Mémoire sur la détermination de l'orbite d'une planète, à l'aide de formules qui ne renferment que les dérivées du premier ordre des longitude et latitude géocentriques *

M. Augustin Cauchy

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The Memoir that Lagrange has given in the *Connassance des Temps* for the year 1821 reduces the determination of the orbit of a star to the resolution of one equation of the seventh degree, which contains the values of the geocentric longitude and latitude corresponding to six observations made in the neighborhood of three diverse epochs; the first observation being supposed very near to the second, the third to the fourth, and the fifth to the sixth. If one admits that the interval contained between two neighboring observations become infinitely small, the equation of Lagrange will contain simply, with the geocentric longitude and latitude relative to the three epochs, the corresponding values of the derivatives of these two variables, differentiated a single time each with respect to time. Now the derivatives of the first order of the geocentric longitudes and latitudes being able to be determined with much more precision than their derivatives of higher order, it is clear that the method cited by Lagrange offers an advantage [776] which will be very valuable in practice, if one would be able to form and to resolve easily the equation of the seventh degree mention above. In order to arrive to it, it would suffice to find an easy way to obtain an approximate value of the unknown; and I myself had first proposed to resolve this last problem, especially for the case where it is a question of a planet, that is to say an orbit very different from a parabola, and to which, in consequence, the method of Olbers could not be applied. But, after having obtained the desired solution, I have been agreeably surprised to see that the principles of which I made use, being directly applied to the search for the elements of the orbit, furnished, for the approximate determination of these elements, a new method very simple and very easy to follow. This method is based on the use of formulas which contain only some derivatives of the first order, and that I am going to establish in a few words.

Let, at the end of time t,

r be the distance from a planet to the sun;

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τ its distance to the earth;
ρ the projection of τ onto the plane of the ecliptic;
ϕ, θ the geocentric longitude and latitude of the planet;
p the longitude of the planet, measured in the plane of the orbit starting from the ascendant node;
ψ the mean anomaly of the planet;
R the distance from the earth to the sun;
ϖ the heliocentric longitude of the earth;
$\chi = \phi - \varpi$ the elongation;
$\mathscr{R} = R\sin\chi$ the projection of R onto a straight perpendicular to ρ.

Let further

ι be the inclination of the orbit of the planet;
$\mho$ the longitude of the ascendant node;
a the semimajor axis of the orbit;
ε the eccentricity;
$K = \lambda^2 a^3$ the attractive force of the sun;
$H = \sqrt{Ka(1-\varepsilon^2)}$ the double of the area described by the radius vector r in a unit of time;
$-\frac{c}{\lambda}$ the epoch of the passage to perihelion;
$\mathfrak{p}$ the value of p at this epoch;

U, V, W the algebraic projections on the area H onto the coordinate and [777] rectangular planes of x, y, of z, x and of z, y, the plane of x, y being the one of the ecliptic. Finally, let
$\mathfrak{O} = (U\cos\phi + V\sin\phi + W\tan\theta)\cos\theta$, $\quad \mathfrak{Q} = \mathfrak{O}\cos\varpi + V\sin\varpi$ be the projections of the area H onto the planes perpendicular to the radius vectors ρ and R; and put

$$\alpha = \sin\mho\sin\iota, \quad \beta = \cos\mho\sin\iota, \quad \gamma = c + \mathfrak{p}, \quad \Theta = \ln\tan\theta.$$

One will have
(1)
$$R\cos(\varpi - \mho) + \rho\cos(\phi - \mho) = r\cos p, \quad R\sin(\varpi - \mho) + \rho\sin(\phi - \mho) = r\sin p\cos\iota,$$

$$\rho\tan\theta = r\sin p\sin\iota, \tag{2}$$

$$r = a(1 - \varepsilon\cos\psi), \quad \tan\frac{p - \mathfrak{p}}{2} = \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{\frac{1}{2}}\tan\frac{\psi}{2}, \quad \psi - \varepsilon\sin\psi = \lambda t + c. \tag{3}$$

The orbits of the planets are generally in the planes rather near to the one of the ecliptic, so that $\cos\iota$ differs very little from unity. Hence, one will have, in the formulas (1), to substitute without notable error, the following:

$$R\cos(\varpi - \mho) + \rho\cos(\phi - \mho) = r\cos p, \quad R\sin(\varpi - \mho) + \rho\sin(\phi - \mho) = r\sin p, \tag{4}$$

which will become exact, if the inclination ι is reduced to zero, the angle $\mho$ being then arbitrary. Therefore, in order to obtain some good approximate values of the four elements $a, \varepsilon, c, \mathfrak{p}$, when the question concerns a planet, it suffices to consider the case where ι being null, the variables t, r, p, ψ, ρ and ϕ would be linked among them by the equations (3) and (4). We see therefore how one can, in this last case, determine the elements of the orbit.

One draws from formulas (4)

$$r\sin(\phi - \mho - p) = \mathscr{R}. \tag{5}$$

If one differentiates this last equation, then, by having regard to the two formulas

$$r^2 D_t p = H, \quad rD_t r = \lambda a^2 \varepsilon \sin\psi,$$

and by putting besides, for brevity,

$$D_t \phi = \Phi, \quad D_t \mathscr{R} = \mathfrak{R}, \quad r\cos(\phi - \mho - p) = \frac{1}{\varsigma},$$

one will find

$$\Phi - \varsigma\mathfrak{R} = \frac{H - \lambda a^2 \varepsilon \mathscr{R} \varsigma \sin\psi}{r^2}. \tag{6}$$

[778] Now the eccentricities of the planets being much inferior to unity, one will be able, in a first approximation, to reduce formula (6) to the following:

$$\Phi - \varsigma\mathfrak{R} = \frac{H}{r^2}. \tag{7}$$

One will have besides

$$r^2 = \mathscr{R}^2 + \frac{1}{\varsigma^2}. \tag{8}$$

This put, let $t_\prime$ be a second value of t which is not very different from the first, and we name $\varsigma_\prime$ $, \Phi_\prime$ $, \mathscr{R}_\prime$ $, \mathfrak{R}_\prime$ that which $\varsigma, \Phi, \mathscr{R}, \mathfrak{R}$ will become at the end of time $t_\prime$. As the radius $r = a(1 - \varepsilon\cos\,\psi)$ will vary generally very little in the interval of time represented by $t_\prime - t$, the values of r^2 and of $\frac{H}{r^2}$ corresponding to the two epochs indicated by t and $t_\prime$ will be little different from each other. One will have therefore sensibly

$$\begin{cases} \Phi_\prime - \varsigma_\prime \mathfrak{R}_\prime = \Phi - \varepsilon\mathfrak{R}, \\ \mathscr{R}_\prime^2 + \dfrac{1}{\varsigma_\prime^2} = \mathscr{R}^2 + \dfrac{1}{\varsigma^2}. \end{cases} \tag{9}$$

These last two formulas suffice to the determination of the approximate values of ς, $\varsigma_\prime$. If, in order to be more commodious, one puts

$$\mu = \frac{\varsigma - \varsigma_\prime}{2}, \quad \nu = \frac{\varsigma - \varsigma_\prime}{2};$$

then, by neglecting the square of r, one will have simply

$$\mu = \mathscr{A} + \mathscr{B}\mu^3; \tag{10}$$

the values of $\mathscr{A}, \mathscr{B}$ being

$$\mathscr{A} = \frac{\Phi_{\prime} - \Phi}{\Re_{\prime} - \Re}, \quad \mathscr{B} = \frac{1}{4}\frac{\Re_{\prime} + \Re}{\Re_{\prime} - \Re}(\mathscr{R}_{\prime}^2 - \mathscr{R}^2).$$

The values of ς, $\varsigma_{\prime}$ being known, one will draw from formulas (8) and (7) the approximate values of r and of H, next the approximate value of a, from the formula

$$a = \frac{H^2}{K}. \tag{11}$$

We add that, r being little different from a, one would have been able, although less easily, to deduce a first approximate value of a from formula (7).

It is good to observe that if one put $\Delta t = t_{\prime} - t$, and if besides one names $\Delta\Phi, \Delta\Re$ the increases of Φ and of $\Re$ corresponding to increases [779] Δt of t, formula (10) will give again, very nearly,

$$\mu = \frac{\Delta\Phi}{\Delta\Re} + \frac{1}{2}\Re\frac{\Delta\mathscr{R}^2}{\Delta\Re}\mu^3. \tag{12}$$

In order to draw starting from the preceding formulas, if suffices to know four observations, made in the neighborhood of two diverse epochs; the first observation being supposed very near to the second, and the third to the fourth. Then the values of $\Phi = D_t\phi$ and of $\Re = D_t\mathscr{R}$, corresponding to each epoch, are able to be reduced, without sensible error, to the values which the ratios $\frac{\Delta\phi}{\Delta t}$, $\frac{\Delta R}{\Delta t}$ acquire, when one takes for Δt the interval of time contained between two near observations of the epoch of which there is concern.

We remark yet that if, the inclination ι being null, one supposes, as one is then free to do, $\Omega = 0$, one will be able, from formula (5) reduced to

$$\sin(\phi - p) = \frac{\mathscr{R}}{r}, \tag{13}$$

to deduce the value of the angle p, to which the angle $\psi + \mathfrak{p}$ becomes equal when ε vanishes. Then also, from the same formula joined to the second of the equations (3), one will draw

$$(\cos\psi - \varepsilon)\sin(\phi - p) - (1 - \varepsilon^2)^{\frac{1}{2}}\sin\psi\cos(\phi - \mathfrak{p}) = \frac{\mathscr{R}}{a}; \tag{14}$$

next, by neglecting ε,

$$\sin(\phi - \psi - \mathfrak{p}) = \frac{\mathscr{R}}{a}. \tag{15}$$

If, in this last formula, one replaces t by $t_1 = t + \Delta t$, one will have

$$\sin(\phi_1 - \psi_1 - \mathfrak{p}_1) = \frac{\mathscr{R}_1}{a}. \tag{16}$$

the value of $\Delta\psi = \psi_1 - \psi$ being given by the equation

$$\Delta\psi = \lambda\Delta t = \left(\frac{K}{a^3}\right)^{\frac{1}{2}}\Delta t. \tag{17}$$

Formulas (15), (16) are those which Mr. Binet has made use in order to determine, by aid of two observations, the distance from the sun to a planet of which the orbit is supposed circular. After having deduced, either from these formulas, or from those which we have given above, the approximate values of the distances a, r, with that of the angle $\psi + \mathfrak{p}$ or $\gamma + \lambda t$, and hence the approximate value of γ, one will have, by neglecting the terms proportional to the square or to the higher powers of eccentricity, to draw from formula (6), joined to formulas (3) and (8), a linear equation among the correction δa [780] and the constant a, and the constants $\varepsilon \sin c$, $\varepsilon \cos c$, of which the first approximate values are nulls; and, in order to determine approximately δa, $\varepsilon \sin c$, $\varepsilon \cos c$, it will suffice to recur to three linear equations thus formed.

We add that if to formula (6) one substitutes formula (14), each linear equation will contain the four unknowns δa, $\delta \gamma$, $\varepsilon \sin c$, $\varepsilon \cos c$. Therefore then four equations will be necessary in order to determine these unknowns. But, on the other hand, in order to obtain these four linear equations, it will suffice to make use of only four observations. Besides, the approximate values of $a, \varepsilon, c, \gamma$, and, hence, the value of $\mathfrak{p} = \gamma - c$ being known, one will be able to correct anew these approximate values by aid of formula (14) and of the four given observations.

The values of the constants $a, c, \varepsilon, \mathfrak{p}$, determined as we just said, and those that one will deduce from it for r, p, ψ, by aid of the formulas (3), would be exact, setting aside some perturbations, if the plane of the orbit coincided rigorously with the plane of the ecliptic, and if besides the given observations were not affected of any error. These values will not be exact, but not very different from the truth, if the observed star is a planet for which the inclination of the orbit is not reduced to zero. Then also, ♉ being no longer arbitrary, the value that one will have found for $\mathfrak{p}$ by operating as we just said, will be effectively that of $\mathfrak{p}$+d.

The value of r being known for a given epoch, one will obtain the corresponding values of τ and ρ by aid of the formulas

$$s^2 = r^2 - l^2, \quad \tau = s - k, \quad \rho = \tau \cos\theta, \tag{18}$$

in which one has $k = R\cos\theta\cos\chi$, $l^2 = R^2 - k^2$; next the values of $\mathfrak{O}, \mathfrak{Q}$ by aid of the formulas

$$\mathfrak{O} = P + \Lambda R \rho \sin\theta, \quad \mathfrak{Q} = -\frac{\rho \mathfrak{O}}{R\cos\theta}, \tag{19}$$

the values of P, Λ being $P = R^2 D_t \varpi \sin\theta$, $\Lambda = D_t\phi\cos\chi - D_t\Theta\sin\chi$; next, finally, the values of α, β, by aid of the formulas

(20)

$$\alpha = \frac{\mathfrak{Q}\cos\theta\sin\phi - (\mathfrak{O} - H\sin\theta)\sin\varpi}{H\cos\theta\sin\chi}, \quad \beta = \frac{\mathfrak{Q}\cos\theta\cos\phi - (\mathfrak{O} - H\sin\theta)\cos\varpi}{H\cos\theta\sin\chi},$$

in which one has

$$H = \sqrt{Ka(1-\varepsilon^2)}.$$

Equations (20), by giving the values of α, β, furnish, hence, those of ι and ♉, that one would be able, moreover, to deduce yet from formulas (1), (3), (5), (9) and (11) from pages 701 and 702.[1]

[781]We add that the diverse elements obtained as we just said will be able to be definitely corrected by aid of the formulas established in my Memoir of 15 November.

I myself am not content to establish the general formulas which precede; I have wished to assure myself by experience that they give with a great facility the elements of an orbit, and I have applied them to the planet Hebe. The differences between the values thus obtained from the first approximations, and those to which I have been lead by the method exposed in the Memoir of 20 September, are extremely weak, in the same way as I will demonstrate it in more detail in another article.

[1]"Mémoire sur la détermination et la correction des éléments de l'orbite d'un astre." Comptes Rendus Hebd. Séances Acad. Sci. 25.