Sur une formule d'analyse*

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If one represents by $f(x)$ an entire function of degree n, and if one knows its $n+1$ values

$$f(x^0), \quad f(x'), \quad f(x''), \dots \quad f(x^n),$$

the formula of Lagrange gives this expression of $f(x)$

$$\frac{(x-x')(x-x'')\cdots}{(x^0-x')(x^0-x'')\cdots} f(x^0) + \frac{(x-x^0)(x-x'')\cdots}{(x'-x^0)(x'-x'')\cdots} f(x') + \dots$$

This value of $f(x)$ is able to be represented under different forms; one of the most remarkable is the following

$$A' \sum_{i=0}^{i=n} f(x^i) - A''\psi_1(x) \sum_{i=0}^{i=n} \psi_1(x^i) f(x^i) + A'''\psi_2(x) \sum_{i=0}^{i=n} \psi_2(x^i) f(x^i) - \cdots;$$

where A', A'', A''', ... designate the coefficients of x in the quotients of the continued fraction

$$\cfrac{1}{q_1 + \cfrac{1}{q_2 + \cfrac{1}{q_3 + \cdots}}},$$

resulting from the development of

$$\frac{1}{x-x^0} + \frac{1}{x-x'} + \cdots + \frac{1}{x-x^n}$$

and $\psi_1(x)$, $\psi_2(x)$,... the denominators of the convergent fractions that one draws from it.

This formula has the advantage of giving $f(x)$ under the form of an entire function, of which the terms, in general, present a series sensibly decreasing. In the particular case of

$$x^0 = \frac{n}{n}, \quad x' = \frac{n-2}{n}, \quad x'' = \frac{n-4}{n}, \quad x^{(n)} = \frac{-n}{n},$$

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. November 3, 2011 and n infinitely great, this formula furnishes the development of $f(x)$ according to the values of certain functions, that Legendre has designated by X^m (Exer. Part V, §10), and which are determined here by the reduction of the expression $\log \frac{x+1}{x-1}$ as continued fraction.

But the most precious property of this formula is this here:

If one takes in this formula only the first terms in numbers any m, one finds an approximate value of $f(x)$ under the form of a polynomial of degree $m-1$ and with the coefficients indicated by the *method of least squares*, under the assumption that the values given of

$$f(x^0), \quad f(x'), \quad f(x''), \ldots \quad f(x^n)$$

are affected by errors of like nature.

In a short time, I will have the honor to present to the Academy a Memoir, where one will see, besides, the part that one is able to draw from this formula for Analysis.