Application d'un théorèm nouveau du calcul des probabilités.*

Jules Bienaymé

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"There has appeared in the *Compte rendu* of the before last session (23 August 1875, n° 8, t. LXXXI, p. 351–353 and 377–379) many numerical series of observations which have seemed to me quite proper to show the application of a new theorem of the Calculus of probabilities of which I have given recently the statement to the Mathematical Society (*Bulletin* of this Society, n° 5, t. II, p. 153, session of 3 June 1874). It is around fifteen or twenty years ago, a particular circumstance obligated me to sen by post my formula, which seemed to me of nature to end a scientific discussion; and, at this period, I communicated it to several persons who themselves are able to report it. Here is in what this singular theorem consists: If some observations any whatsoever are ranked in order where they are presented, and not arbitrarily classed, the number of the maxima and of the minima, or of the sequences,[1] that one will count will be contained between the limits

$$\frac{2n-1}{3} - t\sqrt{\frac{16n-29}{45}}$$

and

$$\frac{2n-1}{3} + t\sqrt{\frac{16n-29}{45}}$$

with the well-known approximate probability

$$\frac{2}{\sqrt{\pi}}\int_0^t e^{-x^2}\,dx,$$

n being the number of observations and great enough in order to permit not taking account of the order of $\frac{1}{n}$ in an approximation of this kind. It is necessary to remark that this formula is applied in all rigor only to some observations of which the probability, no matter what besides, is infinitely small for each, or in some observations of which

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. January 10, 2010

[1]If one represents the observations as the ordinates of a polygon, the name of *sequence* is applied to the series of contiguous sides of this polygon, which are ascendants or descendants between a maximum and one of the adjacent minima. Thus there will be some sequences of a single side, of two, of three; and there is able to exist one of no more than $n-1$ sides. Exactly one is able to count the point of origin as maximum or minimum and, next, at least a sequence.

the probability is finite, but which is not able to be repeated. When the repetitions are possible, the mean value of the numbers of the maxima and minima, or of the sequences, is modified. For example, for the possible extreme repetition, in the case which leaves to the observation only two values, the mean number of maxima and minima, or of ascendant and descendant sequences, is no longer $\frac{2n-1}{3}$, but only $\frac{n+1}{2}$; so that, whatever be the repetitions, one is able to say that this mean is contained between the half and two-thirds of the number of observations. As the difference of these two values is only $\frac{1}{6}$, one sees that there is place to pay attention to some deviations which, in some other questions, would be able to be regarded as insignificant.

"Moreover, the concern here is only of the theorem relative to the value $\frac{2n-1}{3}$: this is the case which presents itself at each instant in the observations of each kind, in the drawings of lots of each specie, etc. The cases of repetitions are much less frequent, and besides it is often what is arranged in the limits above.

"I pass to the trials that the *Compte rendu* furnished on 23 August. First one will find in the Note of Mr. Chapelas *sur les étoiles filantes du 10 août*, for the right ascensions at the beginning of the trajectory of 225 of these falling stars:

P. 377,	1st	column,	7	maxima or minima	on 13	observations.
	2nd		11		13	
P. 378,	1st		43		64	
	2nd		39		65	
P. 379,	1st		22		35	
	2nd		24		35	
	Total		146	maxima on	225	observations.

"The mean indicated by the formula above would be

$$\frac{2 \times 225 - 1}{3} = 149 + \frac{2}{3}.$$

The deviation of the observations is therefore only $3 + \frac{2}{3}$, a number which does not even require that one make $t = 1$ in the limits $\pm\sqrt{\frac{16n-29}{45}}$, this which reduces them to 8.9, and this which does not raise the probability to 0.8427, let it be a little more than 5 against 1 (16 against 3).

"If now one takes at least the end of the declinations at the beginning of the trajectory of the same stars, one will find:

P. 377,	1st	column,	7	maxima on	13	observations.
	2nd		9		13	
P. 378,	1st		50		64	
	2nd		39		65	
P. 379,	1st		24		35	
	2nd		23		35	
	Total		154	maxima on	225	observations.

"The theoretical mean is $149\frac{2}{3}$, the deviation is therefore only $4\frac{1}{2}$, and, consequently, it is contained in the previously calculated limits.

"The probability that these two values would be contained within the same limits above were *a priori* only the square of the preceding, let it be 0.17 or only $3\frac{1}{2}$ against 1.

"One will recognize likewise for the right ascensions at the end of the trajectory:

P. 377,	1st	column,	4	maxima on	13	observations.
	2nd		4		8	
P. 378,	1st		41		60	
	2nd		39		64	
P. 379,	1st		23		35	
	2nd		24		35	
	Total		135	maxima on	215	observations.

Here the theoretic mean is more than $\frac{2\times 215-1}{3} = 143$. The deviation is raised to 8. But the limits are no longer, for hte same probability, but of $\sqrt{frac16 \times 215 - 2945 = 8.7}$, and however this deviation is found yet contained. This fact merits to be observed, for frequent enough repetitions exist in the series of falling stars, of all necessity.

"Taking finally the declinations at the end of the trajectories, one will find:

P. 377,	1st	column,	9	maxima on	13	observations.
	2nd		6		8	
P. 378,	1st		42		60	
	2nd		39		64	
P. 379,	1st		22		35	
	2nd		23		35	
	Total		141	maxima on	215	observations.

The deviation is only 2, and it is largely contained within the calculated limits.

"*A priori*, if these four means were completely independent, there would not have been odds more than 1 against 1 that they would be each contained within the same limits, which determined $t = 1$, with the probability 0.8427.

"On page 353 of the same number of the *Comptes rendus*, Mr. Le Verrier made known 28 observations of a completely other importance than the preceding. The concern is of the difference between the observations made at Greenwich and at Paris on the heliocentric longitude of Saturn. Here, despite the small number of observations, the theoretic mean $\frac{2\times 28-1}{3} = 18 + \frac{1}{3}$ coincides nearly exactly with the number of maxima and minima observed, which is 18. The small divergences from one observatory to another give place to no particular remark. And, indeed, the theorem is applying to each specie of collection of fortuitous magnitudes, there is nothing to conclude from this that a series satisfied, as the two preceding examples make it. But it is no longer likewise when one raises in the same Communication, pages 351–352, the 22 modern observations of the heliocentric longitude of Saturn made at Greenwich and at Paris. There is found only 9 maxima or minima: this is less than half. It is likewise for the 16 old observations, which offer only 8 maxima. Despite the relative smallness of the numbers 22 and 16, it would seem that any cause had been able alone to reduce systematically the number of maxima or minima observed. Perhaps this cause would merit to be researched. It is to the astronomers to judge it. In this note, there is able to be question only of probabilities; but the astronomical observations do not deviate more than the others in the examination of the theory of probabilities, despite the extreme precision to which they are arrived between the hands of observers so able and of geometers of greatest renown.

"The difference of the values employed in the two calculations of the heliocentric longitude of Saturn, for the mass of Jupiter, produce, as one is able to see, no sensible effect on the 28 observations. It appeared effectively quite small fro this mass rather badly known, despite the elevated number which represents this gross planet. I have already had occasion (*Mémoire sur les errors d'après la méthode des moindres carrés*, presented 27 October 1851 to the Academy, and published in the *Journal* of our illustrious fellow-member, Mr. Liouville, in 1852, next later in the XV$^{\text{th}}$ volume of the *Recueil des Savants étrangers*, I have already had occasion to signal how much the complication of the equations so numerous of which one had deduced this mass rendered small the probability that one had believed to be able to attach. There would be perhaps place to research if the combinations of which one deduced it now are direct enough and embracing rather little of unknowns in order to permit to define a modification as feeble as that of $\frac{1}{1046.77}$ to $\frac{1}{1050}$.

As to the 22 modern observations and to the 16 old observations of the heliocentric latitude of Saturn, if the number of maxima of the 22 modern is 13, this which borders the theoretical mean $14+\frac{1}{3}$, the number of the maxima of the 16 old is only 7. It would seem thence that there would have been a notable change in the art of observing the declinations, a change of which the right ascensions would have been able to profit; but, again one time, these last numbers of observations are so small for the point of view to which the new theorem envisages them, that it is only by title of examples that it has been permitted to make them the subject of some reflections.

"Here is all that which it seems useful to say on the numbers of the before last *Compte rendu*. I would have briefly some other examples which would be perhaps a little more easy to rediscover, but that nevertheless one would be able to procure them without great difficulty.

"And first I will cite the right ascensions and the declinations of the Comet of Olbers, which are reported in chronological order by Bessel (*Untersuchungen über die Bahn des Olbersschen Kometen. Mémoires de l'Académie de Berlin* for 1812-1813. 183 right ascensions would require $121+\frac{2}{9}$ sequences with a deviation of $\pm 8.02 \times t$. The observation gives only 112 sequences. The deviation of $9\frac{2}{3}$ carries a probability superior to 5 against 1, but of little good. Besides, it is not surprising that by multiplying these trials which must fall within the calculated limits only 5 times out of 6 (more exactly 16 times out of 19) one encounters some cases which exit more or less.

"For the declinations, although there are only 166, which furnish for mean $110+\frac{1}{3}$ with a deviation of $+7.64$, one will find in the Memoir of Bessel 106 sequences or 106 maxima and minima. The difference of the theoretic mean is therefore only $+4+\frac{1}{3}$, returning completely within the limits and with a very weak probability.

"In another specie of facts, one is able to take in the journals the results of the drawing executed 20 July last for the loan of 1871 of the city of Paris. The 88 exited obligations demand a mean of $58+\frac{1}{3}$: the real number is 57. One sees that the deviation is reduced to $1+\frac{1}{3}$, despite the smallness of the number of observations, which permitted limits equal to ± 5.53 with the probability already employed of 16 against 3.

"One is able next to take for trial the 255 exited obligations in the drawing of 3 July last, made on the securities so new of the *tramways* of the quarters of the north of Paris. The same probability would draw a mean of 143 with a deviation of 8.7. The real number of sequences is found 140 (*Journal financier* of 1st August.

"In order to terminate finally, one is able next to examine the drawing of 2 August current of the obligations of the cities of Roubaix and Tourcoing, in the number of 376 (*Globe* or *Réforme financière* of 15 August 1875). The theoretic mean is

$$\frac{2 \times 376 - 1}{3} = 250 + \frac{1}{3},$$

with a deviation of ± 11.53.

"The observed number is 245 sequences, which offer only a deviation of $5 + \frac{1}{3}$ and would not require a probability of 1 against 1.

"The examples to cite are presented on all sides and all days, but it is convenient to stop."