Des Valeurs Moyennes*

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If we agree to call *mathematical expectation* of any magnitude, the sum of all the values that it is susceptible to take, multiplied by their respective probabilities, it will be easy for us to establish a very simple theorem on the limits between which will remain contained a sum of any magnitudes.

Theorem.

If one designates by a, b, c, ... the mathematical expectations of the quantities

$$x, \quad y, \quad z, \ldots,$$

and by a_1, b_1, c_1,... the mathematical expectations of their squares

$$x^2, \quad y^2, \quad z^2, \ldots,$$

the probability that the sum

$$x + y + z + \cdots$$

is contained between the limits

$$a + b + c \cdots + \alpha\sqrt{a_1 + b_1 + c_1 + \cdots - a^2 - b^2 - c^2 - \cdots},$$
$$a + b + c \cdots - \alpha\sqrt{a_1 + b_1 + c_1 + \cdots - a^2 - b^2 - c^2 - \cdots},$$

will be always greater than $1 - \frac{1}{\alpha^2}$, whatever be α.

Demonstration.

Let

$$\begin{array}{lllll} x_1, & x_2, & x_3, & \ldots & x_l, \\ y_1, & y_2, & y_3, & \ldots & y_m, \\ z_1, & z_2, & z_3, & \ldots & z_n, \\ \ldots \end{array}$$

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. November 3, 2011 be all the imaginable values of the quantities x, y, z, ... and let

$$\begin{array}{lllll} p_1, & p_2, & p_3, & \dots & p_l, \\ q_1, & q_2, & q_3, & \dots & q_m, \\ r_1, & r_2, & r_3, & \dots & r_n, \\ \dots \end{array}$$

be the respective probabilities of these values, or else the probabilities of the hypotheses

$$\begin{array}{lllll} x = x_1, & x_2, & x_3, & \dots & x_l, \\ y = y_1, & y_2, & y_3, & \dots & y_m, \\ z = z_1, & z_2, & z_3, & \dots & z_n, \\ \dots \end{array}$$

Conformably to these notations, the mathematical expectations of the magnitudes

$$\begin{array}{lll} x, & y, & z, \dots \\ x^2, & y^2, & z^2, \dots \end{array}$$

will be expressed thus:

$$(1) \qquad \begin{cases} a &= p_1x_1 + p_2x_2 + p_3x_3 + \dots + p_lx_l, \\ b &= q_1y_1 + q_2y_2 + q_3y_3 + \dots + q_my_m, \\ c &= r_1z_1 + r_2z_2 + r_3z_3 + \dots + r_nz_n, \\ \dots \end{cases}$$

$$(2) \qquad \begin{cases} a &= p_1x_1^2 + p_2x_2^2 + p_3x_3^2 + \dots + p_lx_l^2, \\ b &= q_1y_1^2 + q_2y_2^2 + q_3y_3^2 + \dots + q_my_m^2, \\ c &= r_1z_1^2 + r_2z_2^2 + r_3z_3^2 + \dots + r_nz_n^2, \\ \dots \end{cases}$$

Now, as the hypotheses what we just made on the quantities x, y, z,... are the only possible, their probabilities will satisfy the following equations:

$$(3) \qquad \begin{cases} p_1 + p_2 + p_3 + \dots + p_l = 1, \\ q_1 + q_2 + q_3 + \dots + q_m = 1, \\ r_1 + r_2 + r_3 + \dots + r_n = 1, \\ \dots \end{cases}$$

It will be easy for us to find, by aid of equations (1), (2) and (3), to what the sum of all the values of the expression is reduced

$$(x_\lambda + y_\mu + z_\nu + \dots - a - b - c - \dots)^2 p_\lambda q_\mu r_\nu \dots$$

if one makes successively

$$\lambda = 1, 2, 3, \ldots l; \quad \mu = 1, 2, 3, \ldots m; \quad \nu = 1, 2, 3, \ldots n; \ldots$$

In fact, this expression being developed gives us

$$\begin{array}{rl} & p_\lambda q_\mu r_\nu \ldots x_\lambda^2 + p_\lambda q_\mu r_\nu \ldots y_\mu^2 + p_\lambda q_\mu r_\nu \ldots z_\nu^2 + \\ + & 2p_\lambda q_\mu r_\nu \ldots x_\lambda y_\mu + 2p_\lambda q_\mu r_\nu \ldots x_\lambda z_\nu + 2p_\lambda q_\mu r_\nu \ldots y_\mu z_\nu + \ldots \\ - & 2(a+b+c+\ldots)p_\lambda q_\mu r_\nu \ldots x_\lambda - 2(a+b+c+\ldots)p_\lambda q_\mu r_\nu \ldots y_\mu - \\ - & 2(a+b+c+\ldots)p_\lambda q_\mu r_\nu \ldots z_\nu - \ldots + (a+b+c+\ldots)2p_\lambda q_\mu r_\nu \ldots \end{array}$$

By giving, in this expression, to λ all the values from $\lambda = 1$ to $\lambda = l$, and by summing the results of these substitutions, we obtain the sum that is here:

$$\begin{array}{rl} & q_\mu r_\nu \ldots (p_1 x_1^2 + p_2 x_2^2 + p_3 x^2 + \ldots + p_l x_l^2) \\ + & (p_1 + p_2 + p_3 + \ldots + p_l) q_\mu r_\nu \ldots y_\mu^2 + (p_1 + p_2 + p_3 + \ldots + p_l) q_\mu r_\nu \ldots z_\nu^2 \\ + & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ + & 2(p_1 x_1 + p_2 x_2 + p_3 x_3 + \ldots + p_l x_l) q_\mu r_\nu \ldots y_\mu + 2(p_1 x_1 + p_2 x_2 + p_3 x_3 + \ldots + p_l x_l) q_\mu r_\nu \ldots z_\nu \\ + & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & +2(p_1 + p_2 + p_3 + \ldots + p_l) q_\mu r_\nu \ldots y_\mu z_\nu \ldots \\ - & 2(a+b+c+\ldots)(p_1 x_1 + p_2 x_2 + p_3 x_3 + \ldots + p_l x_l) q_\mu r_\nu \ldots \\ - & 2(a+b+c+\ldots)(p_1 \quad + p_2 \quad + p_3 \quad + \ldots + p_l \quad) q_\mu r_\nu \ldots y_\mu \\ - & 2(a+b+c+\ldots)(p_1 \quad + p_2 \quad + p_3 \quad + \ldots + p_l \quad) q_\mu r_\nu \ldots z_\nu \\ - & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ + & (a+b+c+\ldots)^2 (p_1 + p_2 + p_3 + \ldots + p_l) q_\mu r_\nu \ldots \end{array}$$

If, by virtue of equations (1), (2) and (3), we set in the place the sums

$$\begin{array}{c} p_1 x_1 + p_2 x_2 + p_3 x_3 + \ldots + p_l x_l, \\ p_1 x_1^2 + p_2 x_2^2 + p_3 x_3^2 + \ldots + p_l x_l^2, \\ p_1 + p_2 + p_3 + \ldots + p_l \end{array}$$

their values a, a_1 and 1, we will obtain the formula that is here:

$$\begin{array}{rl} & a_1 q_\mu r_\nu \ldots + q_\mu r_\nu \ldots y_\mu^2 + q_\mu r_\nu \ldots z_\nu^2 + \ldots \\ + & 2a q_\mu r_\nu \ldots y_\mu + 2a q_\mu r_\nu \ldots z_\nu + \ldots + 2 q_\mu r_\nu \ldots y_\mu z_\nu + \ldots \\ - & 2(a+b+c+\ldots) a q_\mu r_\nu \ldots - 2(a+b+c+\ldots) q_\mu r_\nu \ldots y_\mu - \\ - & 2(a+b+c+\ldots) q_\mu r_\nu \ldots z_\nu - \ldots + (a+b+c+\ldots) 2 q_\mu r_\nu \ldots \end{array}$$

We give in this formula to μ the values

$$\mu = 1, 2, 3, \ldots m,$$

next we sum the expressions which result from these substitutions, and we replace the sums

$$q_1 y_1 + q_2 y_2 + q_3 y_3 + \ldots + q_m y_m,$$
$$q_1 y_1^2 + q_2 y_2^2 + q_3 y_3^2 + \ldots + q_m y_m^2,$$
$$q_1 + q_2 + q_3 + \ldots + q_m,$$

by their values b, b_1 and 1 drawn from equations (1), (2) and (3), we will obtain the following expression:

$$\begin{gathered} a_1 r_\nu \ldots + b_1 r_\nu \ldots + r_\nu \ldots z_\nu^2 + \ldots \\ +2ab r_\nu \ldots + 2a r_\nu \ldots z_\nu + 2b r_\nu \ldots z_\nu + \ldots \\ -2(a+b+c+\ldots) a r_\nu \ldots - 2(a+b+c+\ldots) b r_\nu \ldots \\ -2(a+b+c+\ldots) r_\nu \ldots z_\nu - \ldots + (a+b+c+\ldots)^2 r_\nu \end{gathered}$$

By treating in the same manner $\nu, \ldots$ we will see that the sum of all the values of the expression

$$(x_\lambda + y_\mu + z_\nu + \ldots - a - b - c - \ldots)^2 p_\lambda q_\mu r_\nu \ldots,$$

that one obtains by making

$$\lambda = 1, 2, 3, \ldots l; \quad \mu = 1, 2, 3 \ldots m; \quad \nu = 1, 2, 3, \ldots n;$$

will be equal to

$$\begin{gathered} a_1 + b_1 + c_1 + \ldots \\ +2ab + 2ac + 2bc + \ldots \\ -2(a+b+c+\ldots)a - 2(a+b+c+\ldots)b - 2(a+b+c+\ldots)c - \ldots \\ +(a+b+c+\ldots)^2, \end{gathered}$$

This expression being developed is reduced to

$$a_1 + b_1 + c_1 + \ldots - a^2 - b^2 - c^2 - \ldots$$

Whence we conclude that the sum of the values of the expression

$$\frac{(x_\lambda + y_\mu + z_\nu + \ldots - a - b - c - \ldots)^2}{\alpha^2(a_1 + b_1 + c_1 + \ldots - a^2 - b^2 - c^2 - \ldots)} p_\lambda q_\mu r_\nu \ldots,$$

that one obtains by making

$$\lambda = 1, 2, 3, \ldots l; \quad \mu = 1, 2, 3 \ldots m; \quad \nu = 1, 2, 3, \ldots n; \ldots,$$

will be equal to $\frac{1}{\alpha^2}$. Now, it is evident that by rejecting from this sum all the terms in which the factor

$$\frac{(x_\lambda + y_\mu + z_\nu + \ldots - a - b - c - \ldots)^2}{\alpha^2(a_1 + b_1 + c_1 + \ldots - a^2 - b^2 - c^2 - \ldots)}$$

is inferior to 1, and by replacing it with unity throughout where it is greater than 1, we will diminish this sum, and it will be less than

$$\frac{1}{\alpha^2}.$$

But this sum, thus reduced, will be formed only from the products

$$p_\lambda \cdot q_\mu \cdot r_\nu \ldots,$$

which correspond to the values of x_λ, y_μ, z_ν,... for which the expression

$$\frac{(x_\lambda + y_\mu + z_\nu + \ldots - a - b - c - \ldots)^2}{\alpha^2(a_1 + b_1 + c_1 + \ldots - a^2 - b^2 - c^2 - \ldots)} > 1,$$

and it will represent evidently the probability that x, y, z, ... have some values which satisfy the condition

$$\frac{(x + y + z + \ldots - a - b - c - \ldots)^2}{\alpha^2(a_1 + b_1 + c_1 + \ldots - a^2 - b^2 - c^2 - \ldots)} > 1 \tag{4}$$

This same probability is able to be replaced by the difference

$$1 - P,$$

if we designate by P the probability that the values of the x, y, z... do not satisfy condition (4), or else, that which is the same thing, that these quantities have some values for which the ratio

$$\frac{(x + y + z + \ldots - a - b - c - \ldots)^2}{\alpha^2(a_1 + b_1 + c_1 + \ldots - a^2 - b^2 - c^2 - \ldots)}$$

is not > 1; and consequently, that the sum

$$x + y + z + \ldots$$

remains comprehended between the limits

$$a + b + c \cdots + \alpha\sqrt{a_1 + b_1 + c_1 + \cdots - a^2 - b^2 - c^2 - \cdots},$$
$$a + b + c \cdots - \alpha\sqrt{a_1 + b_1 + c_1 + \cdots - a^2 - b^2 - c^2 - \cdots},$$

Whence it is evident that the probability P must satisfy the inequality

$$1 - P < \frac{1}{\alpha^2},$$

which gives us

$$P > 1 - \frac{1}{\alpha^2},$$

this which it is required to prove.

Let N be the number of quantities x, y, z, $\ldots$; if one puts in the theorem what we just demonstrated

$$\alpha = \frac{\sqrt{N}}{t},$$

and if one divides by N the sum

$$x + y + x + \ldots,$$

and its limits

$$a + b + c \cdots + \alpha\sqrt{a_1 + b_1 + c_1 + \cdots - a^2 - b^2 - c^2 - \cdots},$$
$$a + b + c \cdots - \alpha\sqrt{a_1 + b_1 + c_1 + \cdots - a^2 - b^2 - c^2 - \cdots},$$

one obtains the following theorem concerning the mean values.

Theorem.

If the mathematical expectations of the quantities

$$x, \quad y, \quad z, \ldots \quad x^2, \quad y^2, \quad z^2, \ldots$$

are respectively

$$a, \quad b, \quad c, \ldots \quad a_1, \quad b_1, \quad c_1, \ldots,$$

the probability that the difference between the arithmetic mean of the N quantities$x$, y, z, … and the arithmetic mean of the mathematical expectations of these quantities will not surpass

$$\frac{1}{t}\sqrt{\frac{a_1 + b_1 + c_1 + \ldots}{N} - \frac{a^2 + b^2 + c^2 + \ldots}{N}}$$

will be always greater than

$$1 - \frac{t^2}{N}$$

whatever be t.

As the fractions

$$\frac{a_1 + b_1 + c_1 + \ldots}{N},$$
$$\frac{a^2 + b^2 + c^2 + \ldots}{N}$$

express the means of the quantities

$$a_1, \quad b_1, \quad c_1, \ldots$$
$$a^2, \quad b^2, \quad c^2, \ldots,$$

all the time that the mathematical expectations

$$a, \quad b, \quad c, \ldots$$
$$a_1, \quad b_1, \quad c_1, \ldots$$

will not surpass a certain finite limit, the expression

$$\sqrt{\frac{a_1 + b_1 + c_1 + \ldots}{N} - \frac{a^2 + b^2 + c^2 + \ldots}{N}}$$

will have also a finite value, however great that the number N be, and consequently it depends on us to render the value of

$$\frac{1}{t}\sqrt{\frac{a_1 + b_1 + c_1 + \ldots}{N} - \frac{a^2 + b^2 + c^2 + \ldots}{N}}$$

as small as one will wish, by attributing to t a value sufficiently great. Now, as, whatever be t, the growth of the number N to infinity renders null the fraction $\frac{t^2}{N}$, we conclude, by virtue of the preceding theorem:

Theorem.

If the mathematical expectations of the quantities

$$U_1, \quad U_2, \quad U_3, \ldots$$

and of their squares

$$U_1^2, \quad U_2^2, \quad U_3^2, \ldots$$

do not surpass any finite limit, the probability that the difference between the arithmetic mean of a number N of these quantities and the arithmetic mean of their mathematical expectations will be less than a given quantity, is reduced to unity, when N becomes infinite.

Under the particular hypothesis that the quantities

$$U_1, \quad U_2, \quad U_3, \ldots$$

are reduced to unity or to zero, according as an event E has or has no place in the 1^{st}, 2^{nd}, 3^{rd},... N^{th} trial, we will note that the sum

$$U_1 + U_2 + U_3 + \ldots + U_N$$

will give the number of *repetition* of the event E in N trials, and the arithmetic mean

$$\frac{U_1 + U_2 + U_3 + \ldots + U_N}{N}$$

will represent the ratio of the number of *repetition* of the event E to the number of *trials*. In order to apply to this case our last theorem, we designate by

$$P_1, \quad P_2, \quad P_3, \ldots P_N$$

the probabilities of the event E, in the 1^{st}, 2^{nd}, 3^{rd},... N^{th} trial; the mathematical expectations of the quantities

$$U_1, \quad U_2, \quad U_3, \ldots U_N$$

and of their squares

$$U_1^2, \quad U_2^2, \quad U_3^2, \dots U_N^2$$

will be expressed, following our notation, by

$$\begin{array}{lll} P_1.1 + (1 - P_1).0; & P_2.1 + (1 - P_2).0; & P_3.1 + (1 - P_3).0; \dots \\ P_1.1^2 + (1 - P_1).0^2; & P_2.1^2 + (1 - P_2).0^2; & P_3.1^2 + (1 - P_3).0^2; \dots \end{array}$$

Whence one sees that these mathematical expectations are

$$P_1, \quad P_2, \quad P_3, \dots$$

and that the arithmetical mean of the N first expectations is

$$\frac{P_1 + P_2 + P_3 + \ldots + P_N}{N},$$

that is the arithmetic mean of the probabilities P_1, P_2, P_3,... P_N.

Hence from that, by virtue of the preceding theorem, we arrive to the following conclusion:

When the number of trials becomes infinite, one obtains a probability, as close as one wishes to unity, that the difference between the arithmetic mean of the probabilities of this event, during these trials, and the ratio of the number of repetitions of this event to the total number of trials, is less than any given quantity.

In the particular case where the probability of the event remains the same during all the trials, we have the theorem of Bernoulli.