Mémoire sur le degré d'exactitude avec lequel on peut déterminer les orbites des planètes et des comètes

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§I. — *General considerations.*

In proceeding from the formulas of which I myself am served in my preceding Memoirs, in order to calculate the distances from Mercury and from Hebe to the sun and to the earth, and in being served by observations made in an interval of time during which the perturbations of the movement of a star would remain insensible, one would be able to determine exactly the orbit of this star, if one arrived to obtain the developments of the geocentric longitude and latitude of the same star according to the ascending powers of the time t, or at least the coefficients of the terms which, in these developments, contained the powers of time inferior to the fourth. It is to form these coefficients that the methods of interpolation serve. Besides, the results furnished by these methods would seem to ought to be so much more exact, as the number of observations employed is more considerable. It is nevertheless that which does not happen always, and one must make on this subject an important remark. The older methods of interpolation, for example the methods of Lagrange and Laplace, are able to make a number n observations superior to four contribute to the determination of the coefficients of the first four terms of each development, only under the condition of introducing into the sought development all the powers of time of a degree inferior to n. Now this condition is very little favorable to the precision of the calculations, seeing that the errors of observation are able to occasion, in the determination of the coefficient of a power of t, some errors so much greater as this power is of a higher degree. There results from it that, in the rather ordinary case where the development of a variable would be able, in the interval of time which separates the extreme observations, to be sensibly reduced to its first four terms, the terms following, sensibly null, would appear often to acquire some considerable values, if, in making use of the method of interpolation of Lagrange or of Laplace, one would wish to make more than four observations serve in the determination of the coefficients of the first four terms. There

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. July 9, 2010 is more: the determination of the coefficient of t^2, and especially of the coefficient of t^3, effected by aid of these methods, will be often not very exact, not only when one will make use of four observations only, but also when this number of observations will become superior to four. On the contrary, in the case of which there is concern, a [573] new method of interpolation will be able to make more than four observations agree with advantage in the determination of the first four terms of the development of a variable, provided that one takes care to stop at the instant where the calculation will furnish the differences comparable to the errors of observation.

We have here supposed that, in the development of a variable, the terms proportional to the fourth power of time and to some superior powers were sensibly null, at least in the interval of time which separates from one another the two extreme observations. This circumstance, which assures the exactitude of the results obtained, will be found indicated a posteriori with a great degree of probability, when in following any method of interpolation, for example the method of Lagrange or of Laplace, one will have determined by aid of four observations the first four coefficients, if the development found represented these four observations and all the intermediate observations, with enough exactitude in order that the differences between the observed values and the values calculated from the variable are comparable to the errors of observation. The same circumstance will be found again indicated a posteriori with much probability, if, in making by the new method all the given observations agree in the determination of the first four coefficients, one finds, for the fourth differences, or for the differences of a lesser order, the numerical values comparable to the errors that the observations involve. Finally, the circumstance of which there is question is able to be indicated a priori, in many cases, by a calculation of which I am going to give an idea in a few words.

Experience proves that, for some stars of which the light is very feeble, some errors of observation of four or five sexagesimal seconds do not pass at all the limits of the possible, nor even of the probable. Therefore, if one seeks the developments of the variables, especially of the geocentric longitude and latitude of a star in a series ordered according to the ascending powers of time, it will be perfectly useless to conserve, in these developments, the terms of which the omission would carry at most error of four or five seconds. Besides, the terms of a higher rank, in the developments of the longitude and of the latitude of a star, will be ordinarily some quantities of the same order as the terms of the same rank in the development of the true anomaly; and, in this last development, a limit superior to the coefficient of the fourth power, or of a higher power of time, is able to be determined approximately by diverse methods, for some stars further from us than the sun, especially if [574] the eccentricity in not very near to unity. Therefore, for such stars, one will be able to calculate approximately a superior limit to the interval of time which will separate the middle observation from each of the extreme observations, when these three observations will be near enough one another in order that the omission of the terms proportional to the fourth power, or of some higher powers, produce only an error of four or five seconds. After having calculated this limit, and choosing arbitrarily the mean observation starting from which the time will be computed, one must choose yet the other observations in number equal or superior to three, in a manner that each of them is separated from the middle observation by an inferior interval, or at most equal to the limit of which there is question.

If this condition is not able to be fulfilled, then, in order to obtain a value sufficiently exact of the coefficient of the cube of time, one will be obliged to admit in the sought development some terms proportional to the fourth power of time. One would be able to calculate, always by aid of the same process, a superior limit to the interval of time which must separate the extreme observations from the middle observation, so that the error occasioned in the development of the variable by omission of the terms proportional to the fifth power of time, and to some higher powers, do not pass at all four or five sexagesimal seconds.

We remark finally that after having fixed, on the one hand, the number of those of the given observations which must contribute to the determination of the sought development; on the other, the number of terms of this same development, one will have, if this last number does not surpass four or five, is limited to pushing the evaluation of the coefficient of each of the conserved terms until a decimal digit such, that the omission of the digit following, at the epoch of each of the extreme observations, occasions at the most, in the value of the term of which there is question, an error of one second.

In operating as we just said, one will render much more easily the application of the methods of interpolation, even of the most exact, and, in particular, of that which I have proposed in the determination of the orbits of the stars. Because that which lengthened especially the calculations, it was the determination of a multitude of useless digits that one introduced, because one knew not how to render account well in advance of the influence that the errors of observation would be able to exercise on the results furnished by a given method of interpolation. The principles that I just indicated, and that I am going to develop in the paragraph following, will permit to the astronomers to form a just idea of this influence, and to choose, in acquaintance of cause, the [575] method which will lead more promptly or more surely to the solutions demanded.

§ II. — *On the errors occasioned in the developments of the geocentric longitude and latitude of a star through the errors of observation.*

We suppose that, the epoch of a certain astronomical observation being taken for origin of time t, one wishes to develop, according to the ascending powers of t, the geocentric longitude and latitude of the observed star, by conserving in each development only the sensible terms, and neglecting those of which the omission would produce only an error of four or five sexagesimal seconds, that is to say an error comparable to the errors that the observations involve. We suppose further that, by any means, one is arrived to understand in advance the number n of the terms which must be conserved, beyond the first, and which in each development must follow this first term independent of t. It is clear that if, on the one hand, the neglected terms, and, on the other, the errors of observation were reduced rigorously to zero, one would be able to obtain the exact values of the conserved terms, by making contribute in the determination of their coefficients, by aid of any method of interpolation, the given observations, provided that the number of these observations may be at least equal to $n+1$.

We suppose that the neglected terms were always null, the errors of observation were not null. Then the polynomial that one will obtain, by making any method of interpolation serve in the determination of the sought coefficient, will be composed of two parts, of which the first will be the sought development, the second part being that which this same development becomes when one replaces the particular given values of the variable by the errors of which these particular values are found affected by virtue of the same observations. This put, we imagine that the observation of which the epoch serves as origin to time being placed toward the middle of the interval which separates the extreme observations, one designates it under the name of *mean* observation. We name m the number of observations distinct from the mean observation, and

$$t_1, t_2, \ldots, t_m$$

the positive and negative values of t which correspond to these last observations. Let besides ϕ be the variable of which the development is counted to be able to be exactly represented by the first $n+1$ terms of a polynomial [576] of degree n; let ϕ_0 be the value of ϕ corresponding to the mean observation, and $\Delta\phi = \phi - \phi_0$ the difference of ϕ to ϕ_0. Let further

$$\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m$$

be the errors of which the diverse values of $\Delta\phi$ are affected, by virtue of the given observations, and of which each will be able to be double of the error which involves a single observation, the error of the mean observation and that of any one of the others being able to have been committed in contrary sense. Finally, we name ε the excess of the polynomial which represents the development of the variable ϕ, deduced by a certain method of interpolation on the true value of ϕ. Then ε will be precisely the polynomial that one will deduce by the same method of interpolation, in taking

$$\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m$$

for the values of ε corresponding to the epochs

$$t_1, t_2, \ldots, t_m.$$

Now it is clear that the method of interpolation employed will furnish a development of ϕ more or less exact, according as the extreme limits, positive and negative, between which the value of the polynomial ε will remain contained, will be more or less narrowed. Besides, the degree of the polynomial ε will be the number m of observations distinct from the mean observation, if the method employed is that of Lagrange or Laplace, and, in one and the other case, the diverse terms of which the development of ε will be composed will offer precisely the same values. Therefore, in order to judge of the degree of exactitude which these two methods will furnish, it will suffice to examine that which the formula of interpolation of Lagrange will furnish. We enter, to this subject, in some details.

The value of ε, determined by the formula of interpolation of Lagrange, will be composed of m terms respectively proportional to the particular values of ε. One will have effectively

$$\varepsilon = \varepsilon_1 T_1 + \varepsilon_2 T_2 + \cdots + \varepsilon_m T_m, \tag{1}$$

$T_1, T_2, \ldots, T_m$ being some functions of t, of which each will be determined by an equation of the form

$$T_1 = \frac{t(t-t_2)\cdots(t-t_m)}{t_1(t_1-t_2)\cdots(t_1-t_m)} \tag{2}$$

This put, the numerical values of the coefficients

$$T_1, T_2, \ldots, T_m,$$

[577] and, hence, the numerical value of ε would be able to become very considerable, if one made the value of t correspond to an epoch situated beyond the interval contained between the extreme observations. We admit now the contrary supposition, and we name δ the limit of the errors of observation that one is able to evaluate to four or five sexagesimal seconds.

If the observations given and distinct from the mean observation are in the number of two, then t_1, t_2 will be affected of contrary signs, and one will have

$$\varepsilon = \varepsilon_1 T_1 + \varepsilon_2 T_2,$$
$$T_1 = \frac{t(t-t_2)}{t_1(t_1-t_2)}, \quad T_2 = \frac{t(t-t_2)}{t_2(t_2-t_1)},$$
$$T_1 + T_2 = -\frac{t(t-t_1-t_2)}{t_1 t_2}.$$

Therefore, if the quantities t_1, t_2 being affected with the same sign, the numerical values of these quantities attained the limit 2δ, one will have

$$\varepsilon = \pm\frac{t(t-t_1-t_2)}{t_1 t_2}2\delta.$$

This last value of ε will be able to become considerable for a value of t contained between t_1, t_2, for example for $t = \frac{t_1+t_2}{2}$, if the numerical value of one of the ratios $\frac{t_1}{t_2}, \frac{t_2}{t_1}$ is superior to $3+2\sqrt{2}$. If, in order to fix the ideas, one supposes $t_1 = -10t_2$, the found value of ε will become

$$\varepsilon = \pm\frac{121}{40}2\delta;$$

and, hence, if one puts $\delta = 5''$, one will have sensibly $\varepsilon = \pm 30''$. Therefore, when the given observations are not equidistant, the value of ε, determined by the formula or Lagrange or of Laplace, is able, even in the case where one makes usage of three observations alone, and for an epoch intermediate among those of the given observations, pass notably the limits of the errors of observation.

The inconvenience that we ourselves just signaled becomes more grave, in the case precisely where one seeks to obtain some more exact results in making a greater number of observations contribute to the solution of the problem. In order to demonstrate it, we consider a case which will present itself often in practice. We suppose that an overcast sky is interrupted with clouds, [578] during a certain lapse of time, a series of astronomical observations, separated from one another by an interval of around one day, and taken as soon as the sky is become again tranquil. Then it is easy to see that the numerical value of ε will be able to become very notably superior to the errors of observation. It is that which will arrive, for example, if one uses, beyond the mean observation, six observations of which the epochs are represented by the numbers

$$-8, -7, -6, 1, 2, 3.$$

Then, by attributing to t the value -3 contained among the given values, one will find, by following any method of interpolation, for example the method of Lagrange or of Laplace, and by making all the observations contribute in the determination of ε,

$$\varepsilon = \frac{3}{11}\varepsilon_1 - \frac{15}{14}\varepsilon_2 + \frac{25}{21}\varepsilon_3 - \frac{75}{14}\varepsilon_4 + 3\varepsilon_5 - \frac{20}{33}\varepsilon_6.$$

Therefore if, the mean observation being exact, the errors of the other observations are each of 5 seconds, but alternatively positive and negatives, one will have sensibly

$$\varepsilon = \pm 57.5''.$$

But if one has some reasons to believe that, in the development of the sought variable, one is able to neglect without sensible error the terms proportional to the fourth power of time or to some higher powers, and if then one made all the observations contribute, by my new method, in the determination of ε, then one will obtain a value of ε comparable to the errors of observation, and one will find, in particular, for $t = -3$, no longer $\varepsilon = \pm 57.5''$, but only $\varepsilon = \pm 2''$.

In that which precedes, we have especially considered the values of ε corresponding to some epochs intermediate between those of the extreme observations. If one employed values of t corresponding to some epochs which were situated beyond the extreme observations, without being even very distant from them, the numerical values of ε, and, hence, the errors contained in the value of the variable ϕ, would be able to become very considerable. Thus, for example, if the variable ϕ represented the geocentric longitude of the new planet Hebe, at the epoch of 12 July, and if one made four of the seven observations recalled in the previous session serve in the determination of ϕ, by following any method of interpolation, then one will find, 1° by joining to the observation of 12 August the four [579] following,

$$\Delta\phi = -561.44''t + 30.62''\frac{t^2}{2} - 1.164''\frac{t^3}{6};$$

2° by joining to the observation of 12 August the three preceding,

$$\Delta\phi = -544.49''t + 35.22''\frac{t^2}{2} - 3.45''\frac{t^3}{6}.$$

Here the difference among the values of ϕ is enormous; and this difference represented, excepting sign, by the polynomial

$$16.95t + 4.60\frac{t^2}{2} + 4.614''\frac{t^3}{6},$$

is raised already to $917.4''$ for $t = 9.06847$, that is to say at the epoch of the last observation.