Sur les valeurs limites des intégrales*

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Liouville, *Journal de mathématiques pure et appliquées*, 2nd Series, T. XIX, (1874) p. 157–160.

Read to the Congress of the French Association for the Advancement of the Sciences, at Lyon. Session of 27 August 1873.

In a very interesting Memoir, for more than one reason, that Mr. Bienaymé has read to the Academy of Sciences, in 1833, and that one finds printed in the *Comptes rendus*, and reproduced in the *Journal de Mathématiques pures et appliquées* de Mr. Liouville (2nd series, T. XII, 1867), under the title: *Considérations à l'appui de la découverte de Laplace sur la loi de probabilité dans la méthode des moindres carrés*, the illustrious scholar gives a method which merits a very particular attention.

This method consists in the determination of the limit value of the integral

$$\int_0^a f(x)\,dx$$

according to the values of the integrals

$$\int_0^A f(x)\,dx, \qquad \int_0^A xf(x)\,dx, \qquad \int_0^A x^2 f(x)\,dx, \ldots$$

where $A > a$ and $f(x)$ an unknown function, subject only to the condition to keep the sign $+$ between the limits of integration. The simple and rigorous demonstration of the law of Bernoulli, that one finds in my Note under the title: *Des valuers moyennes*, is only one of the results that one draws easily from the method of Mr. Bienaymé, and according to which he has arrived himself to demonstrate a proposition on the probabilities, whence the law of Bernoulli arises directly.

By seeking to draw all the part possible out of the limit values of the integral

$$\int_a^b f(x)\,dx$$

the values of the integrals

$$\int_A^B f(x)\,dx, \qquad \int_A^B xf(x)\,dx, \qquad \int_A^B x^2 f(x)\,dx, \ldots, \qquad \int_A^B x^m f(x)\,dx,$$

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. November 8, 2011 where one has

$$A < a, \qquad B > b,$$

and where $f(x)$ remains positive, I am arrived to understand that these researches lead to some theorems of a new kind, concerning the development of the expression

$$\int_A^B \frac{f(x)}{z-x}\,dx$$

into continued fraction, which enjoy a so great role in the theory of series. Here is, for example, one of these theorems:

If $\frac{\phi(z)}{\psi(z)}$ is one of the convergent fractions of

$$\int_A^B \frac{f(x)}{z-x}\,dx,$$

that one finds by developing this expression into continued fraction

$$\cfrac{1}{\alpha z + \beta + \cfrac{1}{\alpha_1 z + \beta_1 + \cfrac{1}{\alpha_2 z + \beta_2 + \cdots}}},$$

and that

$$z_1, \quad z_2, \ldots, \quad z_l, \quad z_{l+1}, \ldots, \quad z_{n-1}, \quad z_n, \ldots, \quad z_m$$

are the roots of the equation

$$\psi(z) = 0,$$

ranked according to their magnitude: all the time that the function $f(x)$ remains positive between the limits $x = A$, $x = B$, the value of the integral

$$\int_{z_l}^{z_n} f(x)\,dx$$

surpasses the sum

$$\frac{\phi(z_{l+1})}{\psi'(z_{l+1})} + \frac{\phi(z_{l+2})}{\psi'(z_{l+2})} + \cdots + \frac{\phi(z_{n-2})}{\psi'(z_{n-2})} + \frac{\phi(z_{n-1})}{\psi'(z_{n-1})}$$

and remains below this here:

$$\frac{\phi(z_l)}{\psi'(z_l)} + \frac{\phi(z_{l+1})}{\psi'(z_{l+1})} + \cdots + \frac{\phi(z_{n-1})}{\psi'(z_{n-1})} + \frac{\phi(z_n)}{\psi'(z_n)}$$

As example of the problems that one arrives to resolve by this method, I will cite the one here:

Being given the length, the weight, the location of the center of gravity and the moment of inertia of a straight line material of density unknown and variable from one point to another, to find the nearest limits of the weights of a section of this weight.

By supposing that the concern is to evaluate the weight of a section of the line counted from one of its ends, of which the distance from the center of gravity is equal to d, and by designating by l, p the length and the weight of the entire line, by k its moment of inertia about the axis passing through its center of gravity and perpendicular to that, by x, z the length and the weight of the section in question, one arrives to this solution:

As long as x is below

$$d - \frac{k}{(l-d)p},$$

the weight z remains comprehended between

$$0 \quad \textit{and} \quad \frac{kp}{(d-x)^2 p + k};$$

in the case where x surpasses

$$d + \frac{k}{dp},$$

this weight remains between the limits

$$p \qquad \textit{and} \qquad \frac{(d-x)^2 p^2}{(d-x)^2 p + k};$$

finally, if x remains comprehended between

$$d - \frac{k}{(l-d)p} \quad \textit{and} \quad d + \frac{k}{dp},$$

the value of this weight is comprehended between the quantities

$$\frac{(x-d)(l-d)p + k}{lx} \quad \textit{and} \quad \frac{(l+d-x)(l-d)p - k}{l(l-x)}.$$