Méthodes nouvelles pour la détermination des orbites des corps célestes, et, in particulier, des comètes*

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In the calculations relative to the determination of the orbit which describes a celestial body, for example a comet, one must distinguish two kinds of quantities. The one, namely the geocentric longitude and the latitude of the comet, and their derivatives taken with respect to time, are immediately furnished by observations, or, at least, are deduced from them, for a given epoch, with an exactitude so much greater, as the number of observations made at some neighboring epochs is more considerable. The comet being counted to describe a conic section, and the quantities of which I just spoke, or many among them, being supposed known, the other quantities, for example the distance from the comet to the earth, or rather the projection of this distance onto the plane of the ecliptic, the inclination of the orbit, the direction of the line of the nodes, etc., are deduced from the equations of movement, by aid of approximate or exact formulas. Among the approximate formulas, one must note those which Lambert, Olbers, Legendre, and, in last place, MM. de Gasparis and Michal have given. Among the [888] exact formulas, one must distinguish those to which Lagrange, Laplace and M. Gauss have arrived. Lagrange and Laplace have brought the problem back to the resolution of an equation of the seventh degree. That which Mr. Gauss has found is of the eighth degree, but it is able to be reduced, as Mr. Binet has remarked, in a Memoir which the *Journal de l'École Polytechnique* contains, to the equation already mentioned of the seventh degree. Besides this equation, as Mr. Gauss has acknowledged it, offers four or six imaginary roots. We add that the coefficients that it contains are able to be determined, at least approximately, by aid of three observations of the comet. But as, in the case where three roots are real, two different orbits are able to satisfy the question, there results from it that, in order to obtain, in all cases, an orbit completely determined, one must suppose at least four observations made in some neighboring epochs, or rather the quantities from which the approximate values are able to be calculated by aid of these four observations. I have sought, in admitting this assumption, a simple way to resolve the problem. Astronomers will learn, I hope, with pleasure, that one is able, in each case, to reduce it to the resolution of a single equation of first degree.

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I will add that in supposing known the single quantities of which the approximate determination is able to be effected by aid of three observations, I bring the problem back to the resolution of a single equation of the third degree.

ANALYSIS

We take for the x, y plane the plane of the ecliptic, for the semiaxes of the positive x and y, the straight lines drawn from the center of the sun to the first points of Aries and of Cancer, and we suppose the positive z measured on a perpendicular to the plane of the ecliptic on the side of the north pole. Let besides

x, y, z	be the coordinates of the planet or the comet that one considers;
r	the distance from this comet to the sun;
x, y	the coordinates of the earth;
R	the distance from the earth to the sun;
ϖ	the heliocentric longitude of the earth;
α, θ	the longitude and geocentric latitude of the comet;
ι	the distance from the earth to the comet;
ρ	the projection of this distance onto the plane of the ecliptic.

One will have

(1) $$x = \mathrm{x} + \rho \cos \alpha, \quad y = \mathrm{y} + \rho \sin \alpha, \quad z = \rho \tan \theta,$$

[889] and

(2) $$x = R \cos \varpi, \quad \mathrm{y} = R \sin \varpi.$$

Moreover, by taking for unity of mass the mass of the sun, and for unity of distance the mean distance of the earth to the sun, one will have further

(3) $$D_t^2 x + \frac{x}{r^3} = 0, \quad D_t^2 y + \frac{y}{r^3} = 0, \quad D_t^2 z + \frac{z}{r^3} = 0,$$

and

(4) $$D_t^2 \mathrm{x} + \frac{\mathrm{x}}{R^3} = 0, \quad D_t^2 \mathrm{y} + \frac{\mathrm{y}}{R^3} = 0.$$

Now, from formulas (3), joined to equations (1) and (4), one draws

(5) $$D_t \rho = A\rho, \quad D_t^2 \rho + \frac{\rho}{r^3} = B\rho, \quad \frac{1}{r^3} - \frac{1}{R^3} = C\rho,$$

the values of the coefficients A, B, C being determined by the system of formulas

(6) $$\begin{cases} C\mathrm{x} + [B - (D_t \alpha)^2] \cos \alpha - (D_t^2 \alpha + 2AD_t \alpha) \sin \alpha = 0. \\ C\mathrm{y} + [B - (D_t \alpha)^2] \sin \alpha - (D_t^2 \alpha + 2AD_t \alpha) \cos \alpha = 0. \end{cases}$$

(7) $$B\Theta + 2AD_t \Theta + D_t^2 \Theta = 0,$$

and the value of Θ being

$$\Theta = \tan\theta. \tag{8}$$

Besides one will draw, from formulas (1) and (2),

$$r^2 = R^2 + 2R\rho\cos(\alpha - \varpi) + (1 - \Theta^2)\rho^2. \tag{9}$$

Knowing the movement of the earth, one knows hence, at any epoch, the values of the quantities x, y, R, ϖ. On the other hand, the values of the quantities α, θ, and the derivatives of these quantities differentiated with respect to time, are able to be deduced, for a given epoch, from observations made at some neighboring epoch, with an exactitude so much greater, as the number of observations is more considerable. One is able to arrive by aid of the formula of interpolation due to Newton and employed by Laplace, or better still, by aid of those which I have given in a Memoir lithographed at Prague in 1837, and reprinted in the Journal of M. Liouville.[1]

The values of

$$\alpha, \quad D_t\alpha, \quad D_t^2\alpha; \quad \theta, \quad D_t\theta, \quad D_t^2\theta,$$

[890] being known, equations (6) and (7) will determine the coefficient A, B, C, and one will be able to draw since then, from formulas (5) and (9), the values of

$$\rho, \quad r, \quad D_t\rho, \quad D_t^2\rho.$$

If one considers in particular the last of equations (5), it will suffice to eliminate ρ or r from it by aid of formula (9) in order to obtain the equation in r or ρ that Lagrange and Laplace give, and which is of the seventh degree.

We imagine now that in the first of the equations (5) one joins its derivative

$$D_t^2\rho = AD_t\rho + \rho D_tA;$$

and will conclude from it

$$D_t^2\rho = (A^2 + D_tA)\rho. \tag{10}$$

Besides, the two last equations (5) give

$$D_t^2\rho = \left(B - \frac{1}{R^3} - C\rho\right)\rho. \tag{11}$$

By equating one to the other the two preceding values of $D_t^2\rho$, one will find

$$C\rho = B - A^2 - D_tA - \frac{1}{R^3}. \tag{12}$$

Such is the equation of the first degree which will furnish immediately the value of the unknown ρ.

[1] *Translator's note*: Mémoire sur l'interpolation.

In order to draw practically from equation (12) the value of ρ, that is to say the distance of a comet or a planet from the earth, or rather the projection of this distance onto the plane of the ecliptic, it is necessary to know at least four complete observations, so that one may be able to calculate at least approximately the derivatives of the third order of α and of θ, contained in the value of D_tA.

Moreover, when the question is of a comet, and when the orbit is assumed parabolic, one is able, from formulas (5) and (9) joined to the equation of lively forces, to deduce easily a new equation which, being only of the third degree with respect to the unknown ρ, no longer contains the derivatives of the third order $D_t^3\alpha$, $D_t^3\theta$. One will arrive, in fact, by operating as it follows.

Let a be the semimajor axis of the described orbit. One will have generally

$$\frac{1}{a} = \frac{2}{r} - (D_tx)^2 - (D_ty)^2 - (D_tz)^2. \tag{13}$$

Besides, from equation (13), joined to formulas (1) and to the first of the [891] formulas (5), one will draw

$$\frac{2}{r} = \frac{1}{a} + \mathscr{A} + \mathscr{B}\rho + \mathscr{C}\rho^2, \tag{14}$$

the values of $\mathscr{A}, \mathscr{B}, \mathscr{C}$ being

$$\begin{cases} \mathscr{A} = (D_t\mathrm{x})^2 + (D_t\mathrm{y})^2, \\ \mathscr{B} = (A\cos\alpha - \sin\alpha\, D_t\alpha)D_t\mathrm{x} + (A\sin\alpha + \cos\alpha\, D_t\alpha)D_t\mathrm{y}, \\ \mathscr{C} = A^2 + (D_t\alpha)^2 + (A\Theta + D_t\Theta)^2. \end{cases} \tag{15}$$

If the described orbit is reduced to a parabola, so that one has $\frac{1}{a} = 0$, equation (14) will give simply

$$\frac{2}{r} = \mathscr{A} + \mathscr{B}\rho + \mathscr{C}\rho^2. \tag{16}$$

On the other hand, the last of equations (5), presented under the form

$$\frac{1}{r^3} = \frac{1}{R^3} + \mathscr{C}\rho, \tag{17}$$

and combined, by way of multiplication, with equation (9), will give

$$\frac{1}{r} = \left(\frac{1}{R^3} + C\rho\right)[R^2 + 2R\rho\cos(\alpha - \varpi) + (1 + \Theta^2)\rho^2].$$

Therefore, in regard to equation (16), one will have

$$2\left(\frac{1}{R^2} + C\rho\right)[R^2 + 2R\rho\cos(\alpha - \varpi) + (1 + \Theta^2)\rho^2] = \mathscr{A} + \mathscr{B}\rho + \mathscr{C}\rho^2. \tag{18}$$

Such is the equation of the third degree, by aid of which one will deduce easily the value of the distance ρ, of the values of α, θ and of their derivatives of the first and of the second order, when the given star will be a comet of which the orbit will be obviously parabolic.

It is good to observe that if, in naming ω the speed of the comet, one puts

$$r^2 = \mathscr{R}, \quad \omega^2 = \Omega; \tag{19}$$

one will have, in regard to formulas (9) and (15),

$$\begin{cases} \mathscr{R} = R^2 + 2R\rho\cos(\alpha - \varpi) + (1 + \Theta^2)\rho^2, \\ \Omega = \mathscr{A} + \mathscr{B}\rho + \mathscr{C}\rho^2, \end{cases} \tag{20}$$

[892] and, by virtue of formula (13),

$$\frac{2}{r} = \frac{1}{a} + \Omega. \tag{21}$$

Now, by differentiating the first of equations (19) and formula (21), one will find

$$2rD_t r = D_t\mathscr{R}, \quad 2r^{-2}D_t r = -D_t\Omega,$$

and, hence,

$$\frac{1}{r^3} = -\frac{D_t\Omega}{D_t\mathscr{R}}. \tag{22}$$

From this last formula, combined with equation (17), one draws

$$\left(\frac{1}{R^3} + C\rho\right) D_t\mathscr{R} + D_t\Omega = 0. \tag{23}$$

Besides, $\mathscr{R}$ and Ω will be determined as functions of ρ by formulas (20), and, by virtue of these formulas, joined to the first of equations (5), $D_t\mathscr{R}$, $D_t\Omega$ will be, in the same way $\mathscr{R}$ and Ω, entire functions of ρ, of the second degree. Therefore equation (23) will be of the third degree in ρ; and this equation, which will subsist, in the same case where the given star will cease to be a comet, and where the orbit will cease to be parabolic, will be able to be substituted with advantage in equation (18). We add that equation (23), as equation (18), contains only, with the angles α, θ, their derivatives of the first and of the second order, that is to say of the quantities of which the approximate values are able to be determined by aid of three observations.