# MÉMOIRE SUR L'INTERPOLATION*

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In the applications of analysis to Geometry, to Physics, to Astronomy ... two sorts of questions present themselves to resolve, and the question is 1° to find the general laws of figures and of phenomena, that is to say the general form of the equations which exist among the diverse variables, for example, among the coordinates of curves and surfaces, among velocities, time, the spaces traversed by moving things, etc.; 2° to fix in numbers the values of the parameters or arbitrary constants which enter into the expression of these same laws, that is to say the values of the unknown coefficients which comprise the found equations. Among the variables one distinguishes ordinarily, as one knows, those which are able to vary independently from one another, and that one names for this reason variables independent, away from those which are deduced from it by the resolution of diverse equations, and which are named functions of independent variables.

We consider in particular one of these functions, and we suppose that it is deduced from the independent variables through an equation or formula which contains a certain number of coefficients. A parallel number of observations or experiences, of which each will furnish a particular value of the function corresponding to a particular system of values of the independent variables, will suffice for the numerical determination of all these coefficients; and, this determination made, one will be able to obtain without difficulty new values of the function corresponding again to new systems of values of the independent variables, and to resolve thus that which one calls the problem of interpolation.

For example, if the ordinate of a curve is found expressed as function of the abscissa through an equation which contains three parameters, it will suffice to know three points of the curve, that is to say three particular values of the ordinate corresponding to three particular values of the abscissa, in order to determine the three parameters; and, this determination effected, one will be able without pain to trace the curve through points by calculating the coordinates of a number as great as one will wish of new points situated on the arcs of this curve contained among the given points. Thus, envisioned in all its extent, the problem of interpolation consists in determining the coefficients or arbitrary constants which contain the expression of the general laws of the figures or of the phenomena, according to a number at least equal to given points,

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or to observations, or to experiences. In one crowd of questions the arbitrary constants enter in the first degree only in the equations which contain them. It is precisely that which happens when a function is developable into a convergent series ordered according to the ascendant or descendant powers of an independent variable, or else again according to the sine or cosine of the multiples of one same arc.

Then the question is to determine the coefficients of those of the terms of the series which one is not able to neglect without having to fear that there results from it a sensible error in the values of the function. In the small number of formulas which have been proposed for this object, one must distinguish a formula drawn from the calculus of finite differences, but applicable only to the case where the diverse values of the independent variable are equidifferent among them, and the formula of Lagrange applicable, whatever be these values, to some series ordered according to the ascendant powers of the independent variable. However this last formula itself is complicated more and more in measure as one wishes to conserve in the development of the function into series a greater number of terms; and that which is more troublesome, is that the approximate values of the diverse orders corresponding to the diverse cases where one would conserve into the series a single term, next two terms, next three terms... is obtained by some calculations nearly so independent from one another, so that each new approximation, far from being rendered easy by those which precede it, demand on the contrary more time and more work.

Struck with these inconveniences, and led by my researches on the dispersion of light to occupy myself anew with the problem of interpolation, I have had the good luck to encounter for the solution of this problem a new formula which, under the double relation of the certitude of the results and of the facility with which one obtains them, appears to me to have over the other formulas some advantages so incontestable, that I scarcely doubt that it is soon of a general use among the persons adopted to the culture of the physical and mathematical sciences.

In order to give an idea of this formula, I suppose that a function of $x$, represented by $y$, is developable into a convergent series ordered according to the ascendant or descendant powers of $x$, or else further according to the sines or cosines of arcs multiples of $x$, or even more generally according to other functions of $x$ that I represent by

$$\phi(x) = u, \quad \chi(x) = v, \quad \psi(x) = w, \quad \dots$$

so that one has

$$y = au + bv + cw + \cdots \tag{1}$$

$a, b, c, \dots$ designating some constant coefficients. The question is to know, 1° how many terms one must conserve in the second member of equation (1) in order to obtain a value of $y$ sufficiently closely, of which the difference with the exact value is insensible and comparable to the errors which involve the observations; 2° to fix in numbers the coefficients of the terms conserved, or, that which reverts to the same, to find the approximate value of which we just spoke. The data of the problem are a sufficiently great number of values of $y$ represented by

$$y_1, \quad ,y_2, \quad \dots, \quad y_n,$$

corresponding to a parallel number $n$ of values of $x$ represented by $x_1, x_2, \ldots x_n$, consequently also to a parallel number of values of each of the functions $u, v, w, \ldots$ values that I will represent likewise by

$$u_1, \quad ,u_2, \quad \ldots, \quad u_n,$$

for the function $u$, by

$$v_1, \quad ,v_2, \quad \ldots, \quad v_n,$$

for the function $v$, etc.

Thus, in order to resolve the problem, one will have among the unknown coefficients $a, b, c, \ldots$ the $n$ equations of the first degree

$$(2) \qquad \begin{cases} y_1 = au_1 + bv_1 + cw_1 + \cdots, \\ y_2 = au_2 + bv_2 + cw_2 + \cdots, \\ \\ y_n = au_n + bv_n + cw_n + \cdots, \end{cases}$$

which, if one designates by $i$ any one of the whole numbers

$$1, \quad 2, \quad \ldots, \quad n,$$

will be found entirely contained in the general formula

$$(3) \qquad y_i = au_i + bv_i + cw_i + \cdots$$

One will effect the first approximation by neglecting the coefficients $b, c, \ldots$, or, that which reverts to the same, by reducing the series to its first term. Then the general value approximated from $y$ will be

$$(4) \qquad y = au;$$

and, in order to determine the coefficient $a$, one will have the system of equations

$$(5) \qquad y_1 = au_1, \quad y_2 = au_2, \quad \ldots \quad y_n = au_n.$$

The diverse values of $a$, which one is able to deduce from these equations (5) considered each in part, or combined among themselves, would be all precisely equals if the particular values of $y$, which we suppose given by observation, were rigorously exact. But it is not so in practice where the observations comport with errors contained between certain limits; and then it matters to combine among them the equations (5) in a manner, in the cases more unfavorable, the influence exercised on the value of the coefficient $a$ by the errors committed on the values of $y_1, y_2, \ldots y_n$ may be the least possible. Now the diverse combinations that one is able to make on the equations (5) in order to draw from it a new equation of first degree, with respect to $a$, furnishes all the values of $a$ contained in the general formula

$$(6) \qquad a = \frac{k_1 y_1 + k_2 y_2 + \cdots + k_n y_n}{k_1 u_1 + k_2 u_2 + \cdots + k_n u_n},$$

that one obtains by adding member to member the equations (5) after having them respectively multiplied by some constant factors $k_1, k_2, \ldots, k_n$.

There is more; as the value of $a$ determined by equation (6) does not vary when one makes the factors $k_1, k_2, \ldots, k_n$ vary simultaneously in the same ratio, it is clear that among these factors, the greatest (setting aside the sign) can always be counted reduced to unity. We remark finally that, if one names

$$\varepsilon_1, \quad \varepsilon_2, \quad \ldots, \quad \varepsilon_n,$$

the errors committed respectively in the observations on the values of

$$y_1, \quad ,y_2, \quad \ldots, \quad y_n,$$

the preceding formula (6) will furnish for $a$ an approximate value, of which the difference with the true will be

$$a = \frac{k_1\varepsilon_1 + k_2\varepsilon_2 + \cdots + k_n\varepsilon_n}{k_1u_1 + k_2u_2 + \cdots + k_nu_n}. \tag{7}$$

It is necessary now to choose $k_1, k_2, \ldots, k_n$ of such sort that, in the most unfavorable cases, the numerical value of the expression (7) is the least possible.

We represent by

$$Su_i$$

the sum of the diverse numerical values of $u_i$, that is to say that which becomes the polynomial

$$\pm u_1 \pm u_2 \pm \cdots \pm u_n$$

when one disposes of each sign in a manner to render each term positive. We represent by $S\varepsilon_i$ not the sum of the numerical values $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$, but that which the sum $Su_i$ becomes, when one replaces each value of $u_i$ by the corresponding value of $\varepsilon_i$.

If one reduces to $+1$ or to $-1$ each of the coefficients $k_1, k_2, \ldots, k_n$, in choosing the signs in a manner that, in the denominator of the fraction

$$\frac{k_1\varepsilon_1 + k_2\varepsilon_2 + \cdots + k_n\varepsilon_n}{k_1u_1 + k_2u_2 + \cdots + k_nu_n}$$

all the terms are positives, this fraction will be reduced to

$$\frac{Se_i}{Su_i}; \tag{8}$$

and it will offer a numerical value all the more equal to the ratio

$$\frac{E}{Su_i},$$

if one designates by $E$ the sum of the numerical values of $\varepsilon_i$, or, that which reverts to the same, the numerical value of $Se_i$ in the most unfavorable case. On the other hand, by attributing to $k_1, k_2, \ldots, k_n$ some unequal values of which the greatest (setting aside the sign) is unity, one will obtain for denominator of the fraction a quantity of which the numerical value will be evidently inferior to $Su_i$, while the numerical value of the numerator will be able to be raised to the limit $E$;

this which will happen effectively if the errors $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ are all null, with the exception of that which will be multiplied by an equal factor, with sign excepted, to unity. There results from it that the greatest error to fear on the value of $a$ determined by the formula

$$a = \frac{k_1 y_1 + k_2 y_2 + \cdots + k_n y_n}{k_1 u_1 + k_2 u_2 + \cdots + k_n u_n}.$$

will be the least possible if one puts generally

$$k_1 = \pm 1,$$

by choosing the signs in a manner that in the polynomial

$$k_1 u_1 + k_2 u_2 + \cdots + k_n u_n$$

all the terms are positives. Then this formula will give

$$a = \frac{Sy_i}{Su_i}, \tag{9}$$

$Sy_i$ being that which the sum $Su_i$ becomes when one replaces each value of $u_i$ in it by the value corresponding to $y_i$, and the equation $y = au$ will become

$$y = \frac{u}{Su_i} Sy_i. \tag{10}$$

In one makes for brevity

$$\alpha = \frac{u}{Su_i}. \tag{11}$$

one will have simply

$$y = \alpha Sy_i. \tag{12}$$

If one supposed generally $u = 1$, the equation $y = au$, reduced to

$$y = a,$$

would express that the value of $y$ is constant; and as one would have then

$$\alpha = \frac{u}{Su_i} = \frac{1}{n},$$

the formula $y = \alpha Sy_i$ would give

$$y = \frac{1}{n} Sy_i.$$

Therefore then one should take for the approximate value of $y$ the arithmetic mean among the observed values; and the greatest error to fear would be smaller for this approximate value than for each other.

This property of arithmetic means, joined to the ease with which one calculates them, justifies completely the usage where one is to accord to them preference in the evaluation of the arbitrary constants which are able to be determined directly by observation.

Let now $\Delta y$ be the rest which must complete the approximate value of $y$ furnished by the equation

$$y = \alpha S y_i. \tag{12}$$

so that one has

$$y = \alpha S y_i + \Delta y. \tag{13}$$

We put likewise

$$v = \alpha S v_i + \Delta v, \quad w = \alpha S w_i + \Delta w, \quad \cdots \tag{14}$$

One will deduce from the formula $y_i = a u_i + b v_i + c w_i + \cdots$,

$$S y_i = a S u_i + b S v_i + c S w_i + \cdots; \tag{15}$$

then from this last, multiplied by $\alpha$, and subtracted from equation (1),

$$\Delta y = b \Delta v + c \Delta w + \cdots \tag{16}$$

Let there be besides $\alpha_i$, $\Delta y_i$, $\Delta v_i$, $\Delta w_i, \ldots$ that which the values of $\alpha$, $\Delta y$, $\Delta v$, $\Delta w, \ldots$ become, drawn from equations (11), (13) and (14), when one replaces $x$ by $x_i$, $i$ being one of the whole numbers $1, 2, \ldots, n$. If the values of

$$\Delta y_1, \quad \Delta y_2, \quad \ldots, \quad \Delta y_n$$

are very small, and comparable to the errors which include the observations, it will be useless to proceed to a second approximation, and one will be able to hold to the approximate value of $y$ furnished by the equation $y = \alpha S y_i$.

If the contrary holds, it will suffice, in order to obtain a new approximation, to operate on formula (16) which gives $\Delta y = b \Delta v + \cdots$, as in the first approximation one has operated on formula (1) $y = au + \cdots$.

This put, we designate by

$$S' \Delta v_i$$

the sum of the numerical values of $\Delta v_i$, and by

$$S' \Delta y_i, \quad S' \Delta w_i, \quad \ldots$$

the polynomials in which the sum $S' \Delta v_i$ is changed when one replaces each value of $\Delta v_i$ by the value corresponding to $\Delta y_i$ or to $\Delta w_i, \ldots$; let be finally

$$\beta = \frac{\Delta v}{S' \Delta v_i} \tag{17}$$

if one is able, without sensible error, to neglect in series (1) the coefficient $c$ of the third term and those of the following terms, one would take for approximate value of $\Delta y$

$$\Delta y = \beta S' \Delta y_i. \tag{18}$$

Let $\Delta^2 y$ be the remainder of the second order which must complete this approximate value, and we make consequently

$$\Delta y = \beta S' \Delta y_i + \Delta^2 y. \tag{19}$$

We put likewise

$$\Delta w = \beta S' \Delta w_i + \Delta^2 w, \quad \dots; \tag{20}$$

one will deduce successively, from formula (16),

$$\Delta y_i = b \Delta v_i + c \Delta w_i + \cdots \tag{21}$$

$$S' \Delta y_i = b S' \Delta v_i + c S' \Delta w_i + \cdots; \tag{22}$$

then this last, multiplied by $\beta$ and subtracted from equation (16),

$$\Delta^2 y = c \Delta^2 w + \cdots \tag{23}$$

Let there be besides $\beta_i, \Delta^2 y_i, \Delta^2 w_i, \dots$, that which the values of $\beta$, $\Delta^2 y, \Delta^2 w, \dots$ become, deduced from equations (17), (19) and (20), when one replaces $x$ by $x_i$, $i$ being one of the whole numbers $1, 2, \dots, n$. If the values of

$$\Delta^2 y_1, \quad \Delta^2 y_2, \quad \dots, \quad \Delta^2 y_n$$

are very small and comparable to the errors which involve the observations, it will be useless to proceed to a new approximation, and one will be able to be held to the approximate value of $\Delta y$ furnished by equation (18).

If the contrary takes place, it will suffice, in order to obtain a third approximation, to operate on formula (23) which gives $\Delta^2 y$, as one has operated in the first approximation on formula (1). In continuing in that way, one will obtain the following rule:

The unknown $y$, function of the variable $x$, being supposed developable into a convergent series

$$au + bv + cw + \cdots \tag{I}$$

where $u, v, w, \dots$, represent some given functions of the same variable, if one knew $n$ particular values of $y$ corresponding to $n$ particular values

$$x_1, \quad x_2, \quad \dots, \quad x_n$$

of $x$, if besides one names $i$ any one of the whole numbers $1, 2, \dots, n$, and $y_i$, $u_i$, $v_i$, $\dots$, that which $y$, $u$, $v, \dots$, become when one replaces $x$ by $x_i$; then, in order to obtain the general value of $y$ with a sufficient approximation, one will determine first the coefficient $a$ by aid of the formula

(II) $$u = \alpha S u_i,$$

in which $Su_i$ designates the sum of the numerical values of $u_i$, and the difference of the first order $\Delta y$ by aid of the formula

(III) $$y = \alpha S y_i + \Delta y.$$

If the particular values of $\Delta y$, represented by $\Delta y_1, \Delta y_2, \ldots, \Delta y_n$, are comparable to the errors of observation, one will be able to neglect $\Delta y$ and to reduce the approximate value of $y$ to

$$\alpha S y_i.$$

In the contrary case, one will determine $\beta$ by aid of the formulas

(IV) $$v = \alpha S v_i + \Delta v, \qquad \Delta v = \beta S' \Delta v_i,$$

$S'\Delta v_i$ being the sum of the numerical values of $\Delta v_i$, and the difference of the second order $\Delta^2 y$ by aid of the formula

(V) $$\Delta y = \beta S' \Delta y + \Delta^2 y.$$

If the particular values of $\Delta^2 y$, represented by $\Delta^2 y_1, \Delta^2 y_2, \ldots, \Delta^2 y_n$, are comparable to the errors of observation, one will be able to neglect $\Delta^2 y$ and to reduce in consequence the approximate value of $y$ to $\alpha S y_i + \beta \alpha S' \Delta y_i$.

In the contrary case, one will determine $\gamma$ by the formulas

(VI) $$w = \alpha S w_i + \Delta w, \quad \Delta w = \beta S' \Delta w_i + \Delta^2 w, \quad \Delta^2 w = \gamma S'' \Delta^2 w_i,$$

$S''\Delta^2 w_i$ being the sum of the numerical values of $\Delta^2 w_i$, and the difference of the third order $\Delta^3 y$ by the formula

(VII) $$\Delta^2 y = \gamma S'' \Delta^2 y_i + \Delta^3 y, \quad \ldots$$

Thus, finally, by supposing the coefficients $\alpha, \beta, \gamma, \ldots$, determined by the system of these equations, etc., one ought to calculate the differences of the diverse orders represented by

$$\Delta y, \quad \Delta^2 y, \quad \Delta^3 y, \quad \ldots,$$

or rather their particular values corresponding to the values $x_1, x_2, \ldots, x_n$ of the variable $x$, until this that one arrives to a difference of which the particular values are comparable to the errors of observation. Then it will suffice to equate to zero the value of this difference drawn from the system of equations (III), (V), (VII), …, in order to obtain the value of $y$ with a sufficient approximation. This general value will be therefore

$$y = \alpha S y_i, \quad \text{or} \quad y = \alpha S y_i + \beta S' \Delta y_i, \quad \text{or} \quad \text{etc.},$$

according as one will be able, without sensible error, to reduce the series to its first term, or to its first two terms … Therefore, if one names $m$ the number of conserved terms, the problem of interpolation will be resolved by the formula

$$y = \alpha S y_i + \beta S' \Delta y_i + \gamma S'' \Delta^2 y_i + \cdots,$$

the second member being prolonged to the term which contains $\Delta^{m-1} y_i$.

It is good to observe that from the preceding formulas one draws not only

$$S\alpha_i = 1; \quad S\beta_i = 0, S'\beta_i = 1; \quad S\gamma_i = 0, S'\gamma_i = 0, S''\gamma_i = 1; \quad \ldots;$$

but also

$$S\Delta v_i = 0; \quad S\Delta w_i = 0, \quad S\Delta^2 w_i = 0, \quad S'\Delta^2 w_i = 0, \quad \ldots$$

and

$$\begin{aligned} &S\Delta y_i = 0; \qquad S\Delta^2 y_i = 0, \qquad S'\Delta^2 y_i = 0; \\ &S^3\Delta y_i = 0, \qquad S'\Delta^3 y_i = 0 \qquad S''\Delta^3 y_i = 0, \qquad \ldots \end{aligned}$$

These last formulas are as many equations of condition to which the particular values of $\alpha, \beta, \gamma, \ldots$ must satisfy, as those of the differences of the diverse orders of $u, v, w, \ldots, y$; and there results from it that one is able to commit into the calculation of these particular values no error of numbers without being cautioned by the single fact that the equations of condition cease to be verified.

In summary, the advantages of the new formulas of interpolation are the following:

1° They apply themselves to the development into series, whatever be the law according to which the different terms are deduced from one another, and whatever be the values equidifferent or not of the independent variable.

2° The new formulas are of very easy application, especially when one employs logarithms for the calculations of the ratios $\alpha, \beta, \gamma, \ldots$ and of the products of these ratios by the sums of the diverse values of the functions or of their differences. Then, in fact, all the operations are reduced to some additions or to some subtractions.

3° By aid of our formulas the successive approximations are executed with a greater and greater facility, seeing that the differences of the diverse orders go generally by diminishing.

4° Our formulas permit to introducing at the same time into the calculation the numbers furnished by all the given observations, and to increase thus the exactness of the results by making agree to this end a very great number of experiences.

5° They offer yet this advantage, that at each new approximation, the values that they furnish for the coefficients $a, b, c, \ldots,$ are precisely those for which the greatest error to fear is the least possible.

6° Our formulas indicate by themselves the moment where the calculation must stop, by furnishing then from the differences comparable to the errors of observation.

7° Finally the quantities that they determine satisfy some equations of condition which do not permit to commit the slightest fault of calculation, without that it be perceived nearly immediately.

One will find in the new exercises of mathematics numerous applications of our formulas of interpolation.