Formules pour la détermination des orbites des planètes et des comètes *

Mr. Augustin Cauchy

Comptes Rendus Hebd. Séances Acad. Sci. 26, pp. 57–61, 133–136, 157–162, 236–240.
OC I, 10 (393, 396, 397, 398), 457–462, 467–481.

Session of 17 January

In order to obtain a very rapid determination and very exact determination of the elements of the orbit of a star, it suffices to apply the principles exposed in the preceding session, to the formulas given by Lagrange in the Memoir of 1780, or rather to the equations into which these formulas are transformed, when the observations are brought together indefinitely near. One is able besides to resolve easily these last equations, when one begins by determining approximately the unknowns, by aid of the linear equation which exists among the three algebraic projections of the area which the radius vector drawn from the sun to the observed star describes (*see* the session of 27 December 1847). Then the sole quantities which will enter in to the calculation will be some particular values of certain variables and of their derivatives of the first order. Besides, these particular values will be able to be deduced from four observations of the star, joined to the very simple formulas that I am going to indicate.

ANALYSIS

Let $t_1, t_2, t_3, \ldots$ be diverse particular values attributed to time t; [58] let, moreover, $\phi = f(t)$ be a constant function of time t, and we put

$$f(t,t_1) = \frac{f(t_1)-f(t)}{t_1-t}, \quad f(t,t_1,t_2) = \frac{f(t_2,t_1)-f(t_1,t)}{t_2-t}, \quad \text{etc.}$$

Then $f(t,t_1)$, $f(t,t_1,t_2)$,... will be that which Mr. Ampere has named *interpolating functions* of the diverse orders, descended from one another, and one will have

$$(1) \qquad \begin{aligned} f(t) &= f(t_1)+(t-t_1)f(t,t_1) \\ &= f(t_1)+(t-t_1)f(t_1,t_2)+(t-t_1)(t-t_2)f(t,t_1,t_2) \\ &= \text{etc.} \end{aligned}$$

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. July 11, 2010

Then also the quantities

$$(2)\qquad \begin{cases} f(t,t_1), & f(t_1,t_2), & f(t_2,t_3),\ldots, \\ & f(t,t_1,t_2), & f(t_1,t_2,t_3),\ldots, \\ & & f(t_1,t_2,t_3,t_4),\ldots \\ & & \text{etc.}; \end{cases}$$

will be those that Laplace has designated by

$$(3)\qquad \begin{cases} \delta\phi, & \delta\phi_1, & \delta\phi_2,\ldots, \\ & \delta^2\phi, & \delta^2\phi_1,\ldots, \\ & & \delta^3\phi,\ldots, \\ & & \text{etc.}; \end{cases}$$

Finally, the functions

$$(4)\qquad f(t,t),\quad f(t,t,t),\quad f(t,t,t,t),\ldots$$

will be reduced respectively to the following:

$$(5)\qquad f'(t),\quad \frac{1}{2}f''(t),\quad \frac{1}{2.3}f'''(t),\ldots;$$

and each of the interpolating functions contained in the first, the second, the third, ... horizontal line of the table (2) or (3) will be a mean value of the first, of the second, of the third, ... term of the sequence (4) or (5), that is to say a value of this term corresponding to a value of time t, contained between the smallest and the largest of those which contribute to the formation of the interpolating function. Therefore, if one knows m particular values of the function $f(t)$ corresponding to some [59] values of t contained between certain limits, the formation of the interpolating functions will furnish immediately $m-1$ particular values of $f'(t)$, $m-2$ particular values of $\frac{1}{2}f''(t)$, etc., corresponding to some values of t always contained between the limits of which there is concern. There is more: I show that, in the case where the sequence (4) is rapidly decreasing, and where the times $t_1, t_2, t_3, \ldots$, arranged according to their order of magnitude, are not very different from one another, the interpolating functions

$$(6)\qquad f(t_1,t_2),\quad f(t_1,t_2,t_3),\quad f(t_1,t_2,t_3,t_4),\ldots,$$

represent very nearly the values of the terms of the sequence (5) corresponding to the values of t, expressed by the ratios

$$(7)\qquad \frac{t_1+t_2}{2},\quad \frac{t_1+t_2+t_3}{3},\quad \frac{t_1+t_2+t_3+t_4}{4}.$$

One will draw from formula (1)

$$(8)\qquad \begin{cases} f(t) = f(t_1) + (t-t_1)f(t_1,t_2) + (t-t_1)(t-t_2)f(t,t_1,t_2), \\ f'(t) = f(t_1,t_2) + (2t-t_1-t_2)f(t,t_1,t_2) + (t-t_1)(t-t_2)f(t,t,t_1,t_2), \end{cases}$$

If, in these last formulas, one supposes $t = \frac{t_1+t_2}{2}$, then by making, for brevity,

$$t - t_1 = t_2 - t = i,$$
$$f(t,t_1,t_2) = k, \quad f(t,t,t_1,t_2) = l,$$

one will find

$$\text{(9)} \qquad \begin{cases} f(t) = \dfrac{f(t_1) - f(t_2)}{2} - i^2 k, \\ f'(t) = \dfrac{f(t_2) - f(t_1)}{t_2 - t_1} - i^2 l. \end{cases}$$

If, in a first approximation, one neglects the terms i^2k, i^2l, one will have simply

$$\text{(10)} \qquad f(t) = \frac{f(t_1) - f(t_2)}{2},$$

$$\text{(11)} \qquad f'(t) = \frac{f(t_2) - f(t_1)}{t_2 - t_1};$$

and the errors committed, by virtue of the substitution of formulas (10) into formulas (9), will be represented by the numerical values of the products

$$i^2 k, \quad i^2 l.$$

[60] But when the sequence (4) or (5) will be a rapidly decreasing sequence, one will be able, as we have seen, to calculate approximately diverse particular values of the functions $f'(t)$, $\frac{1}{2}f''(t)$ corresponding to divers values of t. Therefore then also one will know some approximate values of the coefficients k, l which will be sensibly equal to the values of $f'(t)$ and of $\frac{1}{2}f''(t)$ corresponding to the value of t represented by the ratio $\frac{t_1+t_2}{2}$.

In the application of the preceding formulas to astronomy, $f(t)$ will be able to represent, for example, the geocentric latitude and longitude of the observed star. We suppose, in order to fix the ideas, that $\phi = f(t)$ represents the geocentric longitude. Then, of the two functions $f(t)$, $f'(t)$, the second will be that of which the particular values, drawn from the observations, will be generally less exact, and, consequently, that which it will be acceptable to determine with a greater precision. Besides, if one names δ the error which is able to result, for a particular value of $f(t)$, from the inexactness of the observations, the second member of formula (11) will be able to be, for this reason, affected of an error represented, excepting sign, by the ratio $\frac{2\delta}{t_2-t_1} = \frac{\delta}{i}$. Therefore the sum of the errors which will arise, 1° from the inexactness of formula (11), 2° from the inexactness of the observations, will be able to be raised, setting aside the sign, to the limit

$$\frac{\delta}{i} + i^2 l.$$

Now this sum will become a minimum, when the first error being the half of the second, the interval i will verify the condition

$$\text{(12)} \qquad i = \left(\frac{\delta}{2l}\right)^{\frac{1}{3}}.$$

This last equation determines the value that it agrees to assign to the interval $2i$ contained between two observations admitted to contribute to the determination of the plane of the orbit of a star. If one wishes, for example, to apply the formula (12) to the planet Hebe, by starting from the observations made in the month of July 1847, one will find for the mean value of l, a number little different from 0.05; and, hence, formula (12) will give

$$(13) \qquad i = (10\delta)^{\frac{1}{3}}.$$

If, in order to fix the ideas, one takes $\delta = 4''$, one will have around $i = 3.4''$. Therefore [61] then i must be from 3 to 4 days, and the interval $2i$ contained between two consecutive observations, from 6 to 8 days, if it is possible.

We will remark in finishing that, in the values of $f(t)$, $f'(t)$, furnished by formulas (10) and (11), one will be able, if one wishes, to correct approximately the errors produced by the inexactness of these formulas, since one will know the approximate values of the products i^2k, i^2l. But it will not be likewise with the errors produced by the inexactness of the observations. One will know only the probable limits of these last errors; but their sign will remain unknown in this last calculation.

We remark further that, if to the sum of the errors arising from the two causes indicated above one substituted the sum of the squares of these errors, one would obtain, in place of formula (13), the following

$$(14) \qquad i = \left(\frac{\delta}{l\sqrt{2}}\right)^{\frac{1}{3}} = 1.12\left(\frac{\delta}{3l}\right)^{\frac{1}{3}}$$

from which one would deduce a value of i little different from that which equation (12) furnishes.

Suite – Session of 24 January

[133] I have researched in the last session the value that it is acceptable to assign to the interval contained between two observations admitted to contribute to the determination of the orbit of a star. This interval one time found, it remains to know what will be the unknowns of which it will be acceptable to fix first the values. Now all the methods employed for a first determination of the unknowns introduce into the calculations the derivatives of the variables, or at least their approximate values represented by some ratios of the differences relative to some very small increases of time; and as these derivatives are calculated with so much less precision as they are of higher order, it is important to make so that the solution of the problem depends uniquely, if it is possible, on formulas which contain only the derivatives or some differences of the first order. This condition is fulfilled for the formulas that Lagrange has given in the Memoir of 1780, as proper to furnish the inclination of the orbit of the star observed with the longitude of the ascendant node, by aid of three pairs of observations which, taken two by two, are very near to one another. [134] It is true that the formulas of Lagrange are only approximate; but one is able to convert them into rigorous formulas by supposing that two neighboring observations are brought together indefinitely. Then the formula of which Lagrange is part is transformed into an equation of the second

degree among the three algebraic projections of the area described in the unit of time by the radius vector drawn from the sun to the star what one considers. It is true further that the elimination of two unknowns among three similar equations has led Lagrange to a final equation of the seventh degree, which, as I have shown, is able to be lowered to the sixth. But one avoids the resolution of this final equation by resolving simultaneously by the linear method the three equations that it replaces, after having obtained the approximate values of the three unknowns by aid of the linear equation which exists among them. Then one obtains, for the determination of any orbit, a simple and easy method which gives immediately with a great exactitude the semi-parameter and the position of the plane of the orbit, by aid of the givens furnished by four observations alone.

ANALYSIS

Let, at the end of time t,

ϕ, θ be the geocentric longitude and latitude of the observed star;
ϖ the heliocentric longitude of the earth;
R the distance from the earth to the sun.

Let further

H be the area described, in the unit of time, by the radius vector r drawn from the star;

U, V, W the algebraic projections of this area onto the planes of the coordinates, the plane of the ecliptic being taken for the plane of x, y;

C the area described, in the unit of time, by the radius R.

We put besides, for brevity,

$$\mu = \frac{\cos\phi}{\tan\theta}, \quad \nu = \frac{\sin\phi}{\tan\theta},$$
$$\mathscr{L}\cos\Pi = D_t\mu, \quad \mathscr{L}\sin\Pi = D_t\nu,$$
$$\mathscr{M} = \mathscr{L}\cos(\varpi - \Pi), \quad \mathscr{N} = \mathscr{L}\sin(\varpi - \Pi),$$
$$\mathscr{A} = \mathscr{N}R^2.$$

By bringing together two indefinitely near observations, on will reduce the [135] formula of Lagrange to the equation

$$(1) \qquad (\mu U + \nu V + W)(\mu U + \nu V + W - C) = \mathscr{R}(U\cos\varpi + V\sin\varpi).$$

Three equations similar to those will determine the values of three unknowns U, V, W. It is true that the elimination of W and of $W - C$ would furnish a final equation of the sixth degree in $\frac{U}{V}$. But, instead of resolving this final equation, one is able to apply the linear method to the simultaneous resolution of the three equations which produce it, after having determined the approximate values of the unknowns by aid of the linear equation which exists among them. If one puts, for brevity,

$$2v = D_t \mathrm{l}\mathscr{R} + \cot(\varpi - \Pi) D_t\varpi,$$
$$\mathscr{P} = \mu - \frac{D_t\mu}{v}, \quad \mathscr{Q} = \nu - \frac{D_t\nu}{v},$$

the letter l indicating a hyperbolic logarithm, and the arc ϖ being expressed in parts of the radius taken for unity, the linear equation of which there is question will be

$$\mathscr{P}U+\mathscr{Q}V+W-C=0. \tag{2}$$

Hence, if one puts further

$$\mathscr{S}=\frac{\mathscr{Q}\cos\varpi-\mathscr{P}\sin\varpi}{D_t\mu\Delta\mathscr{Q}-D_t\nu\Delta\mathscr{P}},$$

$\Delta\mathscr{P}$, $\Delta\mathscr{Q}$ being the increases in $\mathscr{P}$ and $\mathscr{Q}$ corresponding to the increases Δt of time t, one will have

$$\frac{U}{\Delta\mathscr{Q}}=-\frac{V}{\Delta\mathscr{P}}=\frac{W-C}{\mathscr{Q}\Delta\mathscr{P}-\mathscr{P}\Delta\mathscr{Q}}=\nu\frac{S-C}{D_t\mu\Delta\mathscr{Q}-D_t\nu\Delta\mathscr{P}}. \tag{3}$$

This last equation determines immediately the first approximate values of the three unknowns U, V, W.

It is good to remark that equation (1) is able to be immediately furnished by the elimination of τ between the two formulas

$$\left\{\begin{array}{l}\tau(U\cos\phi\cos\theta+V\sin\phi\cos\theta+W\sin\theta)=-R(U\cos\varpi+V\sin\varpi),\\ U\cos\phi\cos\theta+V\sin\phi\cos\theta+(W-C)\sin\theta=-\mathscr{N}R\tau\sin^2\theta,\end{array}\right. \tag{4}$$

τ being the distance from the earth to the observed star. Besides, in order to establish directly equations (4), it suffices to determine the cosines of the angles that the direction of the radius τ forms with the linear moments of the [136] absolute and apparent speeds of this star, the center of the sun being taken for origin of the moments.

The *Exercices d' Analyse et de Physique mathématique* will offer more ample developments on the formation and the application of formulas (1), (2), (3). I will demonstrate at the same time the relations which exist among these formulas and those which have been given by other authors or by myself, especially with the formulas of MM. de Gasparis and Michal.

Suite – Session of 31 January

[157] The formulas that I have mentioned in the session of 24 January furnish, as I have said, the means to obtain with a great exactitude the plane of the orbit of a planet or a comet, with the semi-parameter, by aid of the data furnished by four observations. There remains to examine what is the method that it is agreeable to follow, and what are the formulas that it is agreeable to use, when the given observations, or at least those of which one is proposed to make use, are in the number of three alone.

Under this last hypothesis, it appears useful to begin by fixing the distance from the observed star to the sun. Now, as one knows, the value of this distance is furnished by an equation of the seventh degree, in which enter some derivatives of the first and of the second order corresponding to a certain epoch enter. Besides, the choice of this epoch is not, very far from it, without importance, and from this choice is able to depend total success of the operation. In fact, the degree of approximation with which one will obtain the sought distance will depend especially on the degree of exactitude of the derivatives of the second order. Besides, by virtue of the principles established in the session of 17 January, if one considers any one of the variable quantities as function of time t, the interpolating function of the second order corresponding to three given observations will represent sensibly the half of the derivative of the second order, not for any one value of t, but especially for that which is the arithmetic mean among the epochs of the three observations. It is therefore this last value of t which must be chosen by preference, and the epoch that it will indicate will be that for which one will be able to hope to obtain with a sufficient exactitude the distance from the sun to the observed star. Besides, this distance being known, the diverse elements of the orbit would be able to be determined approximately, and next corrected by aid of the formulas established in the session of 15 November.[1] Then, the definitive results of the calculation being deduced by the linear method applied to some formulas which will no longer contain any derivative, the degree of precision of the found elements will depend uniquely on the degree of exactitude of the three observations employed.

I will make here, in passing, some remarks which will not be without utility.

[158] It is important to render very easy the formation and the resolution of the equation of the seventh degree, in which the unknown is the distance from the earth to the observed star. One will see in this Memoir that one is able to deduce immediately this fundamental equation of the single consideration of the corresponding centrifugal force, not to the absolute speed of the star, but to the apparent speed of the point where the radius vector drawn from the earth to the star encounters a plane parallel to the plane of the ecliptic. If, besides, as Mr. Binet has done, one applies to the found equation the theorem of Rolle, one will obtain without difficulty the limits between which will fall the two real and positive roots proper to verify this equation, and, hence, these roots themselves.

In a preceding session, I have remarked that the final equation, to which Lagrange is arrived in the Memoir of 1780, is able to be lowered from the seventh degree to the sixth. In order to demonstrate directly the possibility of this lowering, it suffices to observe that the equations of the tenth degree, among which the elimination is effected, is able to be verified by two systems of values of the unknowns. There results from it that the final equation of the eight degree, to which one will be led by elimination, will contain two strange roots to the question. It will be able to be lowered from the eight degree to the sixth, conformably to the remark that I just recalled.

ANALYSIS

§ I. — *On some formulas of mechanics*.

We consider a material point A which is moved freely in space, and let, at the end of time t,

x, y, z	be the coordinates of this point with respect to three axes rectangular among them;
r	the distance from the same point to the origin O of the coordinates;
ω	its speed;
P	the accelerative force applied to the point of which there is concern.

[1] See "Mémoire sur la détermination et la correction des éléments de l'orbite d'un astre." *Comptes Rendus Hebd. Séances Acad. Sci.* 25.

The algebraic projections of the speed ω and of the force P onto the axes of the coordinates will be respectively

$$D_t x, D_t y, D_t z; \quad D_t^2 x, D_t^2 y, D_t^2 z.$$

This put, we imagine first that the material point is moved in a plane parallel to the plane of x, y; and we name ρ the radius of curvature of the [159] described curve. One will have

$$\frac{D_t x D_t^2 y - D_t y D_t^2 x}{\omega^3} = \pm \frac{1}{\rho}; \tag{1}$$

and if one supposes the radius of curvature ρ measured starting from the described curve, the cosines of the angles formed by the direction of this radius with the positive semi-axes of x and y will be respectively equal to the products

$$-\frac{\omega^2}{\rho} \frac{D_t y}{D_t x D_t^2 y - D_t y D_t^2 x}, \quad \frac{\omega^2}{\rho} \frac{D_t x}{D_t x D_t^2 y - D_t y_t^2 x}.$$

We consider now the general case where the material point is moved in any manner in space; and we put, in this case,

$$x = \mu z, \quad y = \nu z.$$

Then μ, ν will be the trigonometric tangents of the angles formed by the projections of the radius vector onto the coordinate planes of x, z and of y, z with the positive semiaxis of z. Then also the algebraic projections $D_t^2 x, D_t^2 y, D_t^2 z$ of the accelerative force P would be able to be presented under the forms

$$\mu D_t^2 z + 2D_t \mu D_t z + z D_t^2 \mu, \quad \nu D_t^2 z + 2D_t \nu D_t z + z D_t^2 \nu, \quad D_t^2 z.$$

Hence, the force P will be able to be decomposed into two others Q, R, of which the first, corresponding to the algebraic projections

$$\mu D_t^2 z, \quad \nu D_t^2 z, \quad D_t^2 z,$$

will be directed according to the radius vector r, while the second R, corresponding to the algebraic projections

$$2D_t \mu D_t z + z D_t^2 \mu, \quad 2D_t \nu D_t z + z D_t^2 \nu, \quad 0$$

will be contained in the plane drawn through the point A parallel to the plane of x, y. There is more: we name ABC the trace of this last plane on the cone which has for summit the origin O, and for base the curve described by the material point. Let besides a be the point where the straight line OA encounters the plane drawn parallel to the plane of x, y, but at the distance 1, on the side of positive z; and we name abc the trace of this same plane on the cone of which there is concern. The force R will be decomposed itself into two others R', R'', of which the one R', corresponding to the algebraic projections

$$2D_t \mu D_t z, \quad 2D_t \nu D_t z,$$

[160] will be directed according to the tangent drawn through the point A to the curve ABC; while the force R'', corresponding to the algebraic projections

$$zD_t^2\mu, \quad zD_t^2\nu,$$

will be the product of z by the force S, corresponding to the algebraic projections $D_t^2\mu$, $D_t^2\nu$, that is to say by the force S to which the movement of the point a supposed free on the curve abc will be able to be attributed. Finally, if one names $\mathfrak{r}$ the radius of curvature of the curve abc, and ♉ the speed of the point a on this curve, one will have

$$(2) \qquad \text{♉}^2 = (D_t\mu)^2 + (D_t\nu)^2,$$

$$(3) \qquad \frac{D_t\mu D_t^2\nu - D_t\nu D_t^2\mu}{\text{♉}^2} = \pm\frac{1}{\mathfrak{r}},$$

and the cosines of the angles formed by the direction of the forces R'', S with the positive semi-axes of x and y will be

$$-\frac{\text{♉}^2}{r}\frac{D_t\nu}{D_t\mu D_t^2\nu - D_t\nu D_t^2\mu}, \quad \frac{\text{♉}^2}{r}\frac{D_t\nu}{D_t\mu D_t^2\nu - D_t\nu D_t^2\mu}.$$

Therefore, since the algebraic projections of the force R on the axes of x and y are

$$zD_t^2\mu, \quad zD_t^2\nu,$$

one will have simply

$$R'' = \pm\frac{\mathrm{d}^2}{r}z.$$

But $\frac{\text{♉}^2}{r}$ represents precisely the centrifugal force due to the speed ♉. One is able therefore to enunciate the following proposition, that it is, moreover, easy to establish without calculation.

Theorem. — We suppose that a material point A, of which the rectangular coordinates, brought back to the origin O, are x, y, z, is moved freely in space, by virtue of a certain accelerative force P. We suppose besides that two planes parallel to the plane of y, z are drawn, one through the point A, the other through the point a situated on the radius OA, at the distance 1 from the plane of y, z. Finally, let ABC, abc be the traces of these two planes on the cone which the straight line OA describes, and we decompose the force P into two others Q, R, directed, one according to the radius vector OA, the other according to a straight line contained in a plane perpendicular to the axis of z. If one projects the force R onto the radius of curvature of the curve ABC, the ratio of the projection to the ordinate z will be represented, excepting the sign, by the centrifugal force due to the apparent speed of the point a onto the curve abc, for an observer placed at point O.

§ II. — *On the fundamental equation by aid of which the distances from a planet or from a comet to the earth or to the sun is determined.*

We conserve the notations adopted in the session of 24 January. Let besides K be the attractive force of the sun, measured at the unit of distance, r the distance from the center of the sun to the center A of the observed star, and x, y, z the coordinates of the point A. The action of the sun on the observed star will have for algebraic projections on the coordinate axes

$$-\frac{Kx}{r^3}, \quad -\frac{Ky}{r^3}, \quad -\frac{Kz}{r^3};$$

while the action of the sun on the earth will have for algebraic projections

$$-\frac{K\mathrm{x}}{R^3}, \quad -\frac{K\mathrm{y}}{R^3}, \quad 0;$$

x, y being the coordinates of the earth. Besides, the relative coordinates

$$x-\mathrm{x}, \quad y-\mathrm{y}, \quad z-\mathrm{z}$$

are the algebraic projections of the distance τ from the earth to the observed star. Therefore, if one names P the resultant of the attraction $\frac{K}{r^2}$, and of a force equal but directly opposed to the attraction $\frac{K}{R^2}$, and if one decomposes the force P applied to the point A into two directed forces $\mathrm{P}', \mathrm{P}''$, the one according to the radius vector τ, the other according to a straight line contained in a plane perpendicular to the axis of z; the algebraic projections of the force P'' onto the coordinate axes will be

$$K\mathrm{x}\left(\frac{1}{R^3}-\frac{1}{r^3}\right), \quad K\mathrm{y}\left(\frac{1}{R^3}-\frac{1}{r^3}\right), \quad 0.$$

Besides, the apparent movement of the star for an observer placed in the center O of the earth will be able to be attributed to the force P; and, if one names ABC the trace of the plane drawn through the point A parallel to the plane x, y on the cone described by the straight line OA, the projection of the force P'' on the normal to the curve ABC must be equal, excepting sign, to the centrifugal force due to the apparent speed from the point A onto the same curve. On the other hand, the radius of curvature of this last curve will be equal, excepting sign, to

$$\mathfrak{r}z,$$

the value of $\mathfrak{r}$ being that which formulas (2) and (3) of § I determine. [162] This put, the theorem enunciated in paragraph I will furnish the equation

$$\tau = A\left(\frac{1}{R^3}-\frac{1}{r^3}\right), \tag{1}$$

the value of A being determined by the formula

$$\frac{1}{A} = \frac{\sin\theta}{KR}\,\frac{D_t\mu D_t^2\nu - D_t\nu D_t^2\mu}{\sin\varpi\, D_t\mu - \cos\varpi\, D_t\nu}, \tag{2}$$

in which one will have

$$D_t\mu D_t^2\nu - D_t\nu D_t^2\mu = \pm\frac{1}{\mathfrak{r}}[(D_t\mu)^2+(D_t\nu)^2]^{\frac{3}{2}}.$$

We add that if one names $\pm z$f the centrifugal force due to the apparent speed of the point A on the curve ABC, equation (2) will give simply

$$A = \pm \frac{KR\cos(\widehat{R,f})}{\mathrm{f}}. \tag{3}$$

Thus the value of the coefficient A is deduced immediately from that of the centrifugal force f. Besides, equation (1) established one time, it suffices to eliminate τ from it by aid of the formula

$$r^2 = R^2 + 2R\tau\cos(\phi - \varpi)\cos\theta + \tau^2,$$

in order to obtain the equation which determines the distance r.

Suite – Session of 21 February

[236] When one wishes, by the aid of three observations, to determine the distance from a planet or from a comet to the sun or to the earth, one arrives, as one knows, to an equation of the seventh degree. If besides one applies to the found equation the theorem of Rolle, by taking for unknown the distance from the observed star to the earth, one concludes from it, as Mr. Binet has remarked, that the equation is no longer able to admit four real roots, of which the one is reduced to zero. But if, instead of taking for unknown the distance to the earth, one takes for unknown the distance to the sun, then the calculation shows that the equation obtained is able to offer no more than three real and positive roots. These three roots are all three superior to the perpendicular dropped from the center of the sun onto the radius vector drawn from the sun to the earth, and all [237] three inferior to a certain limit which the equation of the lively forces furnish. Besides one of these three roots, strange to the question, is reduced to the distance from the sun to the earth; and in order to know if this root is or is not contained between the two others, if suffices to consult the sign of a certain known quantity. Finally, in order to know if the distance from the observed star to the sun is inferior or superior to the distance from the sun to the earth, it suffices to recur to a beautiful remark of Lambert, or, that which reverts to the same, it suffices to examine in what sense is directed the radius of curvature of the curve according to which, in the apparent movement of the star, a plane parallel to the one of the ecliptic cuts the cone described by the radius vector drawn from the earth to the star. By virtue of these remarks, the roots of the equation of the seventh degree which will be able to resolve the proposed question will be reduced to two, or even to one alone; and one will be found thus brought back to the conclusion deduced by Mr. Binet, from the geometric discussion of the equation which determines the distance from the star to the earth. We add that, if the observed star is a comet, the value of the major axis determined by the equation of the lively forces must be very considerable, and that then this equation will furnish a very simple way, not only to recognize the true solution, but also to correct the value of the distance from the sun to the earth furnished by the first approximation.

I will make yet here an important remark. The two general equations that I have given in the session of 27 December[2] last contain only, with the semi-parameter, the

[2] See "Sur deux formules générales, dont chacune permet de calculer rapidement des valeurs très approchées des élémens de l'orbite d'une planète ou d'une comète." Comptes Rendus Hebd. Séances Acad. Sci. 25.

coordinates of the observed star and the distance from this star to the sun. Now this distance is found linked by an equation of the second degree to the distance from the star to the earth, which is the only unknown on which the coordinates of the star depend. Finally, if the observed star being a comet, one considers it, in a first approximation, as describing a parabola, its distance to the sun will depend uniquely on time and on two elements, of which one will be precisely the semi-parameter, the other being the epoch of the passage of the comet at the perihelion. Therefore, then, the two general equations mentioned above will be able to be counted to contain only two unknowns, namely, the two elements of which there is concern. Therefore they will suffice in order to correct the elements, and the correction thus obtained will be so much more exact, as the two equations do not contain any derivative. We add that one will simplify the calculation by conserving, in the equations of which there is question, only one of the three systems of values of the variables corresponding to the three given observations, and by replacing the two other systems by the two systems of the finite differences of the first and of the second orders formed with the three values of each of these same variables.

ANALYSIS

We conserve the notations adopted in the preceding sessions, and we put, moreover,

$$B = k + \frac{A}{R^3}.$$

The distances r, τ from the observed star to the sun and to the earth will be linked between them, and to the distance

$$s = \tau + k, \tag{1}$$

through the two equations

$$\tau = A\left(\tfrac{1}{R^3} - \tfrac{1}{r^3}\right), \tag{2}$$

$$r^2 = s^2 + l^2, \tag{3}$$

in which R designates the distance from the earth to the sun, and l the perpendicular lowered from the center of the sun onto the radius vector R. Moreover, by naming ω the absolute speed of the observed star, K the attractive force of the sun, and a the semi-major axis of the described orbit, one will have

$$\frac{2}{r} = \frac{1}{a} + \frac{\omega^2}{K}, \tag{4}$$

and one will be able to reduce $\frac{\omega^2}{K}$ to a function of τ, from the force

$$\frac{\omega^2}{K} = \mathscr{A} + 2\mathscr{B}\tau + \mathscr{C}\tau^2, \tag{5}$$

$\mathscr{A}, \mathscr{B}, \mathscr{C}$ being some known quantities. Besides, by virtue of formula (5), the polynomial

$$\mathscr{A} + 2\mathscr{B}\tau + \mathscr{C}\tau^2,$$

essentially positive, will not be able to be lowered from the limit

$$\frac{\mathscr{A}\mathscr{C}-\mathscr{B}^2}{\mathscr{C}}.$$

This put, there results immediately from formulas (3), (4), that the [239] radius vector r is contained between the limits

$$l \quad \text{and} \quad \frac{2\mathscr{C}}{\mathscr{A}\mathscr{C}-\mathscr{B}^2}.$$

We add that, τ being a positive quantity, the distance r, by virtue of formula (2), will be superior or inferior to R, according as the quantity A will be positive or negative.

If one eliminates τ and s among the formulas (1), (2), (3), one will obtain the equation

$$r^2-l^2-\left(B-\frac{A}{r^3}\right)^2=0, \tag{6}$$

to which one satisfies by putting $r=B$. On the other hand, by differentiating equation (6) with respect to r, one obtains the equation derived which is able to be presented under the form

$$r^2-4ABr^3+B^2=0, \tag{7}$$

and which admits no more than two real roots. Therefore, hence, equation (6) will be able to offer no more than three real roots; and as the root R is strange to the question, as besides the problem must offer at least one solution, it is clear that equation (7), of which the last term is positive, will offer precisely two real roots, and equation (6) three real roots. Finally, as in the neighborhood of the value $r=R$, the first member of formula (6) is negative or positive for $r>R$, according as the difference

$$R^5-3Ak$$

is negative or positive, it is clear that the question will offer a unique solution, if one has

$$R^5-3Ak<0, \tag{8}$$

and that, under this hypothesis, equation (7) will offer between the limits l, R, if A is negative, or between the limits R, $\frac{\mathscr{A}\mathscr{C}-\mathscr{B}^2}{3}$, if A is positive, a single root which will be precisely the value sought of R.

If one has, on the contrary,

$$R^5-3lk>0,$$

equation (6) will offer, beyond the root R, two real roots, both [240] superior to R, when A will be negative, both inferior to R when A will be positive; and it will be easy to operate the separation of these two roots, since they will comprehend between them one root of equation (7).