# Mémoire sur quelques propriéties remarquables des fonctions interpolaires, et sur le parti qu'on en peut tirer pour une détermination sûre et facile des éléments de l'orbite d'une planète ou d'une comète *

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In the methods generally employed for the determination of the orbit of a star, one never knows a priori what will be the degree of approximation that the calculated values of the elements of this orbit will present, and likewise, when the calculation is achieved, one is not able ordinarily to form a precise idea of the exactitude of the solution obtained, before having submitted this solution to new tests, and before having deduced from enough numerous observations, by aid of the linear method, the corrections of the elements. It would be therefore to render service to the astronomers that to establish a method which indicated itself the degree of precision of the results that it would furnish, in a manner to expose not at all those who would wish to follow it to make some useless calculations. Some remarkable properties of interpolating functions permit to attain this end. We enter, to this subject, in some details.

The elements of the orbit of a star being in number six, it is necessary, in order to determine them, to establish among these elements at least six equations. Besides, three given values of two functions of these elements and of time, for example, of the geocentric longitude and latitude of the star, would suffice to furnish six equations of this kind. Therefore the solution of the problem will be to deduce from three complete observations. But the solution thus found will not be unique, two different orbits are able to correspond to the system of three given observations: and hence, in the general case, in order to determine without any uncertainty all the elements of the orbit of a star, it will be necessary to know at least four observations.

It is good to observe that being given three values of a variable, for example of the geocentric longitude, considered as function of time $t$, one will be able to deduce from it immediately some values of interpolating functions of the first and of the second order.

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A fourth value of the principal function would permit to calculate beyond a particular value of an interpolating function of the third order; and if the given observations are brought together indefinitely, the interpolating functions of the first, of the second, of the third order, ... will transform themselves into derivatives of these same orders divided by the products 1, 1.2, 1.2.3,... Therefore, in order to be able, without any [30] uncertainty, to determine the elements of the orbit of a star, it will be necessary to know, for a given epoch, with the geocentric longitude and latitude, their derivatives of the first, of the second and of the third order. The mathematical analysis leads to the same conclusion, by showing that these three kinds of derivatives enter into the exact formulas which resolve the problem by reducing it to the resolution of an equation of the first degree. We add that if one considers two distinct epochs, instead of one alone, one will be able further to reduce easily the problem to the first degree (*see* the session of 27 December 1847), by supposing known for each epoch, with the geocentric longitude and latitude of the observed star, their derivatives of the first and of the second order only.

The precision of the results deduced from the exact formulas which we are going to recall, will depend on the degree of approximation with which one will obtain the derivatives of the first and of the second order of the geocentric longitude and latitude. One determines ordinarily these derivatives by aid of certain formulas of interpolation, among which one must distinguish those that Lagrange and Laplace have given. But these last formulas being solely some approximate equations which arise from the omission of terms of which the value is unknown a priori, I have ought to research if it would not be possible to replace them by some more rigorous formulas, which indicated themselves the degree of approximation of the results of the calculation. My researches have led me effectively to some new formulas, of which one is able to give a very just idea by saying that they are, with respect to the formula of interpolation of Laplace, that which is, with respect to the formula of Taylor, the finite equation, substituted to that one by Lagrange, in the theory of analytic functions. My new formulas decompose a derivative of any order, into two parts, of which the first is expressed rigorously by aid of interpolation formulas of the same order and of the higher orders, up to the order $n$; the second part being the product of a certain factor contained between certain limits by a mean quantity among the diverse values that, in the interval of time contained between the extreme observations, the derived function of order $n+1$ is able to acquire. Hence, one will be able to decompose a derivative of any order $m$ into two parts, of which the first will contain two interpolating functions, the one of this same order, the other of order $m+1$ immediately higher; the second part being the product of a factor contained between certain limits by a mean value of the derivative of order $m+2$. Moreover, the first part will be found reduced to a single interpolating function of order $m$, that is [31] to say to an interpolating function formed with $m+1$ different values of the principal function, if to these values correspond $m+1$ values of $t$, of which the arithmetic mean is exactly the epoch for which one wishes to calculate the value of the derivative of order $m$.

I will add that if these diverse values of the principal function are furnished by some observations from which are able to result, for each of these values, an error represented, excepting the sign, by the number $\delta$, the *maximum* error of which the interpolating function of order $m$ will be able to be affected, deduced from $m+1$ particular given values of the principal function, will be the error which one will obtain, by supposing that to these particular values, arranged in the order of time, one substitutes some quantities alternatively positive and negative, but all equal, setting aside the sign, to the number $\delta$.

I will remark finally that by virtue of a theorem recalled in the Memoir of 16 November 1840, an interpolating function of order $m$, deduced from $m+1$ given observations, will always be the quotient that one will obtain by dividing by the product $1.2\ldots m$ a mean value of the derived function of order $m$, that is to say a value that will take this derivative for an epoch mean between those of the extreme observations.

These principles permit us to draw, from the exact formulas mentioned above, some approximate values of the elements of an orbit, in a manner to form to us a just idea of the degree of approximation obtained.

One wishes, for example, to deduce the elements of the orbit from formulas (7) and (8) of the Memoir read at the session of 27 December 1847. It will be necessary to know, for two different epochs, the geocentric longitude and latitude of the observed star, with their derivatives of the first and of the second order; and, hence, four observations at least are necessary, the first three observation being able to be employed when the concern will be of the first epoch, and the last three observations when the concern will be of the second epoch. We consider in particular the three observations which will serve to determine the values of the unknowns corresponding to the first epoch. From the three unknowns which will represent the geocentric longitude, its derivative of the first order, and its derivative of the second order, the last, or the derivative of the second order, being that of which the value will be generally less exact, will be also that which it will be convenient to determine with a greater precision. Besides, this derivative will have for approximate value an interpolating function of the second order, deduced from the first three observations. We add that the difference from this approximate value, to the true value, [32] will be proportional to a mean value of the derivative of the fourth order only, if one chooses for first epoch the arithmetic mean among the epochs of the three observations. We remark finally that, the interval of the two extreme observations remaining the same, the influence of the errors of observation on the value of the interpolating function will be the least possible, if the intermediate observation is separated from two others by some intervals sensibly equal. We adopt these hypotheses, and we name $i$ the interval of time which will separate the second observation from each of the two others. The error that one will commit by taking for value of the derived function of the second order the value of the interpolating function of the second order, drawn from three observations, will be composed of two proportional parts, the one to the square of $i$, the other to the square of $\frac{1}{i}$, the coefficients of $i^2$ and of $\frac{1}{i^2}$ being, on the one hand, $2\delta$, on the other, a mean value $l$ of the derived function of the fourth order; and the total error, divided by 24, will be the least possible, when these two parts will be equals, setting aside the sign. This equality will furnish a simple means of determination for the value which it will be convenient to attribute to the interval $i$. In fact, one is able first from five, six, seven observations… extended from one another, to deduce the values of the interpolating functions of the first, of the second, of the third and of the fourth order; and those which represent precisely the mean values of the derivatives of the same orders, respectively divided by the numbers $1, 2, 6, 24$, or at least these mean values affected with very small errors, which are produced by the observations, and of which the limits are known. One will know therefore an approximate value of $l$, and it is likewise not necessary that here the approximation is considerable; because, in order to fulfill the indicated condition, $i$ must be reciprocally proportional to the fourth root of $l$; and, hence, the value of the interval will not be diminished or increased from a fifth, if $l$ is doubled or reduced to the half of its value.

The value of $i$ being calculated as we just said, for the case where the principal function is reduced to the geocentric longitude, one will choose three observations in a manner in this that the first and the third are separated from the second by some intervals of time equal to $i$, or at least by some intervals as near to $i$ as it will be possible; and then our formulas will furnish with the geocentric longitude and latitude their derivatives of the first and of the second order relative to a first epoch, which will be the arithmetic mean among the epochs of the three observations, or at [33] least they will furnish the approximate values of these unknowns with a degree of approximation indicated by the same calculation.

The values of the same unknowns, corresponding to a second epoch, will be deduced by the same process, or from a fourth observation joined to two of the first three, or better yet from three new observations; and then formulas (7), (8) from the Memoir of 27 December will furnish the means to determine immediately the orbit of the observed star.

One will remark that, in the preceding method, one begins by fixing the interval of time which must separate from one another two observations admitted to contribute to the determination of an orbit. This fixation dispenses often the calculator from useless work, which he would come to regret forced to redo in whole with the data different from those which served as base to a first calculation.

In fact, the errors, which affect the unknowns of which there is concern here to obtain the values arise, some from the inexactness of the observations, others from the inexactness of the formulas that one employs. From these two sorts of errors, the first increase when one brings together, and the last when one extends the observations. It therefore took place here to seek at what distance two consecutive observations must be placed in order that the total error to fear is a minimum. The advantages that result evidently from the solution of this last problem permits me to hope a favorable welcome from the astronomers for this new work that their kindliness has encouraged me to pursue, and that I myself propose to reproduce with more ample developments in my *Exercices d'analyse et de physique mathématique*.

I will add that one is able still to obtain a very simple determination of the elements of the orbit of a star, by applying the principles exposed above to the formulas given by Lagrange in the Memoir of 1780, or rather to the equations in which these formulas are transformed, when the observations are brought together indefinitely near. It is, moreover, that which I will explain in greater detail in another article.