Application des formules que fournit la nouvelle méthode d'interpolation à la résolution d'un systme d'équations linéares approximatives, et, en particulier, à la correction des élements de l'orbite d'un astre*

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Being given a system of approximate and linear equations of which the number is equal or superior to the one of the unknowns, we suppose that one wishes to determine by an easy calculation, and with a great exactitude, if it is possible, the values of these unknowns. In order to fulfill these two conditions at the same time, it will suffice to recur to the formulas which my new method of interpolation furnishes. By aid of these formulas, one will be able to eliminate successively, from each of the given equations, the first, the second, the third… unknown; and when all the unknowns will have been eliminated, with the exception of one alone, an approximate value of this last will be furnished by each of the remaining equations. We add that the diverse approximate values of the last unknown will be equals among them, if the number of given equations is precisely the number of unknowns, but will be different generally from one another in the contrary case. We observe, finally, that the method in question will furnish, among the diverse approximate values of each unknown, a mean which will be able to be adopted with a great confidence.

The preceding considerations will be able to be usefully applied to the correction of the elements of the orbit of a star. We enter to this subject in some details.

When in following the march indicated in the Memoire of 20 September, one has determined approximately the distances of a star to the sun and to the [651] earth, one is able to deduce immediately, by aid of very simple formulas, the six elements of the orbit of this star. However, it is important to remark, by virtue of the calculations effected and of the formulas of which there is concern, the elements of the orbit depend, not only on the geocentric longitudes and latitudes of the observed star, but again on their derivatives of the first and of the second order. If, in order to increase the exactitude of the results, one wishes that the elements of the orbit be finally determined by

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. July 17, 2010 aid of formulas which contain no derivative, it will suffice to recur to the finite equations of the movement of the observed star. These last, transformed into approximate equations, will become linear with respect to the corrections of the six elements, and will be able then to be resolved as it has been said above. Besides, the number of the equations to resolve will depend on the number of observations given. Each observation furnishes two linear equations among the corrections of the six elements, three observations will be able, in rigor, to be sufficient for the calculation of the sought corrections; but those will be better determined if one makes a more considerable number of observations compete in their determination.

§ I — *General formulas for the resolution of a system of approximate and linear equations.*

Let be given, among n unknowns $x, y, z, \ldots$ some linear and approximate equations

$$(1) \qquad \begin{cases} ax + by + cz + \cdots = k, \\ a_{\prime}x + b_{\prime}y + c_{\prime}z + \cdots = k_{\prime}, \\ a_{\prime\prime}x + b_{\prime\prime}y + c_{\prime\prime}z + \cdots = k_{\prime\prime}, \\ \text{etc.} \end{cases}$$

in number equal or superior to the one of the unknowns. In order to deduce from these equations, by an easy method to follow, and with a great exactitude, if it is possible, the values of $x, y, z, \ldots$, it will suffice to recur to the method that we are going to indicate.

We designate by Sa the sum of the numerical values of the terms of the sequence $a, a_{\prime}, a_{\prime\prime}, \ldots$, and by Sb the sum of the terms of the sequence $b, b_{\prime}, b_{\prime\prime}, \ldots$, taken each with the sign $+$ or with the sign $-$, according as the corresponding term of the sequence $a, a_{\prime}, a_{\prime\prime}, \ldots$ is positive or negative. Let further $Sc, \ldots$ or Sk be that which Sb becomes, when in the sequence $b, b_{\prime}, b_{\prime\prime}, \ldots$ one substitutes the sequence $c, c_{\prime}, c_{\prime\prime}, \ldots$, or the sequence $k, k_{\prime}, k_{\prime\prime}, \ldots$ One will draw from formulas (1)

$$(2) \qquad xSa + ySb + zSc + \cdots = Sk.$$

[652] Let now $\alpha = \frac{a}{Sa}$, $\alpha = \frac{a_{\prime}}{Sa_{\prime}}, \ldots$, and put

$$\begin{aligned} &\Delta b = b - \alpha Sb, && \Delta c = c - \alpha Sc, \ldots, && \Delta k = k - \alpha Sk, \\ &\Delta b_{\prime} = b_{\prime} - \alpha_{\prime} Sb, && \Delta c_{\prime} = c_{\prime} - \alpha_{\prime} Sc, \ldots, && \Delta k_{\prime} = k_{\prime} - \alpha_{\prime} Sk, \\ &\text{etc.} \end{aligned}$$

If from the first, or from the second, or from the third, … of formulas (1), one subtracts formula (2), after having multiplied the two members of this last by α, or by $\alpha_{\prime}$, or by $\alpha_{\prime\prime}$, …, one will obtain, in the place of equations (1), the following:

$$(3) \qquad \begin{cases} y\Delta b + z\Delta c + \cdots = \Delta k, \\ y\Delta b_{\prime} + z\Delta c_{\prime} + \cdots = \Delta k_{\prime}, \\ y\Delta b_{\prime\prime} + z\Delta c_{\prime\prime} + \cdots = \Delta k_{\prime\prime}, \\ \text{etc.} \end{cases}$$

which no longer contains the variable x.

We imagine, at present, that one designates by $S'\Delta b$ the sum of the numerical values of the corresponding terms of the sequence Δb, $\Delta b_\prime$, $\Delta b_{\prime\prime}$, ..., and let $S'\Delta c$, ... or $S'\Delta k$ be the sum of the terms corresponding to the sequence Δc, $\Delta c_\prime$, $\Delta c_{\prime\prime}$, ..., or Δk, $\Delta k_\prime$, $\Delta k_{\prime\prime}$, ..., each taken with the sign $+$ or the sign $-$, according as the term corresponding to the sequence Δb, $\Delta b_\prime$, $\Delta b_{\prime\prime}$, ... is positive or negative. One will draw from equations (3)

$$yS'\Delta b + zS'\Delta c + \cdots = S'\Delta k; \tag{4}$$

next, by putting $\beta = \frac{\Delta b}{S'\Delta b}$, $\beta = \frac{\Delta b_\prime}{S'\Delta b_\prime}$, ... and

$$\begin{array}{lll} \Delta^2 c = \Delta c - \beta S'\Delta c, & \ldots, & \Delta^2 k = \Delta k - \beta S'\Delta k, \\ \Delta^2 c_\prime = \Delta c_\prime - \beta_\prime S'\Delta c, & \ldots, & \Delta^2 k_\prime = \Delta k_\prime - \beta_\prime S'\Delta k, \\ \text{etc.}, & & \end{array}$$

one will draw from formulas (3), joined to formula (4),

$$\left\{ \begin{array}{c} z\Delta^2 c + \cdots = \Delta^2 k, \\ z\Delta^2 c_\prime + \cdots = \Delta^2 k_\prime, \\ z\Delta^2 c_{\prime\prime} + \cdots = \Delta^2 k_{\prime\prime}, \\ \text{etc.} \end{array} \right. \tag{5}$$

By continuing thus, one will substitute successively into equations (1) equations (3), (5), etc.; and when one will have successively eliminated all the unknowns, with the exception of one alone, the remaining equations will furnish, [653] for the last unknown, of the approximate values which will be all equals among them, if the number of equations (1) is equal to the one of the unknowns, but will be able to differ from one another in the contrary case. We add that, in order to determine with a greater exactitude the values of the diverse unknowns, one should generally recur to formulas (2), (4), and similar others, from which one will draw

$$\left\{ \begin{array}{l} x = \dfrac{Sk}{Sa} - y\dfrac{Sb}{Sa} - z\dfrac{Sc}{Sa} \cdots, \\ y = \dfrac{S'\Delta k}{S'\Delta b} - z\dfrac{S'\Delta c}{S'\Delta b} \cdots, \\ z = \dfrac{S''\Delta^2 k}{S''\Delta^2 c} \cdots, \\ \text{etc.} \end{array} \right. \tag{6}$$

These last formulas, not arranged in the order according to which they are written, but in an inverse order, will furnish ordinarily for the last unknown, next for the next to last, etc., of the values so much more exact, as the difference between the number of the given approximate equations and the number of unknowns will be more considerable.

The order according to which the diverse unknowns are eliminated one after the other seem, first of all, to remain entirely arbitrary. But, in order to draw from the calculation some more exact results, or at least some results which merit a greater confidence, it is convenient to choose x, that is to say the first of the unknowns to eliminate, in a manner that the sum Sa is the greatest possible; next then to choose y, that is to say the second of the unknowns to eliminate, in a manner that the sum $S'\Delta b$ is the greatest possible; etc.

One could shorten the calculation, but by diminishing the degree of confidence that its results must inspire, if, in order to eliminate an unknown, one was limited to combine among them, by way of subtraction, the diverse equations, taken two by two, by subtracting the second from the first, the third from the second, and thus in sequence, after having previously reduced to unity, in each equation, the coefficient of the unknown which the concern is to eliminate.

§ II. — *On the determination of the elements of the orbit of a star.*

We conserve the notations of pages 409, 410, by substituting only the letter χ for the letter ψ, so that one has $\chi = \phi - \varpi$. Formulas (1) and (4) on page 410 will give

$$(1) \qquad x = R\cos\varpi + \rho\cos\phi, \quad y = R\sin\varpi + \rho\sin\phi, \quad z = \rho\tan\theta.$$

Let besides, at the end of time t, ψ be the excentric anomaly, and p [654] the heliocentric longitude of the observed star, measured in the plane of the orbit. Let further

$\mathfrak{p}$ be the value of p corresponding to the perihelion;
a, ε the semiaxis and the eccentricity of the orbit;
ι the inclination of the orbit, represented by an acute or obtuse angle, according as the movement of the star is direct or retrograde;
☊ the heliocentric longitude of the ascendant node;
T the duration of the revolution of the star in its orbit;
$\lambda = \frac{2\pi}{T} = \left(\frac{K}{a^3}\right)^{\frac{1}{2}}$ the ratio of the circumference 2π to T;
$-\frac{c}{\lambda}$ the epoch of passage from the star to the perihelion.

The radius vector r and the longitude p will be determined, as function of t, by the known formulas

$$(2) \qquad r = a(1-\varepsilon\cos\psi), \quad \tan\frac{p-\mathfrak{p}}{2} = \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{\frac{1}{2}}\tan\frac{\psi}{2}, \quad \psi - \varepsilon\sin\psi = \lambda t + c.$$

Moreover, if one projects successively the radius vector r on three rectangular axes, of which the first coincides with the line of the nodes, and the third with the axis of z, the three projections will be able to be expressed, either as function of $x, y, z,$ ☊ or as function of r, p, ι; and by equating the two values found for each projection to one another, one will have

$$(3) \qquad x\cos☊ + y\sin☊ = r\cos p, \; y\cos☊ - x\sin☊ = r\sin p\cos\iota, \; z = r\sin p\sin\iota.$$

Let now ω be the speed of the observed star; u, v, w the algebraic projections of this speed onto the axes of x, y, z; H the double of the area described by the radius vector r during the unit of time; U, V, W the algebraic projections of this area on the coordinate planes, and I its absolute projection on the same plane by the line of the nodes perpendicularly to the plane of the ecliptic. One will have

$$(4)\qquad \begin{cases} u = D_t x, \quad v = D_t y, \quad w = D_t z, \\ U = wy - vz, \quad V = uz - wx, \quad W = vx - uy, \\ \omega^2 = u^2 + v^2 + w^2, \quad H^2 = U^2 + V^2 + W^2, \quad I^2 = U^2 + V^2, \end{cases}$$

and the last two of formulas (3) will give

$$(5)\qquad x \sin \mho - y \cos \mho + z \cot \iota = 0, \quad u \sin \mho - v \cos \mho + z \cot \iota = 0;$$

consequently,

$$(6)\qquad \sin \delta = \frac{U}{H}, \quad \cos \delta = -\frac{V}{H}, \quad \cos \iota = \frac{W}{H}, \quad \sin \iota = \frac{I}{H}.$$

[655] Moreover, the equations of the quick forces and of the areas will give

$$\frac{1}{a} = \frac{2}{r} - \frac{\omega^2}{K}, \quad 1 - \varepsilon^2 = \frac{H^2}{Ka}.$$

The formulas which precede are able to be applied in diverse ways to the calculation of the elements of the orbit. Thus, for example, after having deduced $r, \rho, D_t \rho$ from the formulas established in the Memoir of 20 September, and x, y, z, u, v, w, U, V, W, ω, H, I of formulas (1) and (4), one will be able to draw immediately the values of the elements $\mho$, ι, a, ε from formulas (6) and (7), next the value of p from any one of the formulas (3), and the values of ψ, $\mathfrak{p}$, c from equations (2). We add that by virtue of formulas (1) and (3), one will have

$$(8)\qquad \begin{cases} R \cos(\varpi - \mho) + \rho \cos(\phi - \mho) = r \cos p, \\ R \sin(\varpi - \mho) + \rho \sin(\phi - \mho) = r \cos p \cos \iota, \\ \rho \tan \theta = r \sin p \sin \iota, \end{cases}$$

and, hence,

$$(9)\qquad \cot p = \frac{r^2 (D_t \ln \rho + D_t \Theta) - r D_t r}{H}, \quad \sin \iota = \frac{\rho \tan \theta}{r \sin p}.$$

Now these last formulas offer a new way to calculate p and ι.

The calculations which we just indicated are able to be simplified in the following manner:

We make those turn with the coordinate axes which contain the plane of the ecliptic, in a manner to make the axis of the abscissas coincide with the radius vector ρ, and we name $\mathfrak{x}, \mathfrak{y}, \mathfrak{u}, \mathfrak{v}, \mathfrak{U}, \mathfrak{V}$ that which x, y, u, v, U, V then become. The last six of the equations (4) and the formulas (6) will continue to subsist when one will replace x, y, u, v, U, V by $\mathfrak{x}, \mathfrak{y}, \mathfrak{u}, \mathfrak{v}, \mathfrak{U}, \mathfrak{V}$ and $\mho$ by $\mho - \phi$. One will have therefore

$$(10)\qquad \begin{cases} \mathfrak{U} = w\mathfrak{y} - \mathfrak{v} z, \quad \mathfrak{V} = \mathfrak{u} z - w\mathfrak{x}, \quad W = \mathfrak{v}\mathfrak{x} - \mathfrak{u}\mathfrak{y}, \\ \omega^2 = \mathfrak{u}^2 + \mathfrak{v}^2 + w^2, \quad H^2 = \mathfrak{U}^2 + \mathfrak{V}^2 + W^2, \quad I^2 = \mathfrak{U}^2 + \mathfrak{V}^2, \end{cases}$$

$$(11) \qquad \sin(\delta-\phi)=\frac{\mathfrak{U}}{H}, \quad \cos(\delta-\phi)=-\frac{\mathfrak{V}}{H}, \quad \cos\iota=\frac{W}{H}, \quad \sin\iota=\frac{I}{H}.$$

Besides, in equations (10), the values of $\mathfrak{x}, \mathfrak{y}, \mathfrak{u}, \mathfrak{v}, w$ will be determined very simply by aid of the formulas

$$(12) \qquad \mathfrak{x}=\rho+R\cos\chi, \quad \mathfrak{y}=-R\sin\chi, \quad z=\rho\tan\theta,$$

$$(13) \qquad \begin{cases} \mathfrak{u}=D_t\rho+\mathscr{R}\sin(\chi+\Pi), \quad \mathfrak{v}=\rho D_t\phi+\mathscr{R}\cos(\chi+\Pi) \\ w=(D_t\rho+\rho D_t\Theta)\tan\theta, \end{cases}$$

[656] $\mathscr{R}$ and Π being determined themselves by the equations

$$D_tR=\mathscr{R}\sin\Pi, \quad RD_t\varpi=\mathscr{R}\cos\Pi.$$

§ III. — *Correction of the elements of the orbit of the star.*

We suppose after having calculated approximately, by aid of the formulas reported above, the six elements of the orbit of a star, that is to say the six quantities a, ε, c, $\mathfrak{p}$, ♉, ι, one wishes to determine with a great exactitude the very small corrections δa, $\delta\varepsilon$, δc, $\delta\mathfrak{p}$, δ♉, $\delta\iota$, which these elements must incur. It will suffice to recur to the finite equations of movement of the star, and to compare among them the values of r and of p drawn from formulas (2) and (8) of § II. This comparison will furnish, among the corrections δa, $\delta\varepsilon$, δc, $\delta\mathfrak{p}$, δ♉, $\delta\iota$, assumed very small, two linear equations, of which one will not contain $\delta\mathfrak{p}$. Therefore each observation will furnish, among the corrections of the six elements, two distinct equations. By resolving the equations thus obtained by the method exposed in § I, one will obtain easily the six corrections sought, as I will explain in more detail in a new article. If the star has been observed more than four times, the corrections of the single elements a, ε, c, δ, ι will be able to be determined separately by aid of the equations which do not contain $\delta\mathfrak{p}$.