

# Mémoire sur les coefficients limitateurs ou restricteurs\*

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§ I<sup>st</sup> — *General considerations.*

We imagine that being given diverse particular values

$$u_0, u_1, u_2, \dots$$

of a function  $u$  of independent variables  $x, y, z, \dots, t$  with the sum  $s$  of these values of which the number is able to be finite or infinite, one demands that which this sum becomes, when one restricts it to a least number of terms, and when one conserves only the terms corresponding to the values of  $x, y, z, \dots, t$ , which certain conditions confirm. In order to resolve the proposed question, it will suffice evidently to substitute in the function  $u$  the product of this function with a coefficient  $I$  which has the double property of being reduced to unity when the enunciated conditions will be fulfilled and to vanish in the contrary case. This coefficient, which I name, in order to indicate its role, coefficient *limitator* or *restrictor*,<sup>1</sup> will be able besides to assume a great number of diverse forms. We suppose, in order to fix ideas, that a restrictor must either be reduced to unity or vanish, according as the variable  $t$  is positive or negative. This restrictor will be able to be represented by any one of the expressions [151]

$$\frac{1}{2} \left( 1 + \frac{t}{\sqrt{t^2}} \right), \quad \frac{1}{2} \left( 1 + \frac{2}{\pi} \int_0^\infty \frac{\sin t \alpha}{\alpha} d\alpha \right), \quad \frac{1}{2} \left( 1 + \frac{1}{\pi} \int_{-\infty}^\infty \frac{t}{t^2 + \alpha^2} d\alpha \right), \\ \frac{1}{2\pi} \int_{-\infty}^\infty \int_0^\infty e^{\alpha(\lambda-t)i} d\lambda d\alpha, \dots$$

If besides, by adopting the notation that I have proposed in a preceding Memoir, one represents by  $I_t$  any one of the preceding expressions, a restrictor  $I$ , which could be

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<sup>1</sup>In the *Comptes rendus* of 1849, I have indicated the factors of this kind under the name *coefficients limitateurs*. The work *restricteurs*, which is shorter, offers also the advantage of expressing well the role that these coefficients enjoy in the calculation.

reduced to unity only for some real values of  $t$  contained between some limits  $t_I, t_{II}$ , will be able to be expressed by aid of the formula

$$(1) \quad I = I_{t-t_I} - I_{t-t_{II}};$$

and from this formula, combined with the equation

$$I_t = \frac{1}{2} \left( 1 + \frac{t}{\sqrt{t^2}} \right),$$

one will draw immediately

$$(2) \quad I = \frac{1}{2} \left[ \frac{t-t_I}{\sqrt{(t-t_I)^2}} - \frac{t-t_{II}}{\sqrt{(t-t_{II})^2}} \right].$$

On the contrary, by having regard to the equation

$$I_t = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_0^{\infty} e^{\alpha(\lambda-t)i} d\lambda d\alpha,$$

one will find

$$(3) \quad I_t = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{t_I}^{t_{II}} e^{\alpha(\lambda-t)i} d\lambda d\alpha,$$

Similarly,  $v$  being a real function of some variables  $x, y, z, \dots, t$ , a restrictor which could be reduced to unity only for the values of  $v$  contained between two given limits  $v_I, v_{II}$  will be able to be expressed by aid of one of the formulas

$$(4) \quad I = I_{v-v_I} - I_{v-v_{II}},$$

$$(5) \quad I = \frac{1}{2} \left[ \frac{v-v_I}{\sqrt{(v-v_I)^2}} - \frac{v-v_{II}}{\sqrt{(v-v_{II})^2}} \right].$$

$$(6) \quad I_t = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{v_I}^{v_{II}} e^{\alpha(\lambda-t)i} d\lambda d\alpha.$$

It will be equally easy to find a restrictor  $I$  which could be reduced to unity only in the case where the variables  $x, y, z, \dots, t$  verifies at the same time many given conditions. Thus, for example, if  $I$  must be reduced to unity, in the case only where all these variables are positive, one will be able to take

$$(7) \quad I = I_x I_y I_z \dots I_t,$$

and if  $I$  must be reduced to unity, in the case only where two real functions  $v, w$  of these variable are composed, the first between the limits  $v_I, v_{II}$ , the second between the limits  $w_I, w_{II}$ , one will be able to take

$$(8) \quad I = (I_{v-v_I} - I_{v-v_{II}})(I_{w-w_I} - I_{w-w_{II}}),$$

or else further

$$(9) \quad I = \left( \frac{1}{2\pi} \right)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{w_I}^{w_{II}} \int_{v_I}^{v_{II}} e^{\alpha(\lambda-v)i + \beta(\mu-w)i} d\lambda d\mu d\alpha d\beta.$$

[152] The introduction of the restrictors in the calculation permits resolving easily a question which is not without importance, and which we are going to indicate.

We consider  $n$  real variables

$$x, y, z, \dots, v, w,$$

and  $n$  real definite integrals

$$(10) \quad \int_{x_I}^{x_{II}} X dx, \quad \int_{y_I}^{y_{II}} Y dy, \dots, \quad \int_{v_I}^{v_{II}} V dv, \quad \int_{w_I}^{w_{II}} W dw, \quad ,$$

$X$  being a function of  $x$ ,  $Y$  of  $y$ , ...,  $V$  of  $v$ ,  $W$  of  $w$ . The product  $\Pi$  of these integrals will be the multiple integral which presents the formula

$$(11) \quad \Pi = \int_{x_I}^{x_{II}} \int_{y_I}^{y_{II}} \dots \int_{v_I}^{v_{II}} \int_{w_I}^{w_{II}} XY \dots VW dw dv \dots dy dx$$

Besides, by regarding each of the integrals (10) as a sum of infinitely small elements of one of the forms

$$(12) \quad X dx, \quad Y dy, \dots, \quad V dv, \quad W dw,$$

and, hence, the product  $\Pi$  as a sum of partial products of the form

$$(13) \quad XY \dots VW dx dy \dots dv dw,$$

one is able to demand that which the product  $\Pi$  will become, if one takes account solely of the partial products corresponding to some values of  $x, y, \dots, w$ , which fulfill certain conditions, and if, by conserving these, one separates all the others. We imagine, in order to fix the ideas, that the conserved partial products correspond uniquely to the values of  $x, y, \dots, v, w$ , which reduce a certain function  $w$  to a quantity contained between two given limits  $w_I, w_{II}$ . By naming  $P$  the sum of these partial products, and by putting

$$(14) \quad I = I_{w-w_I} - I_{w-w_{II}},$$

one will find

$$(15) \quad P = \int_{x_I}^{x_{II}} \int_{y_I}^{y_{II}} \dots \int_{v_I}^{v_{II}} \int_{w_I}^{w_{II}} IXY \dots VW dw dv \dots dy dx.$$

One will be able besides to give to the restrictor  $I$  the form

$$(16) \quad I = \frac{1}{2} \left[ \frac{w - w_I}{\sqrt{(w - w_I)^2}} - \frac{w - w_{II}}{\sqrt{(w - w_{II})^2}} \right].$$

[153] or else again the form

$$(17) \quad I = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{w_l}^{w_h} e^{\theta(\tau-w)^i} d\tau d\theta.$$

From equation (15), joined to formula (17), one will draw

$$(18) \quad P = \int_{x_l}^{x_h} \int_{y_l}^{y_h} \dots \int_{v_l}^{v_h} \int_{w_l}^{w_h} XY \dots VW \Theta d\tau dv \dots dy dx.$$

the value of  $\Theta$  being

$$(19) \quad \Theta = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{w_l}^{w_h} W e^{\theta(\tau-w)^i} dwd\theta.$$

On the other hand, if one puts, for brevity,

$$\vartheta = \sqrt{(D_w \omega)^2},$$

and if one designates by the notation

$$\begin{aligned} w &= w_h W \\ w &= w_l \vartheta \end{aligned}$$

the sum of the values that the ratio is able to acquire

$$\frac{W}{\vartheta},$$

for the values of  $w$  proper to test the equation

$$(20) \quad \omega = \tau,$$

and contained between the limits  $w_l, w_h$ , formula (19) will give [see the XIX<sup>th</sup> notebook of the *Journal de l'École Polytechnique*]

$$(21) \quad \Theta = \frac{w = w_h W}{w = w_l \vartheta}.$$

Therefore formula (18) will be able to be reduced to the following:

$$(22) \quad P = \int_{w_l}^{w_h} \int_{x_l}^{x_h} \int_{y_l}^{y_h} \dots \int_{v_l}^{v_h} \frac{w = w_h W}{w = w_l \vartheta} XY \dots VW dv \dots dy dx d\tau.$$

On is able, moreover, to establish yet this last equation as it follows.

Let  $w_\tau$  be a value of  $w$  proper to test the equation

$$\omega = \tau,$$

$\tau$  being a quantity contained between the limits  $w_l, w_h$ . If one attributes to  $\tau$  [154] an infinitely small and positive increase  $d\tau$ , the value of  $w$  represented by  $w_\tau$  will receive

a corresponding increase of which the numerical value will be  $\frac{d\tau}{\vartheta}$ , and the part of the integral

$$\int_{w_l}^{w''} W dw$$

corresponding to the same increase will be

$$\frac{W}{\vartheta} d\tau.$$

This put, we imagine that the differentials  $dx, dy, \dots, dv$  are, in the elements (12), some infinitely small quantities of an order superior to the order of the infinitely small quantity represented by  $d\tau$ . Then, in the value of  $P$  given by the formula (15), those of the elements of the integral

$$\int_{w_l}^{w''} IW dw$$

which will correspond to the value  $\tau$  of  $\omega$ , and to the element  $d\tau$  of the difference  $\omega'' - \omega_l$  will be reduced to the products of the form

$$\frac{W}{\vartheta} d\tau.$$

in a way that one will have

$$(23) \quad \int_{w_l}^{w''} IW dw = \int_{\omega_l}^{\omega''} \frac{w = w''}{w = w_l} \frac{W}{\vartheta} d\tau.$$

Now, in regard to this last formula, equation (15) is able to be evidently replaced by equation (22).

We will consider especially the case where the function  $\omega$  is linear with respect to the variables  $x, y, \dots, v, w$ , and of the form

$$(24) \quad \omega = ax + by + \dots + gv + hw.$$

Then, by putting, for brevity,

$$(25) \quad A = \int_{x_l}^{x''} X dx, \quad \mathcal{A} = \int_{x_l}^{x''} X e^{-a\theta x} dx,$$

and naming  $B, \dots, G, H$ , or  $\mathcal{B}, \dots, \mathcal{G}, \mathcal{H}$  that which becomes  $A$  or  $\mathcal{A}$  when to the letters  $x, X$ ,  $a$  one substitutes the letters  $y, Y, b$ , or  $v, V, g$ , or  $w, [155] W, k$ , one will draw from formulas (11) and (15)

$$(26) \quad \Pi = AB \dots GH,$$

$$(27) \quad P = \frac{1}{2\pi} \int_{\omega_l}^{\omega''} \int_{-\infty}^{\infty} \mathcal{A} \mathcal{B} \dots \mathcal{G} \mathcal{H} e^{\theta \tau} \dots d\theta d\tau.$$

If, the functions  $X, Y, \dots, V, W$  being all similar among them, one supposes the integrals (10) all taken between the same limits, one will have

$$A = B = \dots G = H,$$

consequently,

$$(28) \quad \Pi = A^n.$$

Finally, if one supposes

$$x_l = -\infty, \quad x_{ll} = \infty, \quad \omega_l = -v, \quad \omega_{ll} = v,$$

$v$  designating a positive quantity, formulas (25) and (27) will give

$$(29) \quad A = \int_{-\infty}^{\infty} X dx, \quad \mathcal{A} = \int_{-\infty}^{\infty} X e^{-a\theta xi} dx,$$

$$(30) \quad P = \frac{1}{2\pi} \int_{-v}^v \int_{-\infty}^{\infty} \mathcal{A} \mathcal{B} \dots \mathcal{G} \mathcal{L} e^{\theta \tau i} \dots d\theta d\tau.$$

In order to demonstrate an application of the formulas found, we consider in particular the case where one will have

$$(31) \quad X = K e^{-k\alpha^2},$$

$k, K$  designating two positive constants. Then one will have again

$$(32) \quad A = K \sqrt{\frac{\pi}{k}},$$

and one will draw from formulas (28) and (30)

$$(33) \quad \frac{P}{\Pi} = \frac{2}{\sqrt{\pi}} \int_0^{\frac{v}{\sqrt{s}}} e^{-\theta^2} d\theta,$$

the value of  $s$  being

$$(34) \quad s = \frac{a^2 + b^2 + \dots + g^2 + h^2}{k}.$$

If one supposed, on the contrary

$$(35) \quad X = K e^{-k\sqrt{\alpha^2}},$$

[156] one will find

$$(36) \quad A = \frac{2K}{k},$$

and

$$(37) \quad \frac{P}{\Pi} = 2 \int_{-\infty}^{\infty} \mathcal{E} \frac{e^{v\theta i} - 1}{\theta \left( \left(1 + \frac{a^2\theta^2}{k^2}\right) \left(1 + \frac{b^2\theta^2}{k^2}\right) \dots \left(1 + \frac{g^2\theta^2}{k^2}\right) \left(1 + \frac{h^2\theta^2}{k^2}\right) \right)},$$

the sign  $\mathcal{E}^2$  being relative to the variable  $\theta$ .

If, in formula (37) one puts  $n = 2$ , it will give

$$(38) \quad \frac{P}{\Pi} = 1 - \frac{a^2 e^{-\frac{kv}{\sqrt{a^2}}} - b^2 e^{-\frac{kv}{\sqrt{b^2}}}}{a^2 - b^2}.$$

§ II. — *Applications to the calculation of probabilities.*

The use of the restrictors permits resolving very easily a great number of problems relative to the calculus of probabilities, but that one had not at all yet resolved, unless it is in some particular cases, and of which the solution, in these same cases here, had been obtained only with the aid of an analysis difficult to follow. It is that which I am going to show in a few words.

We represent by

$$\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n,$$

$n$  diverse errors that involve  $n$  quantities

$$k_1, k_2, \dots, k_n,$$

determined by aid of certain experiences, or by certain observations. Let  $\varepsilon_l$  be any one of these errors,  $l$  being one of the whole numbers  $1, 2, \dots, n$ . Let further  $\varepsilon$  be a particular value attributed to  $\varepsilon_l$ ,  $d\varepsilon$  an infinitely small increase attributed to  $\varepsilon$ , and

$$l, \quad \kappa$$

two limits inferior and superior between which the error  $\varepsilon$  is certainly contained. Finally we imagine that,  $f(\varepsilon)$  being a function of  $\varepsilon$ , the product

$$f(\varepsilon)d\varepsilon$$

represents the probability of coincidence of the error  $\varepsilon_l$  with a quantity [157] contained between the two infinitely near limits

$$\varepsilon, \quad \varepsilon + d\varepsilon.$$

One will have

$$(1) \quad \int_l^\kappa f(\varepsilon)d\varepsilon = 1.$$

If the quantities  $k_1, k_2, \dots, k_n$  are deduced from observations or experiences of diverse natures, and if they do not involve the same facilities of errors, the form of the function  $f(\varepsilon)$ , and the values of the limits  $l, \kappa$  would be able to vary with the value of  $l$ .

If, in order to fix the ideas, one represents by

$$\phi(\varepsilon), \quad \chi(\varepsilon), \dots, \quad \varpi(\varepsilon),$$

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<sup>2</sup>*Translator's note:* The symbol here is Cauchy's symbol for the residue about the path indicated by the limits.

the successive forms of  $f(\varepsilon)$ , corresponding to the values

$$1, 2, \dots, n$$

of the number  $l$ ; if besides, in formula (1), one writes instead of  $t$  and of  $\kappa$ ,  $t_l$  and  $\kappa_l$ , one will draw successively from this formula

$$(2) \quad \int_{t_1}^{\kappa_1} \phi(\varepsilon_1) d\varepsilon_1 = 1, \quad \int_{t_2}^{\kappa_2} \chi(\varepsilon_2) d\varepsilon_2 = 1, \dots, \quad \int_{t_n}^{\kappa_n} \varpi(\varepsilon_n) d\varepsilon_n = 1,$$

next one will conclude from it

$$(3) \quad \int_{t_1}^{\kappa_1} \int_{t_2}^{\kappa_2} \dots \int_{t_n}^{\kappa_n} \psi d\varepsilon_1 d\varepsilon_2 \dots d\varepsilon_n = 1,$$

the value of  $\psi$  being

$$(4) \quad \psi = \phi(\varepsilon_1) \chi(\varepsilon_2) \dots \varpi(\varepsilon_n).$$

Now it is clear that the element

$$(5) \quad \psi d\varepsilon_1 d\varepsilon_2 \dots d\varepsilon_n,$$

of the multiple integral which forms the first member of equation (3), will be the product of the elements

$$(6) \quad \phi(\varepsilon_1) d\varepsilon_1, \quad \chi(\varepsilon_2) d\varepsilon_2, \dots, \quad \varpi(\varepsilon_n) d\varepsilon_n,$$

contained in the simple integrals which form the first members of equations (2). It will represent therefore the probability of the simultaneous coincidence of the first error with a quantity contained between two infinitely near limits and of the form  $\varepsilon_1$ ,  $\varepsilon_1 + d\varepsilon_1$ , of the second error with a quantity contained between two limits of the form  $\varepsilon_2$ ,  $\varepsilon_2 + d\varepsilon_2$ , ... of the [158]  $n^{\text{th}}$  error with a quantity contained between two limits of the form  $\varepsilon_n$ ,  $\varepsilon_n + d\varepsilon_n$ .

Let now  $\omega$  be a given function of the errors  $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$  and we name  $P$  the probability of coincidence of this function with a quantity contained between the two limits  $\omega_r, \omega_r$ . In order to obtain  $P$ , it will suffice evidently to separate from the integral (3) the elements corresponding to some values of  $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ , which will produce the values of  $\omega$  situated beyond some limits  $\omega_r, \omega_r$ , and to conserve all the others. One will arrive, by multiplying the element (5) of integral (3) by a restrictor  $I$  which has the double property of being reduced to unity when the value of  $\omega$  falls between the limits  $\omega_r, \omega_r$ , and to zero in the contrary case. One will have therefore

$$(7) \quad \int_{t_1}^{\kappa_1} \int_{t_2}^{\kappa_2} \dots \int_{t_n}^{\kappa_n} I \psi d\varepsilon_1 d\varepsilon_2 \dots d\varepsilon_n$$

We add that the value of  $I$  will be able to be deduced from formula (16) or (17) from § I. One will be able therefore to take

$$(8) \quad I = \frac{1}{2\pi} \int_{\omega_r}^{\omega_r} \int_{-\infty}^{\infty} e^{\theta(\tau-\omega)} d\theta d\tau.$$



If the quantities  $k_1, k_2, \dots, k_n$  are deduced from observations or from experiences of the same nature, which involve the same facility of errors, the functions

$$\phi(\varepsilon), \quad \chi(\varepsilon), \dots, \quad \varpi(\varepsilon),$$

will become all equals; and by designating by  $f(\varepsilon)$  any one among them, one will have

$$(9) \quad \psi = f(\varepsilon_1)f(\varepsilon_2)\cdots f(\varepsilon_n).$$

Then also the inferior and superior limits of the integrals (2) will not vary in the passage from one integral to the other, and by designating always these limits by aid of the letters  $\iota, \kappa$ , one will have

$$(10) \quad P = \int_{\iota}^{\kappa} \int_{\iota}^{\kappa} \cdots \int_{\iota}^{\kappa} I\psi d\varepsilon_1 d\varepsilon_2 \dots d\varepsilon_n.$$

If one is not able to assign a priori any inferior or superior limit to the errors  $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ , formula (10) will give

$$(11) \quad P = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} I\psi d\varepsilon_1 d\varepsilon_2 \dots d\varepsilon_n.$$

[159] Finally, if one wishes that the error  $\omega$  fall between two equal limits excepting signs, but affected of contrary signs, and if one puts in consequence

$$\omega_l = -v, \quad \omega_r = v,$$

$v$  being a positive quantity, formula (8) will give

$$(12) \quad I = \frac{1}{2\pi} \int_{-v}^v \int_{-\infty}^{\infty} e^{\theta(\tau-w)i} d\theta d\tau.$$

We will consider especially the case where the error  $\omega$  is a linear function of the errors  $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ , and where one has, hence,

$$(13) \quad \omega_l = \lambda_1 \varepsilon_1 + \lambda_2 \varepsilon_2 + \cdots + \lambda_n \varepsilon_n,$$

$\lambda_1, \lambda_2, \dots, \lambda_n$  being some constant factors. In this case, by putting, for brevity,

$$(14) \quad \mathcal{A} = \int_{\iota}^{\kappa} e^{-\lambda \theta \varepsilon i} f(\varepsilon) d\varepsilon,$$

and by naming  $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$  that which  $\mathcal{A}$  becomes when one replaces successively the letter  $\lambda$  by the factors  $\lambda_1, \lambda_2, \dots, \lambda_n$ , one will draw from formulas (8) and (10)

$$(15) \quad P = \frac{1}{2\pi} \int_{\omega_l}^{\omega_r} \int_{-\infty}^{\infty} \mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n e^{\theta \tau i} d\theta d\tau.$$

If besides one assigns a priori no limit to the errors  $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ , and if one wishes that the error  $\omega$  falls between the limits  $-v, +v$ , formulas (14) and (15) will give

$$(16) \quad \mathcal{A} = \int_{-\infty}^{\infty} e^{-\lambda \theta \varepsilon i} f(\varepsilon) d\varepsilon,$$

$$(17) \quad P = \frac{1}{2\pi} \int_{-v}^v \int_{-\infty}^{\infty} \mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n e^{\theta \tau i} d\theta d\tau.$$

Moreover, equation (1), to which the function  $f(\varepsilon)$  must satisfy will give

$$(18) \quad \int_{-\infty}^{\infty} f(\varepsilon) d\varepsilon = 1.$$

If one supposes, in particular,

$$(19) \quad f(\varepsilon) = K e^{-k\varepsilon^2},$$

[160]  $k, K$  being two positive constants, one will draw from formula (18)

$$K = \sqrt{\frac{k}{\pi}},$$

and equation (17) will give

$$(20) \quad P = \frac{2}{\sqrt{2}} \int_0^{\frac{v}{\sqrt{s}}} e^{-\theta^2} d\theta,$$

the value of  $s$  being determined by the formulas

$$(21) \quad s = \frac{\Lambda}{k}, \Lambda = \lambda_1^2 + \lambda_2^2 + \dots + \lambda_n^2.$$

If one supposes, on the contrary,

$$(22) \quad f(\varepsilon) = K e^{-k\sqrt{s^2}},$$

one will draw from formula (18)

$$K = \frac{k}{2},$$

and equation (17) will give

$$(23) \quad P = 2 \mathcal{E} \int_{-\infty}^{\infty} \frac{e^{v\theta i} - 1}{\theta \left( \left(1 + \frac{\lambda_1^2 \theta^2}{k^2}\right) \left(1 + \frac{\lambda_2^2 \theta^2}{k^2}\right) \dots \left(1 + \frac{\lambda_n^2 \theta^2}{k^2}\right) \right)},$$

the sign  $\mathcal{E}$  of the calculation of the residues being relative to the variable  $\theta$ .

If, in formula (23), one puts  $n = 2$ , it will give

$$(24) \quad P = 1 - \frac{\lambda_1^2 e^{-\frac{kv}{\sqrt{\lambda_1^2}}} - \lambda_2^2 e^{-\frac{kv}{\sqrt{\lambda_2^2}}}}{\lambda_1^2 - \lambda_2^2}.$$

In order to show an application of the formulas which precede, we suppose that one wishes to determine the values of  $m$  unknowns  $x, y, \dots, v, w$  linked to the quantities  $k_1, k_2, \dots, k_n$ , by  $n$  approximate equations of the form

$$(25) \quad a_l x + b_l y + \dots + g_l v + h_l w = k_l,$$

$l$  being any one of the whole numbers  $1, 2, \dots, n$  and  $n$  being superior to  $m$ . In order to obtain the value of  $x$ , it will suffice to multiply each equation by a certain factor  $\lambda_l$ , next to add to one another the diverse formulas thus obtained, by choosing the factors  $\lambda$  in a manner that, in the final equation, [161] the coefficient of  $x$  is reduced to unity, and those of  $x, y, \dots, w$  to zero. If, to be brief, one indicates by aid of the characteristic letter  $S$ , a sum of terms similar to one another, the value of  $x$  will be

$$(26) \quad x = S\lambda_l k_l,$$

the factors  $\lambda_1, \lambda_2, \dots, \lambda_n$  being determined by the formulas

$$(27) \quad Sa_l \lambda_l = 1, \quad Sb_l \lambda_l = 0, \dots, \quad Sh_l \lambda_l = 0;$$

and if one names always  $\varepsilon_l$  the error which involves  $k_l$ , the error  $\xi$  that will involve the error  $x$  will be, by virtue of the formula (26),

$$(28) \quad \xi = S\lambda_l \varepsilon_l,$$

On the other hand, if one is not able to assign a priori to the error  $\varepsilon_l$ , any inferior or superior limit, and if the law of facility of errors is that which formula (19) expresses, the probability of the coincidence of the error  $\xi$  with a quantity contained between the limits  $-v, v$  will be the value of  $P$  that formula (20) gives, and which increases for the decreasing values of the sum

$$(29) \quad \Lambda = S\lambda_l^2.$$

Therefore the value of the most probable  $x$  will be that which will correspond to the minimum value of  $\Lambda$ , and which will be determined by the formula

$$(30) \quad S\lambda_l d\lambda_l = 0,$$

joined to the equations (27) from which one draws

$$(31) \quad Sa_l d\lambda_l = 0, \quad Sb_l d\lambda_l = 0, \dots, \quad Sh_l d\lambda_l = 0.$$

Now, the formulas (30) and (31) must be verified, whatever be among the differentials  $d\lambda_1, d\lambda_2, \dots, d\lambda_n$  those which will remain arbitrary, there results from it that  $\lambda_l$  must be of the form

$$(32) \quad \lambda_l = a_l \alpha + b_l \beta + \dots + h_l \eta,$$

$\alpha, \beta, \dots, \eta$  designating  $m$  new coefficients of which the values will be able to be deduced from the formulas (27). There results from it also that the most probable value of  $x$  will be furnished by the equation

$$(33) \quad x = \alpha X + \beta Y + \dots + \eta W = 0,$$

if, by putting, for brevity,

$$K_l = k_l - a_l x - b_l y - \dots - h_l w,$$

[162] one takes

$$(34) \quad X = Sa_l K_l, \quad Y = Sb_l K_l, \dots, \quad W = Sh_l K_l.$$

If in the variable  $x$  one substitutes one of the variables  $y, z, \dots, w$ , equation (33) will keep the same form, the coefficients  $\alpha, \beta, \dots, \eta$  will no longer be the same; and hence the most probable values of  $x, y, \dots, v, w$  will be generally those which will confirm the formulas

$$(35) \quad X = 0, Y = 0, \dots, V = 0, W = 0,$$

that is to say, those which the method of least squares furnishes.

From that which we just said, there results that the method of least squares, applied to the resolution of linear equations of which the number surpasses the one of the unknowns, will furnish always the most probable results, if the law of facility being the same for the diverse errors which involve the quantities furnished by experiences or observations, one is not able to assign to these errors any inferior or superior limit, and if besides the probability of an error contained between two infinitely near limits is proportional to a Naperian exponential of which the exponent is the product of a negative coefficient with the square of this same error. When these conditions are not fulfilled, the method of least squares is able to furnish for the unknowns  $x, y, \dots, v, w$  some values which differ sensibly from the most probable values. This is effectually that which one is able to conclude from the formulas established in this Memoir, and that which I will explain more in detail in a later article.