Sur Deux Théorèmes Relatifs aux Probabilités*

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§1. In a Memoir, under the title: *Des valeurs moyennes*, we have shown how on obtains some *inequalities*, whence one deduces easily a theorem on the probabilities which contains as particular case the theorem of Bernoulli and the law of large numbers.

This theorem, we have formulated thus:

If the mathematical expectations of the quantities

$$u_1, \quad u_2, \quad u_3, \ldots,$$

and their squares

$$u_1^2, \quad u_2^2, \quad {}_3^2, \ldots$$

do not surpass any finite limit, the probability that the difference between the arithmetic mean of a number n of these quantities and the arithmetic mean of their mathematical expectations will be less than a given quantity, is reduced to unity, when n becomes infinite.

We have been led to this result by seeking to determine the limit values of an integral according to the given values of other integrals which contain under the integral sign $\int$, besides known functions, an unknown function, subject to the single condition of not becoming negative between the limits of integration. By developing the method employed in these researches we arrive, in a particular case, to the following theorem on the integrals.[1]

If the function $f(x)$ remains constantly positive and if one has

$$\int_{-\infty}^{+\infty} f(x)\,dx = 1, \quad \int_{-\infty}^{+\infty} xf(x)\,dx = 0, \quad \int_{-\infty}^{+\infty} x^2 f(x)\,dx = \tfrac{1}{q^2}, \quad \int_{-\infty}^{+\infty} x^3 f(x)\,dx = 0,$$
$$\cdots\cdots$$
$$\int_{-\infty}^{+\infty} x^{2m-2} f(x)\,dx = \tfrac{1.3.5...(2m-3)}{q^{2m-2}}, \quad \int_{-\infty}^{+\infty} x^{2m-1} f(x)\,dx = 0,$$

the value of the integral

$$\int_{-\infty}^{v} f(x)\,dx$$

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[1] "Sur les résidus intégraux qui donnent des valeurs approchées des intègrales." *Acta Mathematica* T. XII, 1888–1889, p. 287–322.

will be comprehended between the limits

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\frac{qv}{\sqrt{2}}} e^{-x^2}\,dx \quad - \quad \frac{3\sqrt{3}(m^2-2m+3)^{\frac{3}{2}}(q^2v^2+1)^3}{2(m-3)^3\sqrt{m-1}},$$
$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\frac{qv}{\sqrt{2}}} e^{-x^2}\,dx \quad + \quad \frac{3\sqrt{3}(m^2-2m+3)^{\frac{3}{2}}(q^2v^2+1)^3}{2(m-3)^3\sqrt{m-1}},$$

for all the real values of v.

We are going to show now, how this theorem on the integrals leads to a theorem on the probabilities, by aid of which the determination of the most certain values of the unknowns, when one has a great number of equations which contain some accidental errors more or less considerable, is brought to the method of *least squares*.

This theorem is able to be formulated thus:

If the mathematical expectations of the quantities

$$u_1, \quad u_2, \quad u_3, \ldots,$$

are all null and if the mathematical expectations of all their powers do not surpass some finite limit, the probability that the sum

$$u_1 + u_2 + u_3 + \ldots + u_n,$$

of a number n *of these quantities, divided by the square root of the double sum of the mathematical expectations of their squares, will be comprehended between any two limits* t *and* t'*, is reduced to*

$$\frac{1}{\sqrt{\pi}}\int_t^{t'} e^{-t^2}\,dx.$$

when the number n *becomes infinite.*

§2. In order to demonstrate this theorem under the most general form, we take $-\infty$ and $+\infty$ for the limits between which are comprehended all the possible values of the quantities

$$u_1, \quad u_2, \quad u_3, \ldots,$$

In designating by

$$\phi_1(x)\,dx, \quad \phi_2(x)\,dx, \quad \phi_3(x)\,dx, \ldots$$

the probabilities that the values of the quantities

$$u_1, \quad u_2, \quad u_3, \ldots,$$

are comprehended between the infinitely near limits

$$x, \quad x + dx,$$

we note:

1) that the functions

$$\phi_1(x)\,dx, \quad \phi_2(x)\,dx, \quad \phi_3(x)\,dx, \ldots$$

are not able to have negative values;

2) that the integrals

$$\int_{-\infty}^{+\infty} \phi_1(u_1)\,du_1, \quad \int_{-\infty}^{+\infty} \phi_2(u_2)\,du_2, \quad \int_{-\infty}^{+\infty} \phi_3(u_3)\,du_3, \ldots,$$

which represent the probabilities that the quantities

$$u_1, \quad u_2, \quad u_3, \ldots,$$

will have any values at all comprehended between the limits $-\infty$ and $+\infty$, will be equal to unity;

3) that the integrals

$$\int_{-\infty}^{+\infty} u_1\phi_1(u_1)\,du_1, \quad \int_{-\infty}^{+\infty} u_2\phi_2(u_2)\,du_2, \quad \int_{-\infty}^{+\infty} u_3\phi_3(u_3)\,du_3, \ldots,$$

which represent the mathematical expectations of the quantities

$$u_1, \quad u_2, \quad u_3, \ldots,$$

according to our hypothesis, must be null;

4) that in general, all the quantities $a_i^{(\mu)}$ defined by the equality

$$a_i^{(\mu)} = \int_{-\infty}^{+\infty} u_i^{\mu}\phi_i(u_i)\,du_i,$$

which represent the mathematical expectations of the different powers of the quantities

$$u_1, \quad u_2, \quad u_3, \ldots,$$

will be, by hypothesis, comprehended between some finite limits.

On the other hand, in designating by

$$f(x)\,dx$$

the probability that the value of the fraction

$$\frac{u_1 + u_2 + \ldots + u_n}{\sqrt{n}}$$

will be comprehended between the infinitely near limits x_1, $x + dx$. we will note that this probability will be given by the equality

$$f(x)\,dx = \int \ldots \int \phi_1(u_1)\phi_2(u_2)\ldots\phi_n(u_n)\,du_1 du_2 \ldots du_n,$$

in which the integration with respect to u_1, u_2,… u_n is extended over all values of these quantities for which the value of the fraction

$$\frac{u_1 + u_2 + \ldots + u_n}{\sqrt{n}}$$

does not exit some infinitely near limits x, $x + dx$.

By multiplying this equality member by member with the equality

$$e^{sx} = e^{\frac{s(u_1+u_2+\ldots+u_n)}{\sqrt{n}}},$$

where s designates any arbitrary constant, and by integrating from $s = -\infty$ to $s = \infty$, we find the equality

$$\int_{-\infty}^{+\infty} e^{sx} f(x)\, dx = \int \ldots \int e^{\frac{s(u_1+u_2+\ldots+u_n)}{\sqrt{n}}} \phi_1(u_1)\phi_2(u_2)\ldots\phi_n(u_n)\, du_1 du_2 \ldots du_n.$$

As in the second member the integration with respect to

$$u_1, \quad u_2 \ldots, \quad u_n$$

is extended over all the values of these quantities between $s = -\infty$ to $s = \infty$, the second member of this equality is reduced to the product of the simple integrals

$$\int_{-\infty}^{+\infty} e^{\frac{su_1}{\sqrt{n}}} \phi_1(u_1)\, du_1 . \int_{-\infty}^{+\infty} e^{\frac{su_2}{\sqrt{n}}} \phi_2(u_2)\, du_2 \ldots \int_{-\infty}^{+\infty} e^{\frac{su_n}{\sqrt{n}}} \phi_n(u_n)\, du_n;$$

we will have therefore the equality

(1)

$$\int_{-\infty}^{+\infty} e^{sx} f(x)\, dx = \int_{-\infty}^{+\infty} e^{\frac{su_1}{\sqrt{n}}} \phi_1(u_1)\, du_1 . \int_{-\infty}^{+\infty} e^{\frac{su_2}{\sqrt{n}}} \phi_2(u_2)\, du_2 \ldots \int_{-\infty}^{+\infty} e^{\frac{su_n}{\sqrt{n}}} \phi_n(u_n)\, du_n;$$

In developing the two members of this equality according to the powers of the arbitrary constant s and by equating the coefficients of the same powers of s, we will find the values of the integrals

$$\int_{-\infty}^{+\infty} f(x)\, dx, \quad \int_{-\infty}^{+\infty} x f(x)\, dx, \quad \int_{-\infty}^{+\infty} x^2 f(x)\, dx, \ldots,$$

which represent the mathematical expectations of the different powers of the quantity

$$x = \frac{u_1 + u_2 + \ldots + u_n}{\sqrt{n}},$$

and which will serve us to determine the limit values of the integral

$$\int_{-\infty}^{v} f(x)\, dx,$$

which represents the probability that the value of the fraction

$$x = \frac{u_1 + u_2 + \ldots + u_n}{\sqrt{n}},$$

will not surpass any quantity v.

§3. By limiting ourselves to the determination of $2m$ integrals

$$\int_{-\infty}^{+\infty} f(x)\,dx, \quad \int_{-\infty}^{+\infty} xf(x)\,dx, \quad \int_{-\infty}^{+\infty} x^2 f(x)\,dx, \ldots, \int_{-\infty}^{+\infty} x^{2m-1} f(x)\,dx,$$

we note that the first member of the equality (1), developed according to the powers of s to the term of order s^{2m-1}, will give the sum

$$\int_{-\infty}^{+\infty} f(x)\,dx + \frac{s}{1}\int_{-\infty}^{+\infty} xf(x)\,dx + \frac{s^2}{1.2}\int_{-\infty}^{+\infty} x^2 f(x)\,dx + \ldots + \frac{s^{2m-1}}{1.2\ldots(2m-1)}\int_{-\infty}^{+\infty} x^{2m-1} f(x)\,dx.$$

In order to determine the corresponding terms in the second member, we are going to set it under the form

$$e^{\log \int_{-\infty}^{+\infty} e^{\frac{su_1}{\sqrt{n}}}\phi_1(u_1)\,du_1 + \log \int_{-\infty}^{+\infty} e^{\frac{su_2}{\sqrt{n}}}\phi_2(u_2)\,du_2 + \ldots + \log \int_{-\infty}^{+\infty} e^{\frac{su_n}{\sqrt{n}}}\phi_n(u_n)\,du_n},$$

and one sees that the development of the second member of (1), exact to the term of order s^{2m-1} inclusively, will be obtained by replacing the logarithms by their developments into series arrested at the terms of the order s^{2m-1}. In order to determine the developments of these logarithms, we will note that the integral

$$\int_{-\infty}^{+\infty} e^{\frac{su_i}{\sqrt{n}}}\phi_i(u_i)\,du_i,$$

will be given by the following approximative expression, exact to the term of order s^{2m-1} inclusively:

$$\int_{-\infty}^{+\infty} \phi_i(u_i)\,du_i \quad + \quad \frac{\int_{-\infty}^{+\infty} u_i\phi_i(u_i)\,du_i}{1.\sqrt{n}}s + \frac{\int_{-\infty}^{+\infty} u_i^2\phi_i(u_i)\,du_i}{1.2}s^2 + \frac{\int_{-\infty}^{+\infty} u_i^3\phi_i(u_i)\,du_i}{1.2.3n\sqrt{n}}s^3 + \ldots$$
$$\ldots + \frac{\int_{-\infty}^{+\infty} u_i^{2m-1}\phi_i(u_i)\,du_i}{1.2.3\ldots(2m-1)n^{m-1}\sqrt{n}}s^{2m-1},$$

and this expression, according to §2, is reduced to

$$1 + \frac{a_i^{(2)}}{2n}s^2 + \frac{a_i^{(3)}}{1.2.3n\sqrt{n}}s^3 + \ldots + \frac{a_i^{(2m-1)}}{1.2.3\ldots(2m-1)n^{m-1}\sqrt{n}}s^{2m-1},$$

where, by hypothesis, the quantities a_i are all finite.

In developing the logarithm of this expression according to the powers of s and by being arrested at the term of order s^{2m-1}, we will find an expression of the form

$$\frac{a_i^{(2)}}{2n}s^2 + \frac{V_i^{(3)}}{n\sqrt{n}}s^3 + \ldots + \frac{V_i^{(2m-1)}}{n^{m-1}\sqrt{n}}s^{2m-1},$$

in which the quantities

$$V_i^{(3)}, \ldots V_i^{(2m-1)}$$

will be some entire functions of the quantities

$$a_i^{(2)}, \quad a_i^{(3)}, \dots \quad a_i^{(2m-1)},$$

not containing n; they will be thus completely finite also.

One will have therefore, with a degree of approximation to the term of the order s^{2m-1} inclusively,

$$\log \int_{-\infty}^{+\infty} e^{\frac{su_i}{\sqrt{n}}} \phi_i(u_i)\, du_i = \frac{a_i^{(2)}}{2n} s^2 + \frac{V_i^{(3)}}{n\sqrt{n}} s^3 + \dots + \frac{V_i^{(2m-1)}}{n^{m-1}\sqrt{n}} s^{2m-1}.$$

By making $i = 1, 2, \dots, n$ and by putting

$$\begin{aligned}
\frac{1}{q^2} &= \frac{a_1^{(2)} + a_2^{(2)} + \dots + a_n^{(2)}}{n}, \\
M^{(3)} &= \frac{V_1^{(3)} + V_2^{(3)} + \dots + V_n^{(3)}}{n}, \\
&\dots\dots\dots\dots \\
M^{(2m-1)} &= \frac{V_1^{(2m-1)} + V_2^{(2m-1)} + \dots + V_n^{(2m-1)}}{n}.
\end{aligned}$$

one finds, with the same degree of approximation,

$$\begin{aligned}
\log \int_{-\infty}^{+\infty} e^{\frac{su_1}{\sqrt{n}}} \phi_1(u_1)\, du_1 \quad &+ \quad \log \int_{-\infty}^{+\infty} e^{\frac{su_2}{\sqrt{n}}} \phi_2(u_2)\, du_2 + \dots + \log \int_{-\infty}^{+\infty} e^{\frac{su_n}{\sqrt{n}}} \phi_n(u_n)\, du_n \\
&= \quad \frac{s^2}{2q^2} + \frac{M^{(3)}}{\sqrt{n}} s^3 + \dots + \frac{M^{(2m-1)}}{n^{m-2}\sqrt{n}} s^{2m-1},
\end{aligned}$$

where the quantities M, as the arithmetic means of the finite quantities

$$\begin{gathered}
V_1^{(3)}, \quad V_2^{(3)}, \dots \quad V_n^{(3)}, \\
\dots\dots\dots\dots \\
V_1^{(2m-1)}, \quad V_2^{(2m-1)}, \dots \quad V_n^{(2m-1)},
\end{gathered}$$

will be also completely finite.

One will have therefore, with a degree of approximation to the term of the order s^{2m-1},

$$\int_{-\infty}^{+\infty} e^{sx} f(x)\, dx = e^{\frac{s^2}{2q^2} + \frac{M^{(3)}}{\sqrt{n}} s^3 + \dots + \frac{M^{(2m-1)}}{n^{m-2}\sqrt{n}} s^{2m-1}}. \tag{2}$$

§4. Whatever be n, we will be able to draw from this equality the values of the $2m$ integrals

$$\int_{-\infty}^{+\infty} f(x)\, dx, \quad \int_{-\infty}^{+\infty} x f(x)\, dx, \dots, \int_{-\infty}^{+\infty} x^{2m-1} f(x)\, dx,$$

by arresting the developments of the two members of this equality at the terms of order s^{2m-1}.

In the particular case $n=\infty$, the equality (2) is reduced to

$$\int_{-\infty}^{+\infty} e^{sx} f(x)\,dx = e^{\frac{s^2}{2q^2}}.$$

since the quantities M, as one has seen, are completely finite.

By developing the two members of this equality according to the powers of s and by being arrested at the terms of order s^{2m-1}, one will have

$$\begin{aligned}\int_{-\infty}^{+\infty} f(x)\,dx \quad + \quad & \frac{s}{1}\int_{-\infty}^{+\infty} xf(x)\,dx + \frac{s^{2m-1}}{1.2\ldots(2m-1)}\int_{-\infty}^{+\infty} x^{2m-1}f(x)\,dx = \\ = \quad & 1+\frac{s^2}{2q^2}+\frac{s^4}{1.2..(2q^2)^2}+\ldots+\frac{s^{2m-2}}{1.2\ldots(m-1)(2q^2)^{m-1}}\end{aligned}$$

and, by equating the coefficients of the same powers of s, we find

$$\int_{-\infty}^{+\infty} f(x)\,dx = 1, \quad \int_{-\infty}^{+\infty} xf(x)\,dx = 0, \quad \int_{-\infty}^{+\infty} x^2f(x)\,dx = \tfrac{1}{q^2}, \quad \int_{-\infty}^{+\infty} x^3f(x)\,dx = 0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\int_{-\infty}^{+\infty} x^{2m-2}f(x)\,dx = \tfrac{1.3.5\ldots(2m-3)}{q^{2m-2}}, \quad \int_{-\infty}^{+\infty} x^{2m-1}f(x)\,dx = 0.$$

As the function $f(x)$ which figures under the sign $\int$, by its nature itself, is not able to have negative values, we conclude, according to the theorem on the integrals cited in §1, that the value of the integral

$$\int_{-\infty}^{v} f(x)\,dx$$

is comprehended between the limits

$$\begin{aligned}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\frac{qv}{\sqrt{2}}} e^{-x^2}\,dx \quad - \quad & \frac{3\sqrt{3}(m^2-2m+3)^{\frac{3}{2}}(q^2v^2+1)^3}{2(m-3)^3\sqrt{m-1}}, \\ \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\frac{qv}{\sqrt{2}}} e^{-x^2}\,dx \quad + \quad & \frac{3\sqrt{3}(m^2-2m+3)^{\frac{3}{2}}(q^2v^2+1)^3}{2(m-3)^3\sqrt{m-1}},\end{aligned}$$

Whence, in noting that the value of the fraction

$$\frac{3\sqrt{3}}{2}\frac{(m^2-2m+3)^{\frac{3}{2}}(q^2v^2+1)^3}{2(m-3)^3\sqrt{m-1}}$$

tends toward zero when the number m increases indefinitely, we draw the equality

$$\int_{-\infty}^{v} f(x)\,dx = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\frac{qv}{\sqrt{2}}} e^{-x^2}dx,$$

By making successively

$$v = \frac{\sqrt{2}}{q} t, \quad v = \frac{\sqrt{2}}{q} t',$$

we find the equalities

$$\int_{-\infty}^{\frac{\sqrt{2}}{q} t} f(x)\, dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{t} e^{-x^2} dx, \quad \int_{-\infty}^{\frac{\sqrt{2}}{q} t'} f(x)\, dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{t'} e^{-x^2} dx,$$

which by subtraction gives us

$$\int_{\frac{\sqrt{2}}{q} t}^{\frac{\sqrt{2}}{q} t'} f(x)\, dx = \frac{1}{\sqrt{\pi}} \int_{t}^{t'} e^{-x^2} dx.$$

The first member of this equality, according to §2, represents the probability that the value of the fraction

$$x = \frac{u_1 + u_2 + \ldots + u_n}{\sqrt{n}},$$

will be comprehended between the limits

$$\frac{\sqrt{2}}{q} t, \quad \frac{\sqrt{2}}{q} t',$$

and consequently the value of the fraction

$$\frac{u_1 + u_2 + \ldots + u_n}{\sqrt{\frac{2n}{q^2}}} = \frac{u_1 + u_2 + \ldots + u_n}{\sqrt{2(a_1^{(2)} + a_2^{(2)} + \ldots + a_n^{(2)})}},$$

will be comprehended between the limits t and t'; by virtue of which this equality which holds for $n = \infty$ shows that the probability that the value of the function

$$\frac{u_1 + u_2 + \ldots + u_n}{\sqrt{2(a_1^{(2)} + a_2^{(2)} + \ldots + a_n^{(2)})}}$$

will be comprehended between any two limits t and t', has for limit, when n increases indefinitely, the value of the integral

$$\frac{1}{\sqrt{\pi}} \int_{t}^{t'} e^{-x^2} dx.$$

In the case of n finite, the probability that the fraction

$$\frac{u_1 + u_2 + \ldots + u_n}{\sqrt{2(a_1^{(2)} + a_2^{(2)} + \ldots + a_n^{(2)})}}$$

will be comprehended between the limits t and t', will differ more or less from its limit value

$$\frac{1}{\sqrt{\pi}} \int_t^{t'} e^{-x^2} dx$$

according as the value of n and of the quantities

$$q, \quad M^{(3)}, \dots \quad M^{(2m-1)},$$

which figure in the equality (2) and of which the values, as one has seen, depend on the values of the mathematical expectations of the different powers of the quantities

$$u_1, \quad u_2, \dots \quad u_n.$$

Without arresting ourselves here at the determination of the superior limit of this difference for n rather great, we will note only that according to the formulas of our Memoir *Sur le développement des fonctions à une seule variable*,[2] this probability for any n will be given by the expression

$$\frac{1}{\sqrt{\pi}} \int_t^{t'} \left[1 - K_3 \left(\frac{q}{\sqrt{2}} \right)^3 \psi_3(x) + K_4 \left(\frac{q}{\sqrt{2}} \right)^4 \psi_4(x) + \dots \right] e^{-x^2} dx,$$

in which the quantities

$$K_3, \quad K_4, \dots$$

are the coefficients of

$$s^3, \quad s^4, \dots$$

in the development of the function

$$e^{\frac{M^{(3)}}{\sqrt{n}} s^2 + \frac{M^{(4)}}{n} s^4 + \dots}$$

according to the powers of s; and the functions

$$\psi_3(x), \quad \psi_4(x), \dots$$

are some polynomials which are obtained by the formula

$$\psi_l(x) = e^{x^2} \frac{d^l e^{-x^2}}{dx^l}.$$

[2] *Bulletin physico-mathématique de l'Académie Impériale des sciences de St.-Pétersbourg*, T. I, p. 193–200.