldots q}\,\overline{\int x^{n+p}\,\overline{1-x}^{m+q}dx}$$

will express the sum of the combinations which give the event demanded, &

$$\frac{m+n\ldots n+1}{1.2\ldots m}\,\overline{\int x^{n}\,\overline{1-x}^{m}dx}$$

the sum of all the possible combinations; the integrals being taken from 1 to 0, the *probability* will be therefore

$$\frac{p+q\ldots p+1}{1.2\ldots q}\,\frac{\overline{\int x^{n+p}\,\overline{1-x}^{m+q}dx}}{\overline{\int x^{n}\,\overline{1-x}^{m}dx}}$$
$$=\frac{p+q\ldots p+1}{1.2.3\ldots q}\,\frac{n+1\ldots n+p.m+1\ldots m+q}{n+m+2\ldots n+m+p+q+1}.$$

3. If $n>m$, & if one demands the *probability* that, in the sequence of events, the number of white balls will surpass that of the blacks, by a determined quantity, one will find, 1.° that this *probability* can never approach indefinitely to 1; 2.° that, following the hypotheses of plurality, it can, after having been increasing, become decreasing; 3.° that after a certain term, it will continue indefinitely to approach the function

$$\frac{\overline{\int x^{n}\,\overline{1-x}^{m}dx}^{\frac{1}{2}}}{\overline{\int x^{n}\,\overline{1-x}^{m}dx}}>\frac{1}{2}$$

the formula $\overline{\int x^{n}\,\overline{1-x}^{m}dx}^{\frac{1}{2}}$ indicating that the integral is taken only from $x=1$ to $x=\frac{1}{2}$. This formula indicates the value of the *probability*, when the number of drawings is supposed infinite.

In the case of $m < n$, the same *probability* is expressed by the same formula; but then one has

$$\frac{\overline{\int x^n \overline{1-x}^m dx}^{\frac{1}{2}}}{\int x^n \overline{1-x}^m dx} < \frac{1}{2}.$$

Likewise

$$\frac{\overline{\int x^n \overline{1-x}^m dx}^{\frac{p}{p+q}}}{\int x^n \overline{1-x}^m dx}.$$

The integral of the formula above being taken from $x = 1$, to $x = \frac{p}{p+q}$, will express the *probability* that, in the sequence, supposed infinity of drawings, the number of the white balls will be to that of the black balls in a ratio greater than that of p to q, a *probability* that is greater or lesser than $\frac{1}{2}$, according as one will have $\frac{n}{n+m} >< \frac{p}{p+q}$. Finally p' being greater than p, the *probability* that the ratio of the number of *A* to that of *N*, will be between the limits $\frac{p'}{p+q}$ & $\frac{p}{p+q}$, will be expressed by

$$\frac{\overline{\int x^n \overline{1-x}^m dx}^{\frac{p'}{p+q}} - \overline{\int x^n \overline{1-x}^m dx}^{\frac{p}{p+q}}}{\int x^n \overline{1-x}^m dx}$$

5.° If one applies these formulas to the natural events of which the order has been known by observation, one will find, 1.° that, if the question is of a constant observed order, one will never have a probability 1, or the certitude that it will continue to be in perpetuity, whatever be the number of observations; 2.° that, if one demands that there is only a fraction $\frac{1}{a}$ of the total number which deviate from this order, the probability will be expressed by $1 - \left(\frac{a-1}{a}\right)^{m+1}$, a probability so much greater, as a is smaller, & m greater.

3.° That, if there is a question only of absolute or proportional plurality, observed between the events, the probability that this plurality will continue indefinitely, will be expressed by

$$\frac{\overline{\int x^{a+b} \overline{1-x}^a dx}^{\frac{1}{2}}}{\int x^{a+b} \overline{1-x}^a dx}$$

in the first case,

$$\frac{\overline{\int x^{ca} \overline{1-x}^a dx}^{\frac{c}{c+1}}}{\int x^{ca} \overline{1-x}^a dx};$$

in the second; therefore the first is so much greater as a & b are great, & the second so much greater as a is great; finally that which it will be between

$$\frac{c'}{c+1} < \frac{c}{c+1} \quad \& \quad \frac{c_\prime}{c+1} > \frac{c}{c+1},$$

is

$$\frac{\overline{\int x^{ca} \overline{1-x}^a dx}^{\frac{c_\prime}{c+1}} - \overline{\int x^{ca} \overline{1-x}^a dx}^{\frac{c'}{c+1}}}{\int x^{ca} \overline{1-x}^a dx}$$

If one wishes only a proportional plurality which is not smaller than that of c to 1, then the probability will be

$$\frac{\overline{\int x^{ca}\,\overline{1-x}^{a}dx}^{\frac{c'}{c+1}}}{\int x^{ca}\,\overline{1-x}^{a}dx}.$$

4. Such are the very narrow limits of the kind of probability which we can have respecting the order of the future events, & the constancy of the observed laws in nature, at least under the two hypotheses; 1.° that the probability of the successive events is always the same; 2.° in this where one considers a class of events as isolated, & the order that one observes as independent of that which is observed in some other events.

Suppose, for example, that, in one same city, one has observed that there are born more boys than girls; that this observation has been confirmed during a great enough number of years without that there have been great variations in the proportion of these numbers, & without that one suspected any considerable change in the constitution of the climate, or of the inhabitants, then one could reasonably regard as constant the probability that there will be born a boy rather than a girl; &, as one does not know *a priori* if, in the laws of nature which determine this production, there is one of them by virtue of which there must exist a constant superiority in favor of one of the sexes. The two suppositions above can be regarded as legitimate. Thus, let $a+b$ be the number of boys, & a the number of girls.

$$\frac{\overline{\int x^{a+b}\,\overline{1-x}^{a}dx}^{\frac{1}{2}}}{\int x^{a+b}\,\overline{1-x}^{a}dx}$$

will express the probability that, all remaining in the same state, there will be born in an indefinite time more boys than girls, or that there is a physical & real cause of this superiority of number. *See* below no. 10 & the following, & the article VÉRITÉ.

5. One has often applied the ordinary calculus of the probabilities to some questions of the public economy where the question of payments of which the period or the duration could depend on this calculus. One could suppose then that the observed ratios in the past events would be rigorously the same in the future events; a hypothesis more or less near to the truth, according as the number of these past events is great, & as the one of the future events is small, but which is extended in a very sensible manner in the case where the ratio of the number of the firsts to that of the seconds is not very great.

It can therefore be useful to apply to these same questions the method which we just exposed.

We are going to give some very simple examples, but sufficient to make sensible the manner to resolve the questions of this kind.

6. Let be, 1.° a man aged n years, & let, out of p men of the same age, one have observed that there are p' who have lived q years, $\overline{n+q}$ is here the last term of the life of the human kind p'' who have lived $q-1$ years. . . p''^{q} who have lived one year only, & let one demand the life annuity which it is necessary to give to this man for a sum 1, the rate of interest being supposed given. Let c be the value which, placed at this interest, is worth 1 at the end of one year, a the life annuity, it is clear that the value of

this pension, for the one who lived one year, will be ca, $a.\overline{c+c^2}$ for the one who lived two, $ac\frac{1-c^q}{1-c}$ for the one who lived q.

Let finally x' be the unknown probability of living q years, x'' that of living $q-1$ years... x''^q that of living one year only, the mean value of the pension will be expressed by

$$\begin{aligned} &a.c\frac{1-c^q}{1-c}\,x'^{p'+1}x''^{p''}\cdots x''^{q^{p''q}} \\ &\quad+ ac.\frac{1-c^{q-1}}{1-c}x'^{p'}x''^{p''+1}\cdots x''^{q^{p''q}} \\ &\quad+ a.c.x^{p'}x''^{p''}\cdots x''^{q^{p''q}+1} \end{aligned}$$

The whole divided by $x'^{p'}x''^{p''}\cdots x''^{q^{p''q}}$. Consequently, in order to have the mean value of all the values of x, it will be necessary to integrate separately the numerator & the denominator a number $q-1$ times, after having made $x''^q = 1-x'\cdots - x''^{q-1}$, & having taken the integrals from $x''^{q-1} = 1-x'\cdots - x''^{q-2}$ to $x''^{q-1} = 0$, from $x''^{q-2} = 1-x'\cdots - x''^{q-3}$, to $x''^{q-2} = 0\ldots$ from $x' = 1$ to $x' = 0$.

Now all these operations being executed, this formula becomes

$$\frac{a\left(c.\overline{p''^q+1}+\overline{c+c}^2.\overline{p''^{q-1}+1}\cdots + c.\frac{1-c^{q-1}}{1-c}\overline{p''+1}+c.\frac{1-c^q}{1-c}\overline{p'+1}\right)}{p+q}$$

By putting back p in place of $p'+p''\cdots+p''^{q-1}+p''^q$, one will deduce from it therefore

$$a = \frac{1.p+q}{c.\overline{p''^q+1}+\overline{c+c}^2.\overline{p''^{q-1}+1}\cdots c.\frac{1-c^{q-1}}{1-c}\overline{p''+1}+c.\frac{1-c^q}{1-c}\overline{p'+1}}$$

instead that, following the ordinary method, one would have had

$$a = \frac{1.p}{c.p''^q+\overline{c+c}^2.p''^{q-1}\cdots c.\frac{1-c^{q-1}}{1-c}p''+c.\frac{1-c^q}{1-c}p'}$$

that is to say, the same formula, if one suppose infinite the numbers $p', p'', \ldots$ whence one sees that, as here $n+q$ being the greatest number of years of human life, can not be a very great number, if the one of the observations is to the contrary a very great number, the error of the ordinary method will be slightly sensible.

7. In this method, as in the ordinary method, if one supposes r persons of the same age, lending each a sum 1, & if one demands the pension a which it is necessary to pay to each, in order that the borrower gives only the interest supposed here given, one will have the same value of a as above; thus, however great that r be, there will be nothing to change in the value of a, only the probability that the borrower has to not pay in reality, according to the future events, an interest neither sensibly above, nor sensibly below, will increase less under the actual hypothesis than under the ordinary hypothesis.

8. If one considers the pensions on all heads, one can follow according to the following method; one will take, as above, for each age n the value a of the life annuity, since one will take in the tables of investments in life annuities of all heads, the number

not of men of each age, but of the unities of a sum placed on the heads of each age, let r', r'', $\ldots r''^{q}$ be these numbers, a', a'', $\ldots a''^{q}$ the corresponding pensions. The value a of the common pension will be expressed by

$$\frac{c'.\overline{r'+1}+a''.\overline{r''+1}\cdots a''^{q}.\overline{r''^{q}+1}}{r'+r''\cdots+r''^{q}+q}$$

This manner to calculate the rate respecting the pensions is not more complicated, & it is more exact than the ordinary method. It will be applied equally to the pensions on two heads, & even with more facility than the ordinary method.

We have here considered only entire years; but the way to make the fractions of years enter into the calculation, as for the payment of pensions, or even as for the ages, has another difficulty only to require some slightly longer calculations.

9. We will choose here for a second example the evaluation of an eventual claim; suppose therefore a single claim, & let p', p'', $\ldots p''^{n}$ express the values of this claim, such as they have been observed during a certain number of years, let the claim p' have been paid in b' years, the claim p'' in b'' years, the claim p''^{n} in b''^{n} years, & let x', x'', $\ldots x''^{q}$ be the probabilities that the following year one must pay p', p'', $\ldots$ or p''^{n}.

It is clear that, setting $1-x'-x''\cdots x''^{n-1}$, instead of x''^{n} the total value of the claim for an infinite number of years will be expressed by

$$\frac{\frac{c}{1-c}\int x'b'\cdots(1-x'\cdots x''^{r-1})b''^{n}(p'x'+p''x''\cdots+p''^{n}.1-x'\cdots x''^{n-1})\,dx'\,dx''\cdots dx''^{n-1}}{\int x'^{b'}x''^{b''}\cdots(1-x'\cdots x''^{n-1})b''^{n}dx'\,dx''\cdots dx''^{n-1}}$$
$$=\frac{c}{1-c}\,\frac{\overline{b'+1}.p'+\overline{b''+1}.p''\cdots+\overline{b''^{n}+1}.p''^{n}}{b'+b''\cdots b''^{n}+n}$$

Following the ordinary method to take some means, this value would be

$$\frac{c}{1-c}\,\frac{b'p'+b''p''\cdots b''^{n}p''^{n}}{b'+b''\cdots b''^{n}}$$

whence one sees that these two values cannot be regarded as very different, at least when the b are very nearly equal or very great, with respect to n, or that finally the p are also very nearly equal.

In order to apply this method to the claim due on a single good, it would be necessary to divide this total sum in the ratio of this good to the one of the total mass of the goods which have produced the claims p', p'', $\ldots p''^{n}$; &, if one has taken it in order to form the mass of the goods of the same nature, & before nearly to be exposed to the same events as the one for which one seeks the value of the claim, this method can be regarded as sufficiently exact.

10. We have regarded until here the probability as being constantly the same in the sequence of events of a like nature. There are some cases where this assumption can appear gratuitous. Suppose now that these events can be independent from one another, as to be subject to preserve the same probability, & we preserve always here the denominations of the second kind, if the probability is constant, that to have the

event *A*, after having obtained *A*, n times, & *N*, m times will be $\frac{n+1}{n+m+1}$; but, if the events are independent, this same probability will be expressed by $\int xdx$, the integral being taken from $x = 1$ to $x = 0$, that is to say, by $\frac{1}{2}$. But the probability to have *A*, n times & *N*, m times under the first hypothesis, is

$$\frac{n+m\ldots n+1}{1.2\ldots m}\int x^n\overline{1-x}^m dx$$

& under the second it is

$$\frac{n+m\ldots n+1}{1.2\ldots m}\overline{\int xdx}^n\,\overline{\int \overline{1-x}dx}^m$$

These two probabilities will be therefore as $\frac{m.m-1\ldots 1}{n+1.n+2\ldots n+m+1}$ to $\frac{1}{2}^{n+m}$ & consequently the mean probability *A* will be

$$\frac{\frac{m+1.m\ldots 1}{n+1.n+2\ldots n+m+2}+\frac{1}{2^{n+m+1}}}{\frac{m.m-1\ldots 1}{n+1.n+2\ldots n+m+1}+\frac{1}{2^{n+m}}}$$

This hypothesis can be applied to some questions, for example, if one has cast a coin $n+m$ times if one has heads m times & tails n times, $n > m$, & if one demands the probability to bring forth heads: this probability will be expressed by the preceding formula, if I have no other reason besides to suppose that the reason for which I have had heads more often than tails, holds to this that the coin is constructed in a manner to fall more often on this side, rather than to regard this superiority as produced by the manner of casting the coin, which is without influence on the following trials. One sees finally that there are some cases where however great that m & n be, & although n has then the superiority to m, one has no *probability* which leads to suppose a constant law.

11. Now consider more closely the nature of the first hypothesis, we will find first that that it is legitimate only in two cases: 1.° when the *probability* of each event is always the same, as when one draws some black or white balls always from one same urn; 2.° when drawing them from different urns, one supposes that these urns have been replenished by taking some balls in a common mass, where they were in a certain ratio. In the first case, it is the *probability* itself which is constant; & in the second, it is only the mean *probability*.

For example, suppose a sequence of packs of cards in number $r+s$, that from r of these packs one has drawn n red & m black cards, & that one demands the *probability* to draw p red & q black cards of $p+q=s$ following packs, one can suppose either that one knows that in each of these packs there are the same number of red cards & of black cards, or else that these packs have all been formed by drawing them at random from one same pile of cards, in this case the *probabilities* are the same in each pack, & one could express this by having p red cards, & q by the preceding formula; but if one knows not in advance the reality of one of these two hypotheses, nor that of the contrary hypothesis, that is to say, that where there is no liaison between the *probability*, relative to each of the piles of cards, & that one demands the expression of the same *probability*, one will take the following method, let $r+s=t$ & x', x'', x''' $\ldots x''^{t}$, t different probabilities of the event *A* & making $\frac{x'+x''+x'''\cdots+x''t}{t} = X$ & $\frac{t-x'-x''-\cdots-x''t}{t} = X'$, X will express the mean *probability* of *A*, X' the mean *probability* of *N* &

$$\frac{s.s-1\ldots p+1}{1.2\ldots q}\,\frac{\int X^{n+p}X'^{m+q}dx'dx''\ldots}{\int X^{n}X'^{m}dxx'dx''\ldots}$$

the *probability* to have p, *A* & q, *N* after having had n, *A* & m, *N*, the integrals being taken successively for each x from 1 to zero.

It is easy to see, by examining this formula, that if $n > m$ one will have a greater *probability* in favor of *A* than in favor of *N*, that this *probability* will be so much greater as m & n will be greater, & that if one seeks the *probability* in the indefinite sequence of future events, the number of *A* will surpass that of *N*, one will have a *probability* so much greater in favor of *A*, as n will surpass m, & that these numbers will be greater, although this *probability* is much smaller than if one had supposed x the same for all the events.

12. In all preceding formulas we have supposed the *probabilities* the same. In some order that the events succeeded themselves, or that which reverts to the same, we have supposed whether variable or constant the value of these *probabilities* was independent of the order in which the events succeeded each other.

But this hypothesis, far from being admitted necessary in every case, appears, in many, contrary to that which the simple reason seems to indicate. Suppose indeed that out of one sequence of twenty thousand successive acts[4] *A* or *N*, the number of *A* has surpassed the one of *N* by 300, & that one demands the *probability* that in two hundred following acts the number of *A* will surpass the number of *N*, it is easy to see that one must naturally regard this conclusion as more probable, if in one hundred sequences of two hundred events each the number of *A* surpasses that of *N*, than if after having surpassed by much in the first ones, this difference moreover was diminished successively, in a manner that, in the following sequences, the plurality has been in favor of *N*. One must therefore in general, & if one has not *a priori* some reason to adopt another hypothesis, to regard the *probability* not only as dependent on the events, but also as dependent on the order that they follow among them. For this one will designate the *probabilities* of *A* & *N*, for the successive events by

	A	B
1	x	$1-x$
2	$\frac{x+x'}{2}$	$\frac{1-x-x'}{2}$
3	$\frac{x+x'+x''}{3}$	$\frac{1-x-x'-x''}{3}$
4	$\frac{x+x'+x''+x'''}{4}$	$\frac{1-x-x'-x''-x'''}{4}$

& thus in sequence: one will take the sequence of products of these *probabilities* corresponding to those of the observed events, & of those which one supposes must succeed

[4] *Translator's note:* Condorcet usually uses the word *événemen* for an event. However, here he introduces the word *fait* for the same purpose. To distinguish the two here, the latter is rendered as act.

them & of which one seeks the *probability*; one will take successively the integral of it with respect to each x from 1 to zero; one will divide this value by a similar product, taken only for the observed events & integrated in the same manner, & the quotient will express the *probability* sought from the given sequence of future events.

13. In this last formula the *probability* depends on the order of the events, & it is rigorously this that it must be, independently of all hypotheses, since the *probability* to have a *probability*, either the same or different for all the events is itself determined only according to the order of past events. But, if one wishes to apply this method to some researches on natural phenomena, it is necessary to make the following two observations: 1.° that the preceding formula, varying for all the possible dispositions, will be with more difficulty subject to some general methods of calculation. 2.° That if one has the x dependent on the time following any one law, there would be still more exactitude to seek, in the sequence of events, a certain combination of these events, a certain law which is absolutely constant, & of which then one can regard the constancy as independent of time; & instead to apply the calculus to the production of the events themselves, this would be in the constancy of this general law which one would apply to it.

14. In following the methods proposed above, one will find that the *probability* deduced from the calculus for the constancy of an observed law is very small; but it is necessary to remark, 1.° that this *probability*, if the number is very great, will give a *probability* rather strong in the constancy of the law observed for a certain period of time; that if during this time the observation teaches that the law has continued to be constant, one will have a greater *probability* than it will be observed in the future during an equal period of time, & thus in sequence. If on the contrary it ceases to be executed, then it will be necessary to seek a new general law constantly executed. 2.° One can moreover consider the *probability* of the constancy of each law, not in a single sequence of events of the same nature, but in the total sequence of observed events.

Let n, for example, be the number of classes of events *A', A"* . . .*A"*^{n} of diverse nature subject to a constant law, & p', $p''\dots p''^{n}$ the one of the events observed in each class, & let one seek the *probability* that in the class of *A"*^{n}, for example, the constant law will continue to be observed.

We will suppose that the *probability* of each event *A', A", A"'* . . .*A"*^{n} is expressed by $\frac{x'+x''\cdots+x''^{n}}{n}$. The assumption of a mean *probability*, the same for these events, is here legitimate, because it is not that of the event *A'* or *A"* by itself which we will consider, but that of the natural event which is itself presented first, rather than the contradictory event.

We will form next for *A'*, the sequence

$$P' = \frac{x' + x''\cdots + x''^{n} + x_{\prime}''^{n+1}}{n+1}$$
$$\frac{x' + x''\cdots + x''^{n} + x_{\prime}''^{n+1} n + r' + 1 - x_{\prime}'^{n+2}}{n+2}$$
$$\frac{x'\cdots + x''^{n} + x_{\prime}''^{n+1} + x_{\prime}'^{n+r-1}}{n+r'-1}$$

for the other *A* of the similar sequence to

$$
\begin{gathered}
P' = \frac{x' + x'' \cdots + x''^{n} + x_{\prime}''^{n+1}}{n+1} \\
\frac{x' + x'' \cdots + x''^{n} + x_{''n}''^{n+1} + x_{''n}'^{n+2} \cdots}{n+2} \\
\frac{x' + x'' \cdots + x''^{n} + x''^{n+1} + x'^{n+2} \cdots + x''^{n+1-1}}{n+r-1}
\end{gathered}
$$

one will take next

$$
\begin{gathered}
Q = \frac{x' + x'' \cdots + x''^{n} + x_{''n}''^{n+1} + x_{''n}''^{n+1}}{n+r} \\
\frac{x' + x'' \cdots + x''^{n} + x_{''n}''^{n+1} + x_{''n}'^{n+r+1} \cdots}{n+1} \\
\frac{x' + x'' + x''^{n} + x_{''}''^{n+1} \cdots + x''^{n+r+s-1}}{n+r+s-1}
\end{gathered}
$$

s being the number of events *A* “^{n} for which one demands that the law is observed, & the *probability* will be expressed by

$$\frac{\int P'P'' \cdots P''^{n}\, Q\, dx'dx'' \cdots}{\int P'P'' \cdots P''^{n}\, dx'dx'' \cdots}$$

the integrals being taken for all the x from 1 to zero. We suppose here some x different for each event *A'*, as this is necessary for the case where the *probabilities* are independent of those of the first events which form the succession.

If one wishes to employ the same method, by supposing the x always the same, then this method would give, for the *probability* of a new event, *A* “$^{n+1}$

$$\frac{r' + r'' + r''' \cdots + r''^{n} + t - n}{r' + r'' + r''' \cdots + r''^{n} + p - n}$$

instead of the

$$\frac{r' + r'' + r''' \cdots + r''^{n} + t}{r' + r'' + r''' \cdots + r''^{n} + p}$$

which the assumption of the same continued in a single event will give.

One sees how often these last results are brought together by the common reason, & they can explicate how we can believe in the permanence not of the act alone, but of the permanence observed in all the analogous acts, or of that which has struck us in all the phenomena of nature.

15. There remains to us to examine still another hypothesis: suppose that the total mass of the observed events *A* & *N* are divided into two sequences s & s', for the one of which one has n events *A*, & m events *N*, while for the other one has n' events *A*, & m' events N; if besides one has some reasons to believe that the value x of the *probability* of *A* is not the same in the two sequences, then, for example, it will not be necessary to resolve the questions respecting the *probability* of future events, as if one

had had $n + n'$ events *A*, & $m + m'$ events N; but to suppose, for the two sequences, two different *probabilities* x & x'. One could make next the calculation by supposing, according to the circumstances, either that the future events can *equally* belong to the two sequences, or that a certain combination of events *A* or *N* must belong to one of these two sequences rather than to the other, according to the manner in which the events have found themselves in the same sequences, or finally that these events belong more or less to one of these sequences, by reason of the number of observations of which they are formed.

One can form still some other hypotheses, of which each must merit preference, according to the nature of the particular questions which one proposes to resolve.

One must observe in general that immediately if the question is to have by the calculus, either a mean value, or a mean *probability*, one must resort to the direct calculation only after having sought to procure all possible knowledge that it is possible to acquire respecting the real value of these quantities, & to not regard as equally possible, as equally probable, only those among which a thorough examination permits not at all to suppose differences; the calculation of the *probabilities* is never indeed but a supplement to our ignorance of real events, of the real laws observed in nature; & it is this which one must never forget in the applications of this calculus.

16. One can propose this general question: the *probability* of an act being given only by observation, to find how often it is necessary to observe in order to have a certain *probability*, that one sequence of a given number of future events will deviate from the result of these observations as in a given ratio also, or in general knowing two of these three elements, to have the third. It is clear that it will be easy to deduce from the formulas exposed above, the equations of which one will have need, in order to resolve these different questions & those of the same genre. We will expose in the article *vérité* some necessary principles in order to apply the calculus of *probabilities* to the discovery & to the proof of natural acts, & we will give in the article *votans* the application of the same principles to the *probability* of the decisions rendered in the plurality of votes. The theory exposed in this third article is yet little known. Messers. Price & Bayes have given the fundamental principles of it in the *Philosophical Transactions* of the years 1764 & 1765. Mr. de Laplace has treated the first analytically, & has made many scholarly applications of it in the *Mémoires de l'académie des sciences*. One will find also some reflections on the same subject in the work that I have published respecting the *probability* of decisions, & in some memoirs inserted in the volumes of the academy, years 1781, 1782 & 1783.