# On elections by ballot*

Jean-Charles de Borda

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## ANALYSIS

In the elections by ballot, one ordinarily employs one of these two methods, either one regards as elected the one of the candidates who has obtained the most votes, or else one prefers the one who after some repeated ballots, is found to muster first more than half, more than two-thirds of the votes.

This second method supposes necessarily that a certain number of the voters finish by choosing the one who they judge most worthy, not among all the competitors, but in the number of those who they believe can muster a sufficient number of votes. Thus, by this manner one will not succeed in choosing the candidate who has the most merit, but to give the position to a man that the plurality judges not unworthy, & one seems to seek less to make the better choice than to be assured of not making a bad.

It is particularly in the first method that there is concern in the Memoir of M. le Chavalier de Borda.[1] He observes first that in the elections made under this form, the apparent wish of the plurality can be contrary to the true wish. For example, if one supposes three candidates who one will call $A$, $B$, $C$, & if there had been eight votes in favor of $A$, seven in favor of $B$, & six in favor of $C$, $A$ obtains the plurality; but by this manner of voting, one knows only that eight persons have preferred $A$ to his two competitors, but one does not know if they prefer $B$ to $C$. One knows that seven prefer $B$ to $A$ & to $C$, but one does not know to whom of $A$ or of $C$ they give preference. Finally, one does not know likewise what is the opinion concerning the merit of $A$ & of $B$ of the six who have voted in favor of $C$. However if the eight voters for $A$, had preferred $C$ to $B$, if the seven voters for $B$, had preferred $C$ to $A$, if the six voters for $C$ had preferred $B$ to $A$, it would follow that there would be thirteen votes against eight for preferring $B$ to $A$, thirteen votes against eight as for preferring $C$ to $A$; thus $A$ should be excluded. But there are also fourteen votes against seven as for preferring $C$ to $B$, $C$ consequently would have obtained the preference. The true wish of the plurality would therefore have been precisely for $C$ who had the least of the votes, & $A$ who had the most of them, is on the contrary the one who the wish of the plurality really places last.

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[1] Jean-Charles de Borda. Born 4 May 1733, died 19 February 1799.

After having shown the fault of the ordinary method, M. de Borda proposes a way to remedy it.

He asks first that the voters each give the list of the candidates, according to the order of merit which they suppose for them, or else that they pronounce on the merit of the candidates compared two by two. Moreover, it is easy to see that this list, according to the order of merit, being given, one can deduce the judgment which the voter has pronounced on the degree of merit of any two competitors.

M. de Borda supposes next to the competitor placed last, a merit which he represents by an indeterminate quantity; the degree of merit of the one who immediately precedes him, will be represented by this quantity, plus another which expresses his superiority; in order to have the merit of the third, one will add again this same quantity; so that the merit of the one who has three or four of his competitors after him, will be expressed by the quantity which expresses the merit of the last, plus, three times or four times the always constant quantity which represents the difference in superiority of merit between two competitors placed immediately the one after the other.

One will have by this method the merit which results for each competitor from the wish of the electors; taking next the sum of each of these values given by each wish, one will have the value which results from the general wish for the merit of each competitor; & the candidate for which this sum is the greatest, is the one in favor of which the wish of the plurality is itself explicated.

The value of merit supposed to the one who is placed last, being accorded for all the electors to all competitors, is equal for each. The value which must be added to it, is proportional to that which one regards as representing the difference of merit between two consecutive competitors, & consequently it does not enter into the comparison which one can make of the respective merits resulting from the election: thus one can regard it as representing the unit or the degree of merit.

Finally the multiple of this degree of merit which corresponds to each competitor, is precisely equal to the number of times that in the successive comparisons made between two competitors, he had obtained the preference; & consequently it is in favor of the one who has obtained it a greater number of times, that the plurality is declared.

In the chosen example preceding, we will find that $A$ having been placed first eight times & the last thirteen times, there will result for him sixteen degrees of merit plus the value common to all. $B$ having been placed first seven times & second six times, there will result for him twenty degrees of merit plus the same quantity. Finally $C$ having been placed first six times & second fifteen times, will have twenty-seven degrees of merit; & one sees that each of these numbers is equal to the number of times that each competitor has been preferred to one of the others.

M. de Borda examines next what plurality it is necessary to require in order that the one who is elected, following the ordinary method, is surely the one who has obtained the true wish in the more exact method which he proposes to substitute for it; for this he takes the most unfavorable distribution of votes for the candidate, that where one of his competitors musters all the votes which are lacking to the first, & where this competitor is put in the second position by all those who refuse him the first, while the one who has obtained the plurality, is put in the last position by all those who have not placed him in the first.

There results from it that, in order to be sure that the election made following the ordinary method, indicates the wish of the plurality, it is necessary that the number of votes obtained by this candidate, is to the total number of electors, in a ratio greater than the one of the number of candidates less one, to their total number. If there are three candidates, it is necessary that he obtain more than two-thirds of the votes; if there are four of them, it is necessary that he obtain more than three-quarters; if the number of candidates is equal or surpasses that of the electors, unanimity is necessary.

M. de Borda observes that the laws of Poland require this unanimity for the election of the King; & any noble Pole can be elected, this is precisely the case where the number of candidates equals & even surpasses that of the electors. This closeness is singular; however one can scarcely suppose that the law had been determined by a design of this kind, & that, in the times when it had been established, one had thought to find the means to be convinced of the true wish of the assembled according to the one of the plurality.

The observations of M. de Borda, on the inconveniences of the method of election, since generally adopted, are very important and absolutely new. He had already developed this idea in a Memoir to the Academy from 1770.

Mémoire sur les Élections au Scrutin.[2]
by M. de Borda
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There is a generally received opinion, & against which I do not know that one has ever made objection, that in an election by ballot, the plurality of the votes always indicates the wish of the electors, that is, that the Candidate who obtains this plurality, is necessarily the one that the electors prefer to his competitors. But I show that this opinion, which is true in the case where the election is made between only two subjects, can induce an error in all other cases.

We suppose, for example, that the election is made among three presented subjects $A$, $B$, $C$; & that the electors are 21 in number: we suppose next that of these 21 electors, there are 13 who prefer subject $B$ to subject $A$, & that 8 alone prefer subject $A$ to subject $B$; that these same 13 electors give also preference to $C$ over $A$, while the eight others give preference to $A$ over $C$; it is clear that then subject $A$ will have, in the collective opinion of the electors, a very marked inferiority, as much with respect to $B$ as with respect to $C$, since each of these latter, compared to subject $A$, has 13 votes, while subject $A$ has only 8; whence it follows evidently that the wish of the electors would give exclusion to subject $A$. Nevertheless it may happen that by making the election in the ordinary manner, this subject had the plurality of votes. In fact, just suppose that in the number of the 13 electors who are favorable to subjects $B$ & $C$, & who give to each preference over $A$, there are 7 who put $B$ above $C$, & 6 who put $C$ above $B$, then, by collecting the suffrages, one will have the following result:

8 votes for $A$.
7 votes for $B$.
6 votes for $C$.

[2]The ideas contained in this memoir, had already been presented to the Academy fourteen years ago, 16 June 1770.

Thus subject $A$ will have the plurality of the votes, although, by hypothesis, the opinion of the electors was contrary to it.

By reflecting on the example reported, one sees that subject $A$ has the advantage in the result of the election, only because the two subjects $B$ & $C$, who are superior to him, have shared very nearly equally the votes of the 13 electors. One can compare them exactly enough to two Athletes, who, after having exhausted their forces the one against the other, would be next vanquished by a third more feeble than each of them.

There results from this that we just stated, that the ordinary method of making elections is very flawed, & the fault comes from this: that in this form of election the electors are not able to make known in a complete enough manner their opinion on the different subjects presented. In fact, if among many subjects $A$, $B$, $C$, $D$, &c. one of the electors gives his vote to $B$, & if another gives his to $C$, the first pronounces only the superiority of $B$, relatively to all his competitors, & says not which position he assigns to $C$ among those who he does not name. Similarly the second, who accords to $C$ preference over all, does not say moreover which position he gives to $B$; however this is not able to be regarded as indifferent, because the one of the two who obtains a position more distinguished among those who one does not name, has, all things being equal besides, a reason for preference over the other, & in general the claim of each subject to the nomination made by the electors, is the result of the different positions which he occupies in the opinion of each elector; whence one sees that in order that a form of election be good, it is necessary that it give to the electors the means to pronounce on the merit of each subject, compared successively to the merits of each of his competitors. Now, there is for this two forms of election which one can adopt equally; in the first, each elector would assign some positions to the presented subjects, according to the degree of merit which he would recognize in each of them; in the second, one would make as many particular elections as there would be combinations among the subjects taken two by two, & that way one could compare successively each subject to all the others. It is easy to see that this last form necessarily derives from the first, & that each would explain, as completely as it is possible, the opinion of the electors on all the presented subjects; but the question is to know how one could conclude the result of the suffrages in these two kinds of election; & this is what I will examine in the continuation of this Memoir.

I will begin with the first kind of election which I will call election by order of merit. We suppose first that there are only three subjects presented, & that each elector has inscribed their three names on a ticket of election, by arranging them following the degree of merit which he attributes to each of them, & let there be

$$\begin{array}{ccccc} A, & A, & B, & C, & \\ B, & C, & A, & B, & \text{\&c.} \\ C, & B, & C, & A, & \end{array}$$

these tickets of election; I consider first one of these tickets, for example, the first in which an elector has given the first position to $A$, the second to $B$, & the third to $C$, & I say that the degree of superiority that this elector has accorded to $A$ over $B$, must be counted the same as the degree of superiority that he has accorded to $B$ over $C$; in fact, as the second subject $B$ is equally susceptible to all the degrees of merit contained among the merits of the two other subjects $A$ & $C$, one has no reason to say that the elector who has adjusted the ranks among the three subjects, had wished to position him more or less nearer to $A$ than to $C$, or, what is the same thing, that he had attributed more superiority to the first over the second, than he had attributed to the second over the third. I say next, that because of the supposed equality among all the electors, each position assigned by one of the electors, must be counted with the same value, & to suppose the same degree of merit as the corresponding position assigned to another subject, or to the same by any other elector.

It follows from this, that if one wishes to represent by $a$, the merit which each elector attributes to the last position, & by $a + b$ that which he attributes to the second, it will be necessary to represent by $a + 2b$ the merit which is proper to the first, & it will be the same in the positions given by the other electors, of which each last will be equally represented by $a$, each second by $a + b$, & each first by $a + 2b$.

We suppose now that there are four subjects presented. One will prove by the same reasoning, that the superiority of the first position over the second, that of the second over the third, & that of the third over the fourth, must be counted equal; & that the corresponding positions given by the different electors, suppose the same degree of merit; whence one will conclude that the merits attributed by the electors to the fourth, third, second & first positions, can be represented by

$$a,\ a + b,\ a + 2b,\ \&\ a + 3b.$$

It will be the same for a more general number of presented subjects.

This put, it will be easy in any election, to compare the value of the suffrages accorded to the different subjects. For this, one will multiply by $a$, the number of last place votes given to each subject; by $a + b$, the number of the last but one votes; by $a+2b$, the number of preceding votes & so forth; one will set in order all these different products for each subject, & the sums of these products will represent the value of the suffrages accorded.

It is easy to see that in the question under consideration, the quantities $a$ & $b$, can be any that one will wish, one can therefore suppose $a = 1$ & $b = 1$, & then the value of the suffrages of each subject, will be represented by multiplying the number of last votes by 1, that of the last but one votes by 2, that of the preceding by 3, & so forth to the number of the firsts, which will be multiplied by the same number as subjects.

We give an example of an election of this kind; we suppose again 21 electors & three presented subjects $A$, $B$, $C$, &

$$\begin{array}{ccccccccccccccccccccc}
A & A & A & A & A & A & A & A & B & B & B & B & B & B & B & C & C & C & C & C & C \\
B & C & C & C & C & C & C & C & C & C & C & C & C & C & C & B & B & B & B & B & B \\
C & B & B & B & B & B & B & B & A & A & A & A & A & A & A & A & A & A & A & A & A
\end{array}$$

the 21 tickets of election. One will have by what we have said, the comparative value of the suffrages by multiplying the first votes by 3, the second votes by 2, & the third by 1, which will give the following results.

$$\begin{array}{ll}
\text{Suffrages of } A & \left\{\begin{array}{rll} 8 & \text{first votes, multiplied by } 3 & = 24 \\ 13 & \text{third votes, multiplied by } 1 & = 13 \end{array}\right\} \quad 37. \\
\text{Suffrages of } B & \left\{\begin{array}{rll} 7 & \text{first votes, multiplied by } 3 & = 21 \\ 7 & \text{second votes, multiplied by } 2 & = 14 \\ 7 & \text{third votes, multiplied by } 1 & = 7 \end{array}\right\} \quad 42. \\
\text{Suffrages of } C & \left\{\begin{array}{rll} 6 & \text{first votes, multiplied by } 3 & = 18 \\ 14 & \text{second votes, multiplied by } 2 & = 28 \\ 1 & \text{third vote, multiplied by } 1 & = 1 \end{array}\right\} \quad 47.
\end{array}$$

whence one sees that the superiority of the suffrages will be in favor of subject $C$, that the second position will be given to subject $B$, and the last to subject $A$.

It is noteworthy that if one had made the election in the ordinary manner, one would have had the following result,

$$\begin{array}{c} 8 \text{ votes for } A, \\ 7 \text{ votes for } B, \\ 6 \text{ votes for } C, \end{array}$$

that is that the plurality would have been for subject $A$, who is the last in the opinion of the electors, & that subject $C$, who is really the first, would have had fewer votes than each of the two others.

We suppose now that one wishes to employ the method of the particular elections, & that there are equally three presented subjects $A$, $B$, $C$; as one can combine these three subjects taken two by two in three different ways, it will be necessary to make three particular elections. Let the results of these elections be as follows.

$$\begin{array}{rl}
\text{1st election between } A \text{ \& } B & \left\{\begin{array}{l} a \text{ votes for } A, \\ b \text{ votes for } B, \end{array}\right. \\
\text{2nd election between } A \text{ \& } C & \left\{\begin{array}{l} a' \text{ votes for } A, \\ c \text{ votes for } C, \end{array}\right. \\
\text{3rd election between } B \text{ \& } C & \left\{\begin{array}{l} b' \text{ votes for } B, \\ c' \text{ votes for } C, \end{array}\right.
\end{array}$$

The concern is to find the comparative value of the suffrages accorded to the three subjects. For this, we will suppose that these elections are the result of an election by order of merit, which is always possible, because by knowing the rank that each subject occupied in the opinion of each elector, one can always determine the number of votes which he must have in an election made between him & any other subject. This put, let $y$, be the number of first votes that subject $A$ would have had in this election by the order of merit; $x$, the number of second votes; & $z$, the number of third votes. It is clear that then the value of the suffrages of subject $A$, would be represented by $3y + 2x + z$; but $y + x + z =$the total number of electors; let therefore this number $= E$, one will have by eliminating $z$, the value of the suffrages of $A$, represented by $2y + x + E$, or simply by $2y + x$, because $E$ is common to all the suffrages. Now, I note that, for each first vote that subject $A$ would have had in the election by order of merit, he must have two votes in the particular elections; namely, one in the election between $A$ & $B$, & one other in the election between $A$ & $C$; that for each second vote which he would have had in the election by order of merit, he will have only one in the particular elections; that for the third vote, he will have none. Whence one concludes that the number of votes which he will have in all the particular elections, namely $a + a'$ will be $= 2y + x$; but we just saw that this quantity $2y + x$ represented the value of the suffrages in the election by order of merit; therefore the quantity $a + a'$ will represent it also in the particular elections, that is that the value of the suffrages accorded to one of the subjects, will be represented by the sum of the votes which he will have had in all the particular elections which concern him; that which applies evidently to the elections made among a greater number of presented subjects.

If one determines the values of $a$, $a'$, $b$, $b'$, $c$, $c'$, according to the supposition that the particular elections are the result of the election by order of merit that I have has reported above, one will find

$$\begin{array}{lll} a = 8, & b = 13, & c = 13, \\ a' = 8, & b' = 8, & c' = 13; \end{array}$$

and consequently, one will have

$$\begin{array}{l} \text{the suffrages of } A \text{ or } a + a' = 16, \\ \text{the suffrages of } B \text{ or } b + b' = 21, \\ \text{the suffrages of } C \text{ or } c + c' = 26; \end{array}$$

which gives among the three suffrages, the same differences which had been found by the first kind of election.

Besides, we will note here that the second form of the election of which we just spoke, will be awkward in practice, when there are presented a great number of candidates, because then the number of particular elections which it will be necessary to make, will be very great. Following this, one must prefer the form of election by order of merit, which is much more expeditive.

I will end this memoir by the examination of a particular question relative to the ordinary manner of making elections. I have shown that in these elections, the plurality of the votes is not always a certain indication of the wish of the electors; but this plurality can be so great that it is not possible that the wish of the electors be for another than for the one who has obtained this plurality. In order to determine in which cases this takes place, let $M$, be the number of presented subjects; $E$, the number of electors; $A$, the subject who has the plurality; $B$, the one who, after subject $A$, has the greatest number of votes; finally $y$, the votes of subject $A$; & $z$, those of subject $B$.

We suppose next that one makes an election by order of merit among all the subjects, it is clear that then subject $A$ will have a number of first votes $= y$, & that subject $B$ will have a number $= z$ of them. Now all that which can happen more unfavorable to subject $A$, will be, that the electors who have not given the first position to him, put him in the last, & that those who have not given the first position to $B$, accord to him all the second. In this case, as the value of the first position is represented by $m$, that of the seconds by $m - 1$, & that of the last by 1, one will have the value of the suffrages of $A = my + E - y$; & that of the suffrages of $B = mz + (m-1).(E-z)$; it will be necessary therefore in order that the result of the election be necessarily in favor of $A$ that one have

$$my + E - y > (mz - 1).(E - z),$$

or

$$y > \frac{z + (m-2).E}{m-1}$$

Let $m = 2$, one will have $y > z$, that is that in the case where the election is made between two subjects alone, the subject who has the plurality of votes, is legitimately elected; thus in this case, but in this latter one alone, the ordinary form of the elections gives an exact result.

We suppose that subject $B$ has all the votes that subject $A$ has not, then one will have $z = E - y$; putting this value into the expression above, one will have $y > E.\frac{m-1}{m}$.

If, in this last expression, one makes $m = 3$,one will have $y = \frac{2}{3}E$, that is that, when there are three subjects presented, it is necessary, in order that one of the subjects be assured to have the wish of the electors, that he has more than two thirds of the votes.

One will find similarly that, when there are four subjects presented, $y$ must be greater than $\frac{3}{4}$ of $E$, & thus in sequence.

Finally, let the number of subjects be equal to the number of electors or greater than this number, the expression above $y > \frac{(m-2).E+z}{m-1}$ will become this $y > E - 1$, that is that then the election can be rigorously decided only by unanimity, a result extraordinary enough which would justify usage that a nation of the North follows in the election of its King.

There remains to me to observe, in ending this Memoir, that all which we have said on the elections, is applied equally to the deliberations made by Societies or Companies; these deliberations are in fact only kinds of elections among different proposed opinions, they are therefore subject to the same rules.