MÉMOIRE
SUR
LA PROBABILITÉ DES RÉSULTATS MOYENS
DES OBSERVATIONS;
DÉMONSTRATION DIRECTE DE LA RÈGLE DE LAPLACE*

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It is not possible to be occupied with public or private economy, and consequently with statistics, without remarking soon on notable differences between the results of operations, even the most exact, and better combined in order to offer an entire concordance. During a long time observers, certain of their experiences, and scandalized by the apparent contradictions which they encountered, have wished to find the explication more or less plausible in the variations of natural causes.

Jacques Bernoulli knew, first, that the greater part of these anomalies the simple effects of chance. He showed how one could submit to the calculus these effects that common opinion regarded as free of all rules, and he founded thus on an unshakable base this part of the *art of conjectures* which must direct the observer into a numerous class of scientific researches.

The researches of social economy or of statistics belong nearly all to this remarkable class, of which the special character consists in the continual usage of the means or the sums of similar phenomena compiled by multiplied observations. It is precisely the ratio which exists between the probable deviation of these means and the multiplicity of the observations that J. Bernoulli has signaled.

Moivre perfected the discovery of Bernoulli. But these two great mathematicians supposed known the possibility of the phenomena, and they limited themselves to deduce from the law of possibility that they assigned, the extent of the deviations of which the observations were able to be susceptible. This was there only one part of the question, and the least applicable part: because the natural laws are unknown, and form precisely the object of the researches. It remained therefore to resolve the problem inverse of the one which alone permitted to render to ever celebrate the name of Bernoulli.

The solution was given only sixty years later, by Bayes, a little known English scholar, without doubt because a premature death interrupted his work, but who appeared to have possessed in a very high degree the qualities of the geometer. Bayes

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. June 21, 2010 arrived, in a manner that Laplace has judged *fine and very ingenious, although a little embarrassed*, to determine, in the case of two events of which the one or the other must necessarily present itself in each observation, the probability that the indicated possibilities through some experiences already made are really contained in some given limits.

Bernoulli had proved that, when the respective probabilities of the two events are, for example, in the ratio of $\frac{3}{5}$ to $\frac{2}{6}$ there are odds more than one thousand against one, out of 25550 trials, the first will be presented more than 14819 times and less than 15841; numbers which would place the observed possibility of the first between $\frac{31}{50}$ and $\frac{29}{50}$, and the observed possibility of the second between $\frac{19}{50}$ and $\frac{21}{50}$.

Bayes will prove reciprocally that, if in 25550 experiences one observes the ratio of $\frac{5}{6}$ to $\frac{2}{5}$ between the numbers of repetition of two events, that is to say the first has been observed 15330 times, and the second 10220 times, there are odds of one thousand against one that the real possibilities fall, the first between $\frac{305}{600}$ and $\frac{295}{600}$, the second between $\frac{195}{600}$ and $\frac{305}{600}$. These fractions would lead respectively to 15588, or to 15072 repetitions of the first event, and to 9962, or 10428 of the second.

It was not useful to repeat these facts. They indicate clearly the nature of the questions which raise the remarked deviations between the observations. They show also what slowness has presided in the progress of this branch of the analysis of hazards, and at what point it was arrested, having the immense impulse that the theory received from the genius of Laplace toward the end of the last century.

Laplace had proposed in all its generality the following problem:

What is the probability of the deviation of which a mean is able to be affected among any great number of different values of a phenomenon or of an event, given by so many observations, when one is not aware of the law of the probabilities of the partial deviations of each observation?

Laplace resolved completely this difficult question, and he has made the most beautiful applications of his method in his admirable *Theorie analytique des probabilités*.

The rule to which he is arrived is quite simple. It prescribes to divide by the number of observations, the square root of the double of the sum of the squares of the differences between the result of each observation and the mean of all. If one represents next the limits of the deviation that one wishes to consider, by the product of the constant thus obtained and a variable factor; the probability that the real deviation is contained in these limits will be read without pain in a table calculated beforehand for all the values of the variable factor, a table which remains the same for a crowd of diverse questions.

Effectively then the probability is expressed by a well-known transcendent, of which a table has been published by Kramp in order to resolve a quite different question, the calculation of the astronomical refractions. Moreover, there is need most of the time only for four or five values of this transcendent; and one is able to conserve them in the memoir.

This rule is so facile, so general, so singular, that it will never lack to excite the astonishment at first of all. Better, when one demonstrates it to persons little versed in mathematics, it seems to them that it is a revelation of a sort of mystery, of a supernatural property of the kind of those that the ancients attributed to the famous numbers of Pythagoras: these persons are not able to be prohibited from seeing something more than human in the algebra, capable to extract such secrets to blind chance.

The men more habituated to the resources of the highest analysis find another subject of surprise in the artifice of which the illustrious inventor himself is served by this rule, in order to eliminate from the calculations the quantities dependent on the unknown law of the probabilities of the deviations of the observations. This uncommon artifice leaves to hover some difficulty on the degree of exactitude of the use of the constant deduced from the squares of the differences between the observations and the mean. It will suffice to repeat in this regard that the sum of these squares appeared substituted in the sum of the products of the squares of all the possible errors by their probabilities, a sum to which the first is not rigorously equal, but of which it differs probably very little when the number of observations is rather great.

Laplace has believed necessary to clarify the usage of this substitution in the *Premier Supplèment à la Théorie des probabilités*. The expressions consigned in this supplement and in the diverse editions of the *Essai sur les probabilités* show the importance that he attached to dissipate the feeble incertitude that was able to enter his analysis.

Next, Mr. Poisson has reprised, with the extreme lucidity which is habitual with him, this difficult passage of the theory of Laplace, and he has demonstrated how the difference between the true constant and the observed constant vanish, in measure as the number of experiences increases.

The study of the beautiful memoirs of Mr. Poisson, on this subject, inserted into the *Connaissance des temps*, and another memoir of Volume IX of the Academy of Sciences, in which the most delicate proceeds of this subtle analysis are applied by him to the examination of the ratio of the births of girls and of boys, this thorough study has made to think that there existed some means to dissipate entirely the slight cloud which was able to subsist yet on the important rule given by Laplace.

If one also wishes to consider that this rule must serve to balance, in some manner, in order to weigh the influence that one is able to accord to the greater part of the observations on which political economy is founded; that it is susceptible, as its author has shown, to be extended to the same facts which hold to the intelligent part, to the moral part of man, and consequently to philosophy; that, besides, the physical sciences properly said have more than one occasion to recur; one will find perhaps some interest to see established directly and especially without this clever elimination, but painful to grasp well, of which Laplace himself is served so happily, the existence of the constant on which the rule reposes all whole. Such is the end of this memoir.

The proceeds which will be employed in order to arrive there are, moreover, only those same of which Laplace has taught the use. It is necessary to repeat again, it is in the *Theorie analytique des probabilités* and in the developments so remarkable given by Mr. Poisson, that the means of demonstration have been drawn.

A rather natural reflection has led there. In order that it is easily seized, it is necessary to repeat that if an event has been observed p times out of a great number $p + p_1$ of trials, the possibility of this event is contained in the limits

$$\frac{p}{p+p_1} \pm c\sqrt{\frac{2pp_1}{(p+p_1)^3}} \tag{1}$$

is equal to the definite integral

$$\frac{2}{\sqrt{\pi}}\int_0^c e^{-t^2}\,dt. \tag{2}$$

This put, it was easy to perceive that to demand the probability of the extent of the deviation between the ratio $\frac{p}{p+p_1}$, and the real ratio of possibility which would be manifested if it were given to man to multiply indefinitely the trials, it is to demand, for a very particular case, the probability of the exactitude of a certain mean furnished by the observations.

Let one represent in fact by γ, γ_1 two arbitrary functions of the observed events, relative respectively, the one to the event A, which is presented p times; and the other to the contrary event B, which has taken place p_1 times; if one names moreover x, x_1, the unknown possibilities of these two events; nothing prevents to imagine that the $(p+p_1)$ trials had had for object to determine the value of the quantity

$$v = \gamma x + \gamma_1 x_1.$$

By taking therefore for this quantity the mean of the products of the functions γ, γ_1, multiplied respectively by the numbers p, p_1 of the events to which they report themselves back, one will be able to demand the probability that the value

$$v' = \frac{\gamma p + \gamma_1 p_1}{p + p_1}$$

is not extended from the real value v by a given quantity.

If now one supposes

$$\gamma = 1, \qquad \gamma_1 = 0,$$

the mean v' is reduced to $\frac{p}{p+p_1}$, the quantity v is reduced to x; and it is clear that the probability which will have been the probability of the difference between the ratio $\frac{p}{p+p_1}$ and the real value of x.

But the probability of this difference is given by formulas (1) and (2), as Laplace has demonstrated. One is able therefore to conclude that these formulas are only a particular case for $\gamma = 1$, $\gamma_1 = 0$, of those which must express the probability of the possible deviations between the mean obtained by observation and the true value of the expression $(\gamma x + \gamma_1 x_1)$.

It is therefore from total necessity that the factor

$$\sqrt{\frac{2pp_1}{(p+p_1)^3}},$$

which enters into formula (1), is precisely the singular constant introduced by Laplace; in which the values given by observation are p times 1, p_1 times 0.

In order to acquire certitude, one will put

$$p + p_1 = n, \qquad \text{whence} \qquad p_1 = n - p,$$

and the radical above will be able clearly to be written:

$$\frac{1}{n}\sqrt{2\left\{p\left(1-\frac{p\times 1+p_1\times 0}{n}\right)^2+p_1\left(0-\frac{p\times 1+p_1\times 0}{n}\right)^2\right\}}$$

One finds therefore precisely the quotient of the root of the double of the sum of the squares of the differences between the observed values (p times 1, p_1 times 0) and the mean $\frac{p\times 1+p_1\times 0}{n}$, by the number of observations: and the radical $\sqrt{\frac{2pp_1}{(p+p_1)^3}}$ is no other than the constant of Laplace.

The introduction of this radical in formula (1) is not only probable: it results, one knows, from a certain analysis. It was therefore permitted to anticipate, after this particular case, that the value of the constant was not less susceptible to be calculated in a certain manner in the general case.

Effectively the decomposition of $\sqrt{\frac{2pp_1}{(p+p_1)^3}}$, one time done, the rigorous demonstration of the rule of Laplace requires no more than the rather simple calculations, although a little long.

The march will be better followed, if one begins by examining the case of which there comes to be question, and which involves only two exclusive events.

Conserving the same denominations,

$$p+p_1=n, \qquad x+x_1=1,$$

and the question is to determine the probability that the unknown quantity

$$v=\gamma x+\gamma_1 x_1. \tag{3}$$

is comprised between the given limits a' and a.

Designating by C the coefficient of the term of the binomial $(x+(1-x))^n$ of which the exponents are p and p', one knows that the probability of the event composed of p repetitions of the event A, and of p_1 of the event B, is

$$Cx^p(1-x)^{p_1}$$

The relation (3) gives

$$x=\frac{v-\gamma_1}{\gamma-\gamma_1}, \qquad 1-x=\frac{\gamma-v}{\gamma-\gamma_1}.$$

Therefore under the hypothesis of an assigned value to v, the probability of the composite event will be

$$C\left(\frac{v-\gamma_1}{\gamma-\gamma_1}\right)^p\left(\frac{\gamma-v}{\gamma-\gamma_1}\right)^{p_1}$$

The probability of the hypothesis of a value of v will be thereafter

$$\frac{(v-\gamma_1)^p(\gamma-v)^{p_1}dv}{\int_\gamma^{\gamma_1}(v-\gamma_1)^p(\gamma-v)^{p_1}dv};$$

the integral being taken for all possible values of v, that is to say from $v = \gamma$ to $v = \gamma_1$, if $\gamma_1 > \gamma$.

Finally, the probability that the real value of v is contained in the limits a' and a, will be found by integrating the preceding expression between these limits.

This probability is

$$\text{(4)} \qquad \frac{\int_{a'}^{a} (v-\gamma_1)^p (\gamma - v)^{p_1}\, dv}{\int_{\gamma}^{\gamma_1} (v-\gamma_1)^p (\gamma - v)^{p_1}\, dv};$$

One knows the general method that Laplace has invented in order to obtain the approximate value of the integrals of this kind. In the following, it is necessary to put

$$\text{(5)} \qquad v = \frac{p\gamma + p_1\gamma_1}{n} + z, \qquad dv = dz,$$

and z will represent the deviation between the real value of v and the mean drawn from the observations.

Each of the terms of the probability (4) will take the form

$$(\gamma-\gamma_1)^n \frac{p^p p_1^{p_1}}{n^n} \int_{b,}^{b} \left(1 + \frac{nz}{p(\gamma-\gamma_1)}\right)^p \left(1 - \frac{nz}{p_1(\gamma-\gamma_1)}\right)^{p_1} dz,$$

$b,$ and b being the limits of z, corresponding to the limits a' and a of v.

Developing the factors $\left(1 + \frac{nz}{p(\gamma-\gamma_1)}\right)$ and $\left(1 - \frac{nz}{p_1(\gamma-\gamma_1)}\right)$ into exponential series, and designating ordinarily by e the base of the Naperian logarithms,

$$\text{(6)} \qquad \begin{aligned} &\left(1 + \frac{nz}{p(\gamma-\gamma_1)}\right)^p = e^{p\ln\left(1+\frac{nz}{p(\gamma-\gamma_1)}\right)} \\ &= e^{\frac{nz}{(\gamma-\gamma_1)} - \frac{1}{2}\frac{n^2z^2}{p(\gamma-\gamma_1)^2} + \frac{1}{3}\frac{n^3z^3}{p^2(\gamma-\gamma_1)^3} - \frac{1}{4}\frac{n^4z^4}{p^3(\gamma-\gamma_1)^4} + \text{etc.}} \\ &\left(1 - \frac{nz}{p_1(\gamma-\gamma_1)}\right)^{p_1} = e^{p_1\ln\left(1-\frac{nz}{p_1(\gamma-\gamma_1)}\right)} \\ &= e^{-\frac{nz}{(\gamma-\gamma_1)} - \frac{1}{2}\frac{n^2z^2}{p_1(\gamma-\gamma_1)^2} - \frac{1}{3}\frac{n^3z^3}{p_1^2(\gamma-\gamma_1)^3} - \frac{1}{4}\frac{n^4z^4}{p_1^3(\gamma-\gamma_1)^4} - \text{etc.}} \end{aligned}$$

and the integral will become

$$(\gamma-\gamma_1)^n \frac{p^p p_1^{p_1}}{n^n} \int_{b,}^{b} dz e^{\left\{ \begin{array}{l} -\frac{n^2z^2}{2p(\gamma-\gamma_1)^2}\left(\frac{1}{p}+\frac{1}{p_1}\right) + \frac{n^3z^3}{3p(\gamma-\gamma_1)^3}\left(\frac{1}{p^2}-\frac{1}{p_1^2}\right) \\ -\frac{n^4z^4}{4p(\gamma-\gamma_1)^4}\left(\frac{1}{p^3}+\frac{1}{p_1^3}\right) + \text{etc.} \end{array} \right\}}$$

Therefore one will put only

$$\frac{n^2z^2}{2p(\gamma-\gamma_1)^2}\left(\frac{1}{p}+\frac{1}{p_1}\right) = t^2,$$

whence

$$z = t\sqrt{\frac{2pp_1(\gamma-\gamma_1)^2}{n^3}}, \quad dz = dt\sqrt{\frac{2pp_1(\gamma-\gamma_1)^2}{n^3}} \tag{7}$$

The transformation used since Laplace consists in representing by t^2 all the exponential series in z: one deduces from it a value of z, ordered according to the powers of t, and hence one transports to the limits of z a part of the difficulties of analysis. But as the question here is to fix rigorously the value of the constant which expresses these limits, it appeared preferable to rid them from this expression of z in a series of which one obtains the terms only with difficulty through the return of the series. By the transformation (7), all this which it is able to be delicate and difficult in the approximation is entirely concentrated in the integral and in the value of the probability.

One arrives thus to the expression

(8)

$$(\gamma-\gamma)^{n+1}\frac{p^p p_1^{p_1}}{n^n}\sqrt{\frac{2pp_1}{n^3}}\int_{c'}^{c} dte^{\left\{\begin{array}{l} -t^2+\frac{4(p_1-p)}{3\sqrt{2npp_1}}t^3-\frac{p^3+p_1^3}{n^2pp_1}t^4 \\ +\frac{8(p^2+p_1^2)(p_1-p)}{5npp_1\sqrt{2npp_1}}t^5-\frac{4(p^5+p_1^5)}{3n^3p^2p_1^2}t^6+\text{etc.}\end{array}\right\}}$$

c' and c representing the limits of t.

The constant factors placed outside of the $\int$ sign will be suppressed in the rest of the calculation, because they would multiply the two terms of the probability (4). Developing the part of the exponential which contains the powers superior to the 2nd, by means of the known formula

$$e^m = 1 + m + \frac{m^2}{1.2} + \frac{m^3}{1.2.3} + \text{etc.},$$

the sought integral is able to be written as follows:

$$\begin{aligned}\int_{c'}^{c} dte^{-t^2}\Big\{1+\frac{4(p_1-p)}{3\sqrt{2npp_1}}t^3-\frac{p^3+p_1^3}{n^2pp_1}t^4+\frac{8(p^2+p_1^2)(p_1-p)}{5npp_1\sqrt{2npp_1}}t^5 \\ -\left(\frac{4(p^5+p_1^5)}{3n^3p^2p_1^2}-\frac{4(p-p_1)^2}{9npp_1}\right)t^6+\text{etc.}\Big\}\end{aligned} \tag{9}$$

It is clear that the coefficients of the terms placed between parentheses will be of the order of the powers of $\frac{1}{\sqrt{n}}$, save some special circumstances of which there is no place to be occupied here.

Moreover, by taking $c' = -c$, the terms which contain some odd powers of t will be destroyed, and there will remain only

$$2\int_0^c dte^{-t^2}\left\{1-\frac{p^3+p_1^3}{n^2pp_1}t^4-\text{etc.}\right\} \tag{10}$$

But seeing that

$$\int_0^t dt\, e^{-t^2} t^{2m} = -\frac{e^{-t^2}}{2}\left(t^{2m-1} + \frac{2m-1}{2}t^{2m-3} + \cdots \right.$$
$$\left. \cdots + \frac{2m-1}{2}\cdot\frac{2m-3}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}t\right) + \frac{2m-1}{2}\cdots\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\int_0^t e^{-t^2}dt,$$

the integral (10) will be partitioned into two parts,

$$(11)\quad 2\left\{1 - \frac{3}{4}\frac{p^3+p_1^3}{n^2pp_1} - \text{ etc.}\right\}\int_0^c e^{-t^2}dt + \frac{e^{-c^2}}{2}\left\{2\frac{p^3+p_1^3}{n^2pp_1}(c^3 + \frac{5}{2}c) + \text{ etc.}\right\}$$

The value of $\frac{e^{-c^2}}{2}$, always fractional, is further for

	$c = 1$	of	0.1839397
but	$c = 2$	reduces it to	0.0091578
	$c = 2 + \frac{1}{2}$	to	0.0009652
	$c = 3$	it falls to	0.0000617

The part of the series multiplied by this very small factor, a part of which the terms are already divided by the powers of the large number $\sqrt{n}$, will be therefore very convergent at the origin: and when n will be very great, one will be able evidently to neglect it without sensible error.

The numerator of the probability (4) will be reduced thus to

$$(12)\qquad 2\left(1 - \frac{3}{4}\frac{p^3+p_1^3}{n^2pp_1} - \text{etc.}\right)\int_0^c e^{-t^2}dt.$$

The denominator is found by integrating the expression (9), for all possible values of t.

γ and γ_1 being the extreme limits of v, one will conclude from the relations (5) and (7) the limits corresponding to t:

$$(13)\qquad l = \sqrt{\frac{pn}{2p_1}} \qquad l' = -\sqrt{\frac{p'n}{2p}}$$

These two quantities are of the order $\sqrt{n}$, and consequently very great; but

$$\frac{1}{2}e^{-t^2}t^m = 0$$

when $t = \infty$; and this expression which becomes so much smaller as t is greater departing from $t = \sqrt{\frac{m}{2}}$, is generally quite small, because one has for

$$t = 3 + \frac{1}{2} \qquad \frac{1}{2}e^{-\frac{49}{4}} = 0.00\,000\,239\,255$$
$$t = 4 \qquad \frac{1}{2}e^{-16} = 0.00\,000\,005\,626$$
$$t = 5 \qquad \frac{1}{2}e^{-25} = 0.00\,000\,000\,000\,694$$

From this consideration there results that if the limits (13), which generally are of the very elevated order $\sqrt{n}$, exceed only the first numbers, the terms furnished by the integration and multiplied by $\frac{1}{2}e^{-t^2}$ will decrease much more rapidly yet in the denominator than in the numerator, even for the powers of t rather elevated. To the degree where one carries ordinarily the approximation, there will not be any error if one neglects these terms, and the integral will be reduced to

$$\left(1 - \frac{3}{4}\frac{p^3 + p_1^3}{n^2 pp_1} - \text{etc.}\right)\int_{l'}^{l} e^{-t^2}dt. \tag{14}$$

One knows besides that if the absolute value of the limits l and l' is in the least superior to 4, one is able to extend them to infinity; since one has for

$$t = 4 \quad \int_4^{\infty} e^{-t^2}dt = 0.00\,000\,001\,366 \tag{15}$$

$$t = 5 \quad \int_4^{\infty} e^{-t^2}dt = 0.00\,000\,000\,000\,136\,254 \tag{16}$$

This extreme part of the entire integral, which becomes 10000 times smaller when t passes from 4 to 5, diminishes much more yet from $t = 5$ to $t = 6$. It suffices therefore that the smallest of the numbers p and p_1 attained 50, provided that one was able to put without sensible error

$$\int_{l'}^{l} e^{-t^2}dt = \int_{-\infty}^{\infty} e^{-t^2}dt = \sqrt{\pi}, \tag{17}$$

π representing the ratio of the circumference to the diameter.

The expression (14) is restored by this approximation excessive to

$$\left(1 - \frac{3}{4}\frac{p^3 + p_1^3}{n^2 pp_1} - \text{ etc.}\right)\sqrt{\pi},$$

and the probability (4) to the definite integral

$$\frac{2}{\sqrt{\pi}}\int_v^c e^{-t^2}dt. \tag{18}$$

This is the probability that the value of $v = \gamma x + \gamma_1 x_1$ is contained between the limits a' and a, which, because of the relations (5) and (7), becomes

$$\frac{\gamma p + \gamma_1 p_1}{n} \pm c\sqrt{\frac{2pp_1(\gamma - \gamma_1)^2}{n^3}}. \tag{19}$$

The expression (18) is equally the probability that the difference between the real value of v and the result of the observations, or the deviation of these observations, is contained between

$$\pm c\sqrt{\frac{2pp_1(\gamma - \gamma_1)^2}{n^3}} \tag{20}$$

It is easy to see that

$$\frac{pp_1(\gamma-\gamma_1)^2}{n} = \frac{pp_1^2+p_1p^2}{n^2}(\gamma-\gamma_1)^2 = p\left(\frac{p_1(\gamma-\gamma_1)}{n}\right)^2 + p_1\left(\frac{p(\gamma-\gamma_1)}{n}\right)^2$$
$$= p\left\{\frac{(n-p)\gamma-p_1\gamma_1}{n}\right\}^2 + p_1\left\{\frac{(n-p_1)\gamma_1-p\gamma}{n}\right\}^2$$

this which changes the expression (20) into

$$(21) \qquad \pm\frac{c}{n}\sqrt{2\left\{p\left(\gamma-\frac{p\gamma+p_1\gamma_1}{n}\right)^2 + p_1\left(\gamma_1-\frac{p\gamma+p_1\gamma_1}{n}\right)^2\right\}}$$

One recognizes under the radical the sum of the squares of the differences between the mean and all the observations; thus, it is proved that, in the case of two simple events, the constant which determines the extent of the deviations is precisely and certainly, that that the genius of Laplace has made to discover by a path so different.

The form (20), under which this constant is offered, must be remarked: it is represented without ceasing, whatever be the number of simple events. It is besides a third way to calculate the constant. It is able to be reunited to those which have been given by Laplace. The form (21), as Mr. Poisson has made to observe, is the most commodious. But that

$$(22) \qquad \pm\frac{c}{n}\sqrt{2\left\{\left(\frac{p\gamma^2+p_1\gamma_1^2}{n}\right)^2 - \left(\frac{p\gamma+p_1\gamma_1}{n}\right)^2\right\}},$$

where enters no more than the mean of the squares of the functions given by the observations, less the square of the mean, is often useful in applications, when the functions γ, γ_1 do not vary. Perhaps it would be found from the circumstances where the form (20) will be it also.

The developments which come to receive the analysis of the particular case of two events will permit shortening certain parts of the demonstration, when there will be a question of the subdivisions more multiplied by the total number of trials among the diverse observed phenomena.

If three phenomena or events A, B, C presented themselves, one would designate again by γ, γ_1, γ_2, the arbitraries which characterize them, by x, x_1, x_2, the unknown possibilities, by p, p_1, p_2, the number of repetitions of each simple event out of the number n trials, and one would have to calculate the probability of the values of the expression

$$(23) \qquad v = \gamma x + \gamma_1 x_1 + \gamma_2 x_2,$$

contained between the given limits a' and a; for which

$$(24) \qquad p+p_1+p_2 = n, \quad x+x_1+x_2 = 1.$$

If $x, x_1,\ x_2$ were known, the probability of the event composed of p times the event A, p_1 times the event B, p_2 times the event C, would be

$$Kx^p x_1^{p_1} x_2^{p_2},$$

by designating by K the coefficient of the term of the polynomial $(x + x_1 + x_2)^n$ in which the exponents are $p,\ p_1,\ p_2$.

The probability of a hypothesis on $x, x_1,\ x_2$, that is to say the probability of a value of v, will be therefore

$$\frac{x^p x_1^{p_1} x_2^{p_2}}{\sum x^p x_1^{p_1} x_2^{p_2}}$$

the sign $\sum$ indicating the sum of all the possible values of the product which it affects.

The values of v must be contained between certain limits a' and a, the sum of the probabilities of these values will be

$$\frac{\sum_{a'}^{n} x^p x_1^{p_1} x_2^{p_2}}{\sum x^p x_1^{p_1} x_2^{p_2}}$$

the sign $\sum$ of the numerator indicating only the systems of values of $x, x_1,\ x_2$ capable of giving for v a value contained between a' and a.

According to the relations (23) and (24), x and x_1 are functions of v and x_2; and there results from it that all the possible values of v are contained between γ and γ_2, the arbitrary γ being the smallest and γ_2 the greatest of the three. One will obtain therefore the probability above by integrating the expression

$$\text{(25)} \qquad \frac{{}_2\int_{a'}^{a} dv dx_2\, x^p x_1^{p_1} x_2^{p_2}}{{}_2\int_{\gamma'}^{\gamma_2} dv dx_2\, x^p x_1^{p_1} x_2^{p_2}}$$

The limits of which the sign $\int$ is affected is returned to the single variable v; and the number 2, placed to the left of this sign, reminds that it is necessary to integrate twice.

The question is thus brought back to the determination of two definite double integrals, and one is consequently completely certain that the solution is independent of the law of probability of the diverse simple events.

x and x_1 being functions of x_2 and v, could be eliminated immediately. There would remain to find the value of the integral

$$\text{(26)} \qquad {}_2\int_{a'}^{a} dv dx_2 \left(\frac{v - \gamma_1 - x_2(\gamma_2 - \gamma_1)}{\gamma - \gamma_1}\right)^p \left(\frac{v - \gamma - x_2(\gamma_2 - \gamma)}{\gamma_1 - \gamma}\right)^{p_1} x_2^{p_2};$$

But it would be much simpler to put first, by extension of the method of Laplace,

$$\text{(27)} \qquad x = \frac{p}{n} + z, \quad x_1 = \frac{p_1}{n} + z_1, \quad x_2 = \frac{p_2}{n} + z_2,$$
$$v = \frac{p\gamma + p_1\gamma_1 + p_2\gamma_2}{n} + u;$$

one will deduce from it

$$\text{(28)} \qquad z + z_1 + z_2 = 0, \quad \gamma z + \gamma_1 z_1 + \gamma_2 z_2 = u.$$

If one calls b' and b the limits of u, the integral (26) will be changed into

$$\frac{p^p p_1^{p_1} p_2^{p_2}}{n^n} \int_{b'}^{b} \left(1+\frac{nz}{p}\right)^p \left(1+\frac{nz_1}{p_1}\right)^{p_1} \left(1+\frac{nz_2}{p_2}\right)^{p_2} dudz,$$

and passing to exponentials, into

$$(29) \qquad \frac{p^p p_1^{p_1} p_2^{p_2}}{n^n} \int_{b'}^{b} dudz_2 e^{\left\{ \begin{array}{l} + n(z+z_1+z_2) - \frac{n^2}{2}\left(\frac{z^2}{p}+\frac{z_1^2}{p_1}+\frac{z_2^2}{p_2}\right) \\ + \frac{n^3}{3}\left(\frac{z^3}{p^2}+\frac{z_1^3}{p_1^2}+\frac{z_2^3}{p_2^2}\right) - \text{etc.} \end{array} \right\}}$$

The first of the relations (28) render null the term in n, in the exponent of e. One will represent by S the series which follows the term in n^2, and the integral to determine will become more commodious for the calculation,

$$(30) \qquad \frac{p^p p_1^{p_1} p_2^{p_2}}{n^n} \int_{b'}^{b} dudz_2 e^{-\frac{n^2}{2}\left(\frac{z^2}{p}+\frac{z_1^2}{p_1}+\frac{z_2^2}{p_2}\right)+\mathrm{S}.}$$

One is able at present to make use of the relations (28) in order to eliminate z and z_1. They give

$$(31) \qquad z = \frac{u - z_2(\gamma_2-\gamma_1)}{\gamma-\gamma_1}, \quad z_1 = \frac{u - z_2(\gamma_2-\gamma)}{\gamma_1-\gamma}.$$

Substituted into the exponent of e, these values lead to

$$\begin{aligned} &\mathrm{S} - \frac{n^2}{2}\left\{\frac{z_2^2}{p_2} + \frac{(u-z_2(\gamma_2-\gamma))^2}{p_1(\gamma_1-\gamma)^2} + \frac{(u-z_2(\gamma_2-\gamma_1))^2}{p(\gamma-\gamma_1)^2}\right\} \\ &= \mathrm{S}\text{-}\frac{n^2}{2(\gamma-\gamma_1)^2}\left\{z_2^2\left(\frac{(\gamma_1-\gamma)^2}{p_2}+\frac{(\gamma-\gamma_2)^2}{p_1}+\frac{(\gamma_2-\gamma_1)^2}{p}\right)\right. \\ &\left. - 2uz\left(\frac{\gamma_2-\gamma}{p_1}+\frac{\gamma_2-\gamma_1}{p}\right) + u^2\left(\frac{1}{p}+\frac{1}{p_1}\right)\right\} \end{aligned}$$

One simplifies the calculation by putting
(32)

$$(\gamma-\gamma_1)^2 = \mathrm{A}_1^2, \qquad \frac{(\gamma_1-\gamma_2)^2}{p}+\frac{(\gamma_2-\gamma)^2}{p_1}+\frac{(\gamma-\gamma_1)^2}{p_2} = \mathrm{A}_2^2$$

$$u\left(\frac{\gamma_2-\gamma_1}{p_1}+\frac{\gamma_2-\gamma_1}{p}\right) = \mathrm{B}_2 \qquad u\left(\frac{1}{p}+\frac{1}{p_1}\right) = \mathrm{C}_2$$

and the integral (30) is reduced to

$$(33) \qquad \frac{p^p p_1^{p_1} p_2^{p_2}}{n^n} \int_{b'}^{b} dudz_2 e^{\mathrm{S}' - \frac{n^2}{2\,\mathrm{A}_1^2}(\mathrm{A}_2^2 z_2^2 - 2\,\mathrm{B}_2 z_2 + \mathrm{C}_2)}$$

S$'$ representing the series of terms in z^3, u^3, etc., that it would be less useful to develop.

But it is important to not forget that each of these terms is multiplied by a factor such as $\frac{n^3}{p^2}$, $\frac{n^4}{p^3}$, etc., according to the degree of the variables that it will be able to contain, and that thus all these terms are of order n.

Under this form (33), one perceives already that the integral taken with respect to z_2, will be known if the limits of z are rather great.

In completing the square in the exponent of e, and making

$$\text{(34)} \qquad y = \frac{n}{\mathrm{A}_1\sqrt{2}}\left(\mathrm{A}_2 z_2 - \frac{\mathrm{B}_2}{\mathrm{A}_2}\right), \quad z_2 = \frac{\mathrm{A}_1\sqrt{2}}{n\,\mathrm{A}_2}y + \frac{\mathrm{B}_2}{\mathrm{A}_2^2},$$

$$dz_2 = \frac{\mathrm{A}_1\sqrt{2}}{\mathrm{A}_2 n}dy;$$

one draws from it

$$\frac{n^2}{2\,\mathrm{A}_1^2}(\mathrm{A}_2^2 z_2^2 - 2\,\mathrm{B}_2 z_2 + \mathrm{C}_2) = y^2 + \frac{n^2}{2\,\mathrm{A}_1^2}\left(\mathrm{C}_2 - \frac{\mathrm{B}_2^2}{\mathrm{A}_2^2}\right),$$

substituting these values, one will obtain

$$\text{(35)} \qquad \frac{p^p p_1^{p_1} p_2^{p_2}}{n^n}\frac{\mathrm{A}_1\sqrt{2}}{n\mathrm{A}_2}\int_{b'}^{b} du dz_2 e^{-y^2 - \frac{n^2}{2\,\mathrm{A}_1^2}\left(\mathrm{C}_2 - \frac{\mathrm{B}_2^2}{\mathrm{A}_2^2}\right) + \mathrm{S}''},$$

S'' designating the expression of S' in y.

If one examines at present the quantity

$$\frac{1}{\mathrm{A}_1^2}\left(\mathrm{C}_2 - \frac{\mathrm{B}_2^2}{\mathrm{A}_2^2}\right) = \frac{1}{\mathrm{A}_2^2}\left(\frac{\mathrm{C}_2\,\mathrm{A}_2^2 - \mathrm{B}_2^2}{\mathrm{A}_1^2}\right)$$

which is a function of u only, the relations (32) will give

$$\mathrm{A}_2^2 = \frac{1}{p_2}\,\mathrm{A}_1^2 + \frac{(\gamma_2-\gamma)^2}{p_1} + \frac{(\gamma_2-\gamma_1)^2}{p}, \qquad \mathrm{C}_2 = u^2\frac{n-p_2}{pp_1},$$

hence

$$\mathrm{C}_2\,\mathrm{A}_2^2 = u^2\frac{n}{pp_1p_2}\,\mathrm{A}_1^2 - u^2\left\{\frac{\mathrm{A}_1^2}{pp_1} - \frac{n-p_2}{pp_1}\left(\frac{(\gamma_2-\gamma_1)^2}{p} + \frac{(\gamma_2-\gamma)^2}{p_1}\right)\right\},$$

$$\mathrm{B}_2^2 = n^2\left\{\frac{(\gamma_2-\gamma)^2}{p_1^2} + \frac{(\gamma_2-\gamma_1)^2}{p^2} + 2\frac{(\gamma_2-\gamma)(\gamma_2-\gamma_1)}{pp_1}\right\}.$$

But the double product

$$\begin{aligned}2(\gamma_2-\gamma_1)(\gamma_2-\gamma) &= (\gamma_2-\gamma)^2 + (\gamma_2-\gamma_1)^2 - ((\gamma_2-\gamma)-(\gamma_2-\gamma_1))^2 \\ &= (\gamma_2-\gamma)^2 + (\gamma_2-\gamma_1)^2 - (\gamma_1-\gamma)^2.\end{aligned}$$

One has thus

$$\begin{aligned}\mathrm{B}_2^2 &= u^2\left\{\frac{(\gamma_2-\gamma)^2}{p_1^2} + \frac{(\gamma_2-\gamma_1)^2}{p^2} + \frac{(\gamma_2-\gamma)^2}{p_1p} + \frac{(\gamma_2-\gamma_1)^2}{pp_1} - \frac{(\gamma_1-\gamma)^2}{pp_1}\right\} \\ &= \frac{u^2}{pp_1}\left\{\frac{(\gamma_2-\gamma)^2(p+p_1)}{p_1} + \frac{(\gamma_2-\gamma_1)^2(p+p_1)}{p} - (\gamma_1-\gamma)^2\right\}\end{aligned}$$

Observing that the positive terms are multiplied by $p + p_1 = n - p_2$, and that the negative term is A_1^2,

$$\frac{\mathrm{B}_2^2}{u^2} = \frac{(n-p_2)}{pp_1}\left(\frac{(\gamma_2-\gamma)^2}{p_1} + \frac{(\gamma_2-\gamma_1)^2}{p}\right) - \frac{\mathrm{A}_1^2}{pp_1}.$$

There results from these modifications, of which the ensemble will be reproduced later, that

$$\mathrm{C}_2\,\mathrm{A}_2^2 - \mathrm{B}_2^2 = u^2\frac{n\,\mathrm{A}_1^2}{pp_1p_2}, \qquad \frac{n^2}{2\,\mathrm{A}_1^2}\left(\mathrm{C}_2 - \frac{\mathrm{B}_1^2}{\mathrm{A}_2^2}\right) = \frac{n^3u^2}{2p_1pp_2\,\mathrm{A}_2^2}$$

and

$$\begin{aligned}&\frac{p^p p_1^{p_1} p_2^{p_2}}{n^n}\frac{\mathrm{A}_1\sqrt{2}}{\mathrm{A}_2 n}\,{}_2\int_{b'}^{b} dudy\, e^{-y^2-\frac{n^2}{2\,\mathrm{A}_1^2}\left(\mathrm{C}_2-\frac{\mathrm{B}_1^2}{\mathrm{A}_2^2}\right)+\mathrm{S}''}\\ &= \frac{p^p p_1^{p_1} p_2^{p_2}}{n^n}\frac{\mathrm{A}_1\sqrt{2}}{\mathrm{A}_2 n}\,{}_2\int_{b'}^{b} dudy\, e^{-y^2-\frac{n^3u^2}{2pp_1p_2\mathrm{A}_1^2}+\mathrm{S}''};\end{aligned}$$

Putting finally

(36) $$\frac{n^3u^2}{2pp_1p_2\,\mathrm{A}_1^2} = t^2, \qquad u = t\sqrt{\frac{2pp_1p_2\,\mathrm{A}_1^2}{n^3}}$$

$$du = dt\sqrt{\frac{2pp_1p_2\,\mathrm{A}_1^2}{n^3}}$$

$$\mathrm{S}'' = \mathrm{S}''' \qquad \text{Limits of } t = c' = c;$$

it remains to integrate

(37) $$\frac{p^p p_1^{p_1} p_2^{p_2}}{n^n}\frac{2\mathrm{A}_1\sqrt{pp_1p_2}}{n^2\sqrt{n}}\,{}_2\int_{c'}^{c} dtdy\, e^{-y^2-t^2+\mathrm{S}'''}.$$

The terms of the series S''' are able to be easily appreciated. It has been established that in the series S', according to the powers of z_2 and u, the coefficient of each term was originally of the order n. When z_2 has been changed (34) into $\left(y\frac{\mathrm{A}_1\sqrt{2}}{n\,\mathrm{A}_2} - \frac{\mathrm{B}_2}{\mathrm{A}_2^2}\right)$ and u into $t\sqrt{\frac{2pp_1p_2\,\mathrm{A}_1^2}{n^3}}$, it is clear, according to the values (32), of $\mathrm{A}_1,\ \mathrm{A}_2,\ \mathrm{B}_2$, that the coefficients of y and of t, in the transformed series S''', have acquired the divisor $\sqrt{n}$, raised to some powers equal to the sum of the exponents of the variables diminished by two units. The term in z_2^3 which was multiplied by $\frac{n^3}{p_2^2}$ has given

$$\frac{n^3}{p_2^2}\,y^3\,\frac{\mathrm{A}_1^3(\sqrt{2})^3}{n^3\,\mathrm{A}_2^3} + \text{ etc.} = y^3\,\frac{1}{p_2^2}\,\frac{\mathrm{A}_1^3(\sqrt{2})^3}{\mathrm{A}_2^3} + \text{ etc.},$$

of which the coefficient is manifestly of the order $\frac{1}{\sqrt{p_2}}$, or in general of the order $\frac{1}{\sqrt{n}}$.

It is likewise of it for each other power: if it is only the divisor $\sqrt{n}$ received some exponents more and more great.

It is necessary therefore already to regard as very small all the coefficients of the series S''' when n is a large number. To this consideration comes to be added another which goes to permit to neglect this series nearly in totality, without sensible error.

One is able effectively to make it exit from the exponent, since always

$$e^{\,\mathrm{S}'''} = 1 + \mathrm{S}''' + \frac{1}{2}\,\mathrm{S}'''^2 + \text{ etc.},$$

and the integral becomes

$$\text{(38)} \qquad \frac{p^p p_1^{p_1} p_2^{p_2}}{n^n}\frac{2\mathrm{A}_1}{n^2}\sqrt{\frac{pp_1p_2}{n}}\,{}_2\int_{c'}^{c} dtdy\, e^{-y^2-t^2}(1 + \mathrm{S}''' + \text{ etc.})$$

But then each term such that

$$\int_{c'}^{c} e^{-t^2}t^s dt\int_0^{\alpha} e^{-y^2}dyy^{2m} =$$
$$= \int_{c'}^{c} e^{-t^2}t^s dt\left\{-\frac{e^{-\alpha^2}}{2}\left(\alpha^{2m-1}+\frac{2m-1}{2}\alpha^{2m-3}+\cdots+\frac{2m-1}{2}\frac{2m-3}{2}\cdots\frac{3}{2}\alpha\right)\right.$$
$$\left.+\ \frac{2m-1}{2}\cdots\frac{1}{2}\int_0^{\alpha} e^{-y^2}dy\right\},$$

or

$$\int_{c'}^{c} e^{-t^2}t^s dt\int_0^{\alpha} e^{-y^2}dyy^{2m+1} =$$
$$= \int_{c'}^{c} e^{-t^2}t^s dt\left\{-\frac{e^{-\alpha^2}}{2}\left(\alpha^{2m}+\frac{2m}{2}\alpha^{2m-2}+\cdots+\frac{2m}{2}\frac{2m-2}{2}\cdots\frac{2}{2}\alpha\right)\right\},$$

and one is able to see previously what was the excessive smallness of the series multiplied by $-\frac{e^{-\alpha^2}}{2}$ as soon as α surpasses 4 or 5. The single term $\int_0^{\alpha} e^{-y^2}dy$ introduces some sensible values when the limits of y are of contrary signs.

It is therefore palpable that if these limits, for any value of t are considerable, the resulting terms of integration of y will become completely insensible even for the high powers of y; because they have, in general, only very small coefficients. There is exception only for the even powers of which the integration leads to

$$\frac{2m-1}{2}\cdot\frac{2m-3}{2}\cdots\frac{3}{2}\cdot\frac{1}{2}\int_{\alpha'}^{\alpha} e^{-y^2}dy = \frac{2m-1}{2}\cdots\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi},$$

when the limits α' and α are of contrary signs and always considerable.

But the relation (34), where $y = \frac{n}{2\,\mathrm{A}_1}\left(z\,\mathrm{A}_2 - \frac{\mathrm{B}_2}{\mathrm{A}_2}\right)$, shows that y is constantly of the very elevated order $\sqrt{n}$, whatever be u or t. There will be therefore to remain in the series S''' only the terms in which y has some even exponents and some limits of different signs.

It is not unuseful to remark, besides, that the odd powers of y or of t disappear by themselves when the double integral is taken between two equal values and of contrary signs of y and of t. Now it is a property of the function designated by y, to have two very elevated limits, most often of contrary signs for a value of t, and changing in sign with t. So that under the hypothesis where $c' = -c$, there remains of the series only the terms in which y and t have at the same time even exponents. Thus no term of the third power subsists, and the terms of the fourth have some coefficients of the very small order $\frac{1}{n}$.

The series S‴ is found therefore reduced by integration of y to two parts: the one, of which all that which precedes explicates the excessive convergence for a great number of terms, is able to be regarded as null from the origin, whatever be t. The other part is composed first of powers of t which have, for themselves, only some very small coefficients of order of $\frac{1}{n}$ and its powers, but which are multiplied by some factors of the form

$$\frac{2m-1}{2}\cdot\frac{2m-3}{2}\cdots\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi},$$

and are able for this reason to acquire some value.

Besides, this second part contains all the even powers of t, which were isolated from y, and of which the coefficients are equally very small.

It conserves therefore a rapid convergence in the first terms, save some particular cases.

When the integration relative to t is effected, this part subsisting alone is divided further into two series of which the first is affected of the factor $\frac{e^{-c^2}}{2}$, which takes some rather feeble values for some small values of t, and increases thus the convergence, to the point even to render insensible the first term if n is a very great number.

It will suffice therefore to consider the second series, which contains only some constant factors, multiplied by $\int_{c'}^{c} e^{-t^2}dt$. In representing by T the sum of these factors, the integral (38) becomes definitely after the integration relative to y,

$$\frac{p^p p_1^{p_1} p_2^{p_2}}{n^n}\frac{2\mathrm{A}_1}{n^2}\sqrt{\frac{pp_1p_2}{n^3}}\int_{c'}^{c} e^{-t^2}dt(\sqrt{\pi}+\mathrm{T}). \tag{39}$$

This expression gives immediately the numerator of the sought probability. The denominator requires the knowledge of the extreme limits of t.

The relations (27) show that v being contained between γ and γ_2, u will be between

$$\begin{aligned}\gamma - \frac{p\gamma + p_1\gamma_1 + p_2\gamma_2}{n} &= -\frac{p_1(\gamma_1-\gamma)+p_2(\gamma_2-\gamma)}{n}\\ \gamma_2 - \frac{p\gamma + p_1\gamma_1 + p_2\gamma_2}{n} &= \frac{p(\gamma_2-\gamma)+p_1(\gamma_2-\gamma)}{n}\end{aligned} \tag{40}$$

The extreme limits of t are deduced from it, and are

$$l' = -\frac{p_1(\gamma_1-\gamma)+p_2(\gamma_2-\gamma)}{\sqrt{2pp_1p_2\mathrm{A}_2^2}}\sqrt{n},\ l = -\frac{p(\gamma_2-\gamma)+p_1(\gamma_2-\gamma)}{\sqrt{2pp_1p_2\,\mathrm{A}_2^2}}\sqrt{n} \tag{41}$$

quantities of the very great order $\sqrt{n}$. Thus the integral relative to all the possible values of v, fulfill to a great reason all the conditions necessary in order to neglect without error the series which vanish in the preceding integral. One will conserve only the constant terms multiplied by $\int_{l'}^{l} e^{-t^2} dt$; and as one is able to extend to infinity the limits of this last integral, it is evident from it for the denominator

$$\frac{p^p p_1^{p_1} p_2^{p_2}}{n^n} \frac{2\,\mathrm{A}_1}{n^2} \sqrt{\frac{pp_1p_2}{n^3}} (\sqrt{\pi} + \mathrm{T})\sqrt{\pi}$$

and for the probability (25)

$$\frac{2}{\sqrt{\pi}} \int_0^c e^{-t^2} dt \tag{42}$$

The values of v, of which this definite integral represents the favorable chances, are contained between

$$v = \frac{p\gamma + p_1\gamma_1 + p_2\gamma_2}{n} \pm c\sqrt{\frac{pp_1p_2\,\mathrm{A}_2^2}{n^3}}$$

or because of the value of A_2^2

$$v = \frac{p\gamma + p_1\gamma_1 + p_2\gamma_2}{n} \pm c\sqrt{2\frac{pp_1(\gamma-\gamma_1)^2 + p_1p_2(\gamma_1-\gamma_2)^2 + p_2p(\gamma_2-\gamma)^2}{n^3}}$$

and the radical

$$\pm c\sqrt{2\frac{pp_1(\gamma-\gamma_1)^2 + p_1p_2(\gamma_1-\gamma_2)^2 + p_2p(\gamma_2-\gamma)^2}{n^3}}$$

expresses the deviations of the real value to the observed value.

The constant, which determines the extent of these deviations, has conserved the symmetric form that it had taken for two events.

Nothing is more easy than to restore it to the form of Laplace. Multiplying by n the terms under the radical, one obtains for the numerator

$$npp_1(\gamma-\gamma_1)^2 + np_1p_2(\gamma_1-\gamma_2)^2 + np_2p(\gamma_2-\gamma)^2,$$

an expression which is transformed into

$$\left\{\begin{array}{c} p^2p_1(\gamma-\gamma_1)^2 + pp_1^2(\gamma-\gamma_1)^2 + pp_1p_2(\gamma-\gamma_1)^2 \\ p_1^2p_2(\gamma_1-\gamma_2)^2 + p_1p_2^2(\gamma_1-\gamma_2)^2 + pp_1p_2(\gamma_1-\gamma_2)^2 \\ p^2p_2(\gamma_2-\gamma)^2 + pp_2^2(\gamma_2-\gamma)^2 + pp_1p_2(\gamma_2-\gamma)^2 \end{array}\right\}$$

$$= \left\{\begin{array}{l} p[p_1(\gamma-\gamma_1) + p_2(\gamma-\gamma_2)]^2 - 2pp_1p_2(\gamma-\gamma_1)(\gamma-\gamma_2) + pp_1p_2(\gamma-\gamma_1)^2 \\ p_1[p(\gamma_1-\gamma) + p_2(\gamma_1-\gamma_2)]^2 - 2pp_1p_2(\gamma_1-\gamma)(\gamma_1-\gamma_2) + pp_1p_2(\gamma_1-\gamma_2)^2 \\ p_2[p(\gamma_2-\gamma) + p_1(\gamma_2-\gamma_1)]^2 - 2pp_1p_2(\gamma_2-\gamma)(\gamma_2-\gamma_1) + pp_1p_2(\gamma_2-\gamma)^2 \end{array}\right\}$$

But the sum of the last six terms is null, because it is the square of the identically null expression

$$\sqrt{pp_1p_2}\,\{(\gamma-\gamma_1) + (\gamma_1-\gamma_2) + (\gamma_2-\gamma)\}\,.$$

Besides each of the first six are able to take the form

$$p[p_1(\gamma-\gamma_1)+p_2(\gamma-\gamma_2)]^2 = p[n\gamma-(p\gamma+p_1\gamma_1+p_2\gamma_2)]^2 = n^2p(\gamma-\mu)^2$$

in representing by μ the mean $\frac{p\gamma+p_1\gamma_1+p_2\gamma_2}{n}$.

One rediscovers thus, for the case of three events, the constant of Laplace; since the substitution gives

$$\begin{aligned}&\sqrt{2\frac{pp_1(\gamma-\gamma_1)^2+p_1p_2(\gamma_1-\gamma_2)^2+p_2p(\gamma_2-\gamma)^2}{}}\\ &=\frac{1}{n}\sqrt{2\left\{p(\gamma-\mu)^2+p_1(\gamma_1-\mu)^2+p_2(\gamma_2-\mu)^2\right\}}\end{aligned}$$

This constant is therefore also the coefficient of the first term of the exponent of e in the integral which furnishes the probability. The approximation carries only on the coefficients of the terms following, which become very small when n is a large number and are able to be neglected without appreciable error. But it could not have any uncertainty on the value and the form of the coefficient of the first, of which no part has been neglected in the course of the preceding analysis.

One senses at present that this analysis, although restricted in appearance in the case of three simple events, will be applied without difficulty in the general case of any number of distinct events.

Let in fact x, x_1, $x_2, \dots x_m$ be the unknown possibilities of m simple events; p,p_1, p_2,... p_m the number of repetitions of each in a number n of trials; γ, γ_1, γ_2, $\dots\gamma_m$ of the arbitrary functions relative to the nature of each event, and ranked by order of magnitude. The concern is to determine the probability P that the real value of

$$v=\gamma x+\gamma_1x_1+\gamma_2x_2+\cdots+\gamma_mx_m \tag{45}$$

is contained between the given limits a' and a. One will have

$$\begin{aligned}p+p_1+p_2\cdots+p_m&=n\\ x+x_1+x_2\cdots+x_m&=1.\end{aligned} \tag{46}$$

It would be superfluous to show that the probability P will be expressed by

$$\frac{{}_m\int_{a'}^{a}x^px_1^{p_1}x_2^{p_2}\cdots x_m^{p_m}dvdx_2dx_3\cdots dx_m}{{}_m\int_{\gamma'}^{\gamma_m}x^px_1^{p_1}x_2^{p_2}\cdots x_m^{p_m}dvdx_2dx_3\cdots dx_m} \tag{47}$$

the index m denoting the number of integrations to effect. The reasoning employed in order to arrive to the analogous expression (22), in the case of three events, is reproduced with a perfect similitude for any number.

One will make $x=\frac{p}{n}+z$, $x_1=\frac{p_1}{n}+z_1$, $x_2=\frac{p_2}{n}+z_2,\dots x_m=\frac{p_m}{n}+z_m$.

$$v=\frac{p\gamma+p_1\gamma_1+\cdots+p_m\gamma_m}{n}+u \tag{48}$$

and one will have

$$z+z_1+z_2+\cdots+z_m=0 \quad \gamma z+\gamma_1z_1+\cdots+\gamma_mz_m=u; \tag{49}$$

whence one deduces

$$
(50)\qquad
\begin{aligned}
z &= \frac{u - z_2(\gamma_2 - \gamma_1) - z_3(\gamma_3 - \gamma_1) - \cdots - z_m(\gamma_m - \gamma_1)}{\gamma - \gamma_1}\\
z_1 &= \frac{u - z_2(\gamma_2 - \gamma) - z_3(\gamma_3 - \gamma) - \cdots - z_m(\gamma_m - \gamma)}{\gamma_1 - \gamma}
\end{aligned}
$$

or, by putting $u - z_3\gamma_3 - z_4\gamma_4 - \cdots - z_m\gamma_m = U_2$
$z_3 + z_4 + \cdots + z_m = Z_2$

$$
(51)\qquad z = \frac{U_2 + \gamma_1 Z_2 - z_2(\gamma_2 - \gamma_1)}{\gamma - \gamma_1} \quad z_1 = \frac{U_2 + \gamma Z_2 - z_2(\gamma_2 - \gamma)}{\gamma_1 - \gamma}
$$

Substituting the values (48) into the integral of the numerator of the probability P (47), and suppressing beyond the $\int$ sign the constant factors

$$
\frac{p^p p_1^{p_1} p_2^{p_2} \cdots p_m^{p_m}}{n^n}
$$

which would be destroyed in the two terms of the probability; it will become

$$
{}_m\int_{b'}^{b} \left(1 + \frac{nz}{p}\right)^p \left(1 + \frac{nz_1}{p_1}\right)^{p_1} \cdots \left(1 + \frac{nz_m}{p_m}\right)^{p_m} dz_2 dz_3 dz_4 \ldots dz_m du
$$

b' and b being the limits of u.

Developing into exponentials, the number e will have for exponent

$$
\begin{aligned}
+n(z + z_1 + z_2 + \cdots + z_m) - \frac{n^2}{2}\left(\frac{z^2}{p} + \frac{z_1^2}{p_1} + \frac{z_2^2}{p_2} + \cdots + \frac{z_m^2}{p_m}\right)&\\
+ \frac{n^3}{3}\left(\frac{z^3}{p^2} + \frac{z_1^3}{p_1^2} + \frac{z_2^3}{p_2^2} + \cdots + \frac{z_m^3}{p_{m^2}}\right) - \text{ etc.}&
\end{aligned}
$$

In this expression, the term in n is clearly null by the first of the relations (49): and if one represents by S the series of powers superior to the second, one will be able to write the integral

$$
(52)\qquad {}_m\int_{b'}^{b} dz_2 dz_3 dz_4 \ldots dz_m du e^{-\frac{n^2}{2}\left(\frac{z^2}{p} + \frac{z_1^2}{p_1} + \frac{z_2^2}{p_2} + \cdots + \frac{z_m^2}{p_m}\right) + \mathrm{S}},
$$

Substituting for z and z_1 their values (50), and designating by K_2 the sum of the terms $-\frac{n^2}{2}\left(\frac{z_3^2}{p_3} + \frac{z_4^2}{p_4} + \cdots + \frac{z_m^2}{p_m}\right)$, which do not contain z_2, one will have to integrate

(53)

$$
{}_m\int_{b'}^{b} dz_2 dz_3 dz_4 \ldots dz_m du e^{\mathrm{K}_2 + \mathrm{S}_1 - \frac{n^2}{2}\left\{\begin{array}{l} \left(\dfrac{\mathrm{U}_2 + \gamma_1 \mathrm{Z}_2 - z_2(\gamma_2 - \gamma_1)}{\gamma - \gamma_1}\right)^2 \dfrac{1}{p} + \\ \left(\dfrac{\mathrm{U}_2 + \gamma_1 \mathrm{Z}_2 - z_2(\gamma_2 - \gamma)}{\gamma_1 - \gamma}\right)^2 \dfrac{1}{p_1} + \dfrac{z_2^2}{p_2} \end{array}\right\}}
$$

It is necessary to order, with respect to z_2, the part of the exponent of e contained between the parentheses; this which gives first:

$$(54)\qquad \begin{aligned} &-\frac{n^2 z_2^2}{2(\gamma-\gamma_1)^2}\left\{\frac{(\gamma_1-\gamma)^2}{p_2}+\frac{(\gamma-\gamma_2)^2}{p_1}+\frac{(\gamma_2-\gamma_1)^2}{p}\right\} \\ &+\frac{2n^2 z_2}{2(\gamma-\gamma_1)^2}\left\{\frac{\gamma_2-\gamma_1}{p}(\mathrm{U}_2+\gamma_1\,\mathrm{Z}_2)+\frac{\gamma_2-\gamma}{p_2}(\mathrm{U}_2+\gamma\,\mathrm{Z}_2)\right\} \\ &-\frac{n^2}{2(\gamma-\gamma_1)^2}\left\{\frac{(\mathrm{U}_2+\gamma_1\,\mathrm{Z}_2)^2}{p}+\frac{(\mathrm{U}_2+\gamma\,\mathrm{Z}_2)^2}{p_1}\right\} \end{aligned}$$

In order to simplify, one will put

$$\begin{gathered} (\gamma-\gamma_1)^2=\mathrm{A}_1^2,\qquad \frac{(\gamma-\gamma_1)^2}{p_2}+\frac{(\gamma_1-\gamma_2)^2}{p}+\frac{(\gamma_2-\gamma)^2}{p_1}=\mathrm{A}_2^2, \\ \frac{\gamma_2-\gamma_1}{p}(\mathrm{U}_2+\gamma_1\,\mathrm{Z}_2)+\frac{\gamma_2-\gamma}{p_2}(\mathrm{U}_2+\gamma\,\mathrm{Z}_2)=\mathrm{B}_2, \\ \frac{(\mathrm{U}_2+\gamma_1\,\mathrm{Z}_2)^2}{p}+\frac{(\mathrm{U}_2+\gamma\,\mathrm{Z}_2)^2}{p_1}=\mathrm{C}_2, \end{gathered}$$

and the quantity (54) will be reduced to

$$(55)\qquad -\frac{n^2}{2\,\mathrm{A}_1^2}\left(\mathrm{A}_2^2 z_2^2-2\,\mathrm{B}_2 z_2+\mathrm{C}_2\right),$$

which becomes

$$-y_2^2-\frac{n^2}{2\,\mathrm{A}_1^2}\left(\mathrm{C}_2-\frac{\mathrm{B}_2^2}{\mathrm{A}_2^2}\right)$$

if one makes likewise as previously.

$$(56)\qquad \frac{n}{\mathrm{A}_1\sqrt{2}}\left(z_2\,\mathrm{A}_2-\frac{\mathrm{B}_2}{\mathrm{A}_2}\right)=y_2,\quad dz_2=dy_2\frac{\mathrm{A}_1\sqrt{2}}{n\,\mathrm{A}_2}.$$

It is good to ascertain immediately that y_2 is of the order $\sqrt{n}$ for all values of u, z_3, etc., which enter into B_2. So that to the limits of z_2, one will be able, as it has already been done, to draw part of the considerable magnitude of the limits of y_2.

By suppressing in advance the constant factor $\frac{\mathrm{A}_1\sqrt{2}}{n\,\mathrm{A}_2}$, which would be introduced before the $\int$ sign; calling S_2 that which the series S becomes, when one has eliminated z and z_1, and when one has replaced z_2 by its value in y_2; the integral is changed into

$$(57)\qquad {}_m\int_{b'}^{b} dz_2 dz_3 dz_4\dots dz_m du\, e^{+\mathrm{K}_2+\mathrm{S}_2-y_2-\frac{n^2}{2\,\mathrm{A}_1^2}\left(\mathrm{C}_2-\frac{\mathrm{B}_2^2}{\mathrm{A}_2^2}\right)},$$

an expression in which it is necessary to transform the following variable, z_3.

The value of K_2 permits to put

$$\mathrm{K}_2=-\frac{n^2}{2}\left(\frac{z_3^2}{p_3}+\frac{z_4^2}{p_4}+\dots+\frac{z_m^2}{p_m}\right)=\mathrm{K}_3-\frac{n^2 z_3^2}{2p_3},$$

and one will be able to write the integral (57) as it follows:

$$(58) \qquad {}_m\int_{b'}^{b} dz_2dz_3dz_4\ldots dz_m du\, e^{+\,\mathrm{K}_3+\mathrm{S}_2-y_2^2-\frac{n^2}{2}\left(\frac{z_3^2}{p_3}-\frac{\mathrm{C}_2\,\mathrm{A}_2-\,\mathrm{B}_2^2}{\mathrm{A}_1^2\,\mathrm{A}_2^2}\right)},$$

Now one will develop $\frac{\mathrm{C}_2\,\mathrm{A}_2-\,\mathrm{B}_2^2}{\mathrm{A}_1^2\,\mathrm{A}_2^2}$, or rather the numerator alone, the variables not entering in the denominator. The relations (51) will give by calling U_3 and Z_3 some sums similar to U_2 and Z_2, but which will contain z_3 no longer:

$$\mathrm{U}_2 = -\gamma_3 z_3 + \mathrm{U}_3, \qquad \mathrm{Z}_2 = z_3 + \mathrm{Z}_3,$$

whence

$$\begin{aligned}
(\mathrm{U}_2+\gamma_1\,\mathrm{Z}_2) &= \mathrm{U}_3+\gamma_1\,\mathrm{Z}_3 - z_3(\gamma_3-\gamma_1),\\
(\mathrm{U}_2+\gamma\,\mathrm{Z}_2) &= \mathrm{U}_3+\gamma_1\,\mathrm{Z}_3 - z_3(\gamma_3-\gamma),\\
(\mathrm{U}_2+\gamma_1\,\mathrm{Z}_2)^2 &= (\mathrm{U}_3+\gamma_1\,\mathrm{Z}_3)^2 - 2z_3(\gamma_3-\gamma_1)(\mathrm{U}_3+\gamma_1\,\mathrm{Z}_3) + z_3^2(\gamma_3-\gamma_1)^2\\
(\mathrm{U}_2+\gamma\,\mathrm{Z}_2)^2 &= (\mathrm{U}_3+\gamma\,\mathrm{Z}_3)^2 - 2z_3(\gamma_3-\gamma)(\mathrm{U}_3+\gamma\,\mathrm{Z}_3) + z_3^2(\gamma_3-\gamma)^2
\end{aligned}$$

There results from it

$$\begin{aligned}
\mathrm{C}_2 &= \frac{[\,\mathrm{U}_3+\gamma_1\,\mathrm{Z}_3 - z_3(\gamma_3-\gamma_1)]^2}{p} + \frac{[\,\mathrm{U}_3+\gamma\,\mathrm{Z}_3 - z_3(\gamma_3-\gamma)]^2}{p_1},\\
\mathrm{B}_2 &= \frac{\gamma_2-\gamma_1}{p}[\,\mathrm{U}_3+\gamma_1\,\mathrm{Z}_3 - z_3(\gamma_3-\gamma_1)] + \frac{\gamma_2-\gamma}{p_1}[\,\mathrm{U}_3+\gamma_1\,\mathrm{Z}_3 - z_3(\gamma_3-\gamma)],
\end{aligned}$$

and by developing the squares

$$\begin{aligned}
\mathrm{C}_2 =& \frac{(\mathrm{U}_3+\gamma_1\,\mathrm{Z}_3)^2}{p} + \frac{(\mathrm{U}_3+\gamma\,\mathrm{Z}_3)^2}{p_1}\\
&- 2z_1\left(\frac{\gamma_2-\gamma_1}{p}(\mathrm{U}_3+\gamma_1\,\mathrm{Z}_3) + \frac{\gamma_2-\gamma}{p_1}(\mathrm{U}_3+\gamma\,\mathrm{Z}_3) + z_3^2\left(\frac{(\gamma_2-\gamma_1)^2}{p}+\frac{(\gamma_2-\gamma)^2}{p_1}\right)\right)\\
\mathrm{B}_2^2 =& \frac{(\gamma_2-\gamma_1)^2(\mathrm{U}_3+\gamma_1\,\mathrm{Z}_3)^2}{p^2} + \frac{(\gamma_2-\gamma)^2(\mathrm{U}_3+\gamma\,\mathrm{Z}_3)^2}{p_1^2}\\
&+ 2\frac{(\gamma_2-\gamma_1)(\gamma_2-\gamma)(\mathrm{U}_3+\gamma_1\,\mathrm{Z}_3)(\mathrm{U}_3+\gamma\,\mathrm{Z}_3)}{pp_1}\\
&- 2z_3\left\{\begin{matrix}\dfrac{(\gamma_2-\gamma_1)(\gamma_3-\gamma_1)}{p}\\ +\,\dfrac{(\gamma_2-\gamma_1)(\gamma_3-\gamma)}{p_1}\end{matrix}\right\}\times\left\{\begin{matrix}\dfrac{\gamma_2-\gamma_1}{p}(\mathrm{U}_3+\gamma_1\,\mathrm{Z}_3)\\ \dfrac{\gamma_2-\gamma}{p_1}(\mathrm{U}_3+\gamma\,\mathrm{Z}_3)\end{matrix}\right\}\\
&+ z_3^2\left\{\frac{(\gamma_2-\gamma_1)(\gamma_2-\gamma)}{p} + \frac{(\gamma_2-\gamma)(\gamma_3-\gamma)}{p_1}\right\}^2,
\end{aligned}$$

If one is reminded at present that

$$\mathrm{A}_2^2 = \frac{(\gamma-\gamma_1)^2}{p_2} + \frac{(\gamma_2-\gamma)^2}{p_1} + \frac{(\gamma_1-\gamma_2)^2}{p},$$

one will have

$$
\begin{aligned}
\mathrm{C}_2\,\mathrm{A}_2^2 - \mathrm{B}_2^2 = {} & \frac{(\mathrm{U}_3 + \gamma_1\,\mathrm{Z}_3)^2}{p}\left\{\frac{(\gamma-\gamma_1)^2}{p_2} + \frac{(\gamma-\gamma_2)^2}{p_1}\right\} + \frac{(\mathrm{U}_3 + \gamma\,\mathrm{Z}_3)^2}{p_1}\left\{\frac{(\gamma-\gamma_1)^2}{p_2} + \frac{(\gamma_1-\gamma_2)^2}{p}\right\} \\
& - 2\frac{(\gamma_2-\gamma_1)(\gamma_2-\gamma)(\mathrm{U}_3 + \gamma_1\,\mathrm{Z}_3)(\mathrm{U}_3 + \gamma_1\,\mathrm{Z}_3)}{pp_1} \\
& - 2z_3\left\{\begin{array}{l} (\mathrm{U}_3 + \gamma_1\,\mathrm{Z}_3)\left[\dfrac{\gamma_3-\gamma_1}{p}\left(\dfrac{(\gamma-\gamma_1)^2}{p_2} + \dfrac{(\gamma-\gamma_2)^2}{p_1}\right) - \dfrac{(\gamma_3-\gamma)(\gamma_2-\gamma)(\gamma_2-\gamma_1)}{pp_1}\right] \\ + (\mathrm{U}_3 + \gamma\,\mathrm{Z}_3)\left[\dfrac{\gamma_3-\gamma}{p}\left(\dfrac{(\gamma-\gamma_1)^2}{p_2} + \dfrac{(\gamma_1-\gamma_2)^2}{p}\right) - \dfrac{(\gamma_3-\gamma_1)(\gamma_2-\gamma_1)(\gamma_2-\gamma)}{pp_1}\right] \end{array}\right\} \\
& + z_3^2\left\{\begin{array}{l} \left(\dfrac{(\gamma_3-\gamma_1)^2}{p} + \dfrac{(\gamma_3-\gamma)^2}{p_1}\right)\left(\dfrac{(\gamma-\gamma_1)^2}{p_2} + \dfrac{(\gamma-\gamma_2)^2}{p_1} + \dfrac{(\gamma_1-\gamma_2)^2}{p}\right) \\ - \dfrac{(\gamma_2-\gamma_1)^2(\gamma_3-\gamma_1)^2}{p^2} - \dfrac{(\gamma_2-\gamma)^2(\gamma_3-\gamma)^2}{p_1^2} - 2\dfrac{(\gamma_2-\gamma_1)(\gamma_2-\gamma)(\gamma_3-\gamma_1)(\gamma_3-\gamma)}{pp_1} \end{array}\right\}
\end{aligned}
$$

or else

$$
\begin{aligned}
\mathrm{C}_2\,\mathrm{A}_2^2 - \mathrm{B}_2^2 = {} & \frac{(\mathrm{U}_3 + \gamma_1\,\mathrm{Z}_3)^2(\gamma-\gamma_1)^2}{pp_2} + \frac{(\mathrm{U}_3 + \gamma\,\mathrm{Z}_3)^2(\gamma-\gamma_1)^2}{p_1p_2} \\
& + \left\{\frac{(\mathrm{U}_3 + \gamma_1\,\mathrm{Z}_3)(\gamma_2-\gamma_1) - (\mathrm{U}_3 + \gamma_1\,\mathrm{Z}_3)(\gamma_2-\gamma)}{pp_1}\right\}^2 \\
& - 2z_3\left\{\begin{array}{l} (\mathrm{U}_3 + \gamma_1\,\mathrm{Z}_3)\dfrac{(\gamma-\gamma_1)^2}{p}\left(\dfrac{\gamma_3-\gamma_1}{p_2} + \dfrac{\gamma_2-\gamma}{p_1}\cdot\dfrac{(\gamma_2-\gamma)(\gamma_3-\gamma_1) - (\gamma_3-\gamma)(\gamma_2-\gamma_1)}{(\gamma-\gamma_1)^2}\right) \\ + (\mathrm{U}_3 + \gamma\,\mathrm{Z}_3)\dfrac{(\gamma-\gamma_1)^2}{p_1}\left(\dfrac{\gamma_3-\gamma}{p_2} + \dfrac{\gamma_2-\gamma_1}{p}\cdot\dfrac{(\gamma_2-\gamma_1)(\gamma_3-\gamma) - (\gamma_2-\gamma)(\gamma_3-\gamma_1)}{(\gamma-\gamma_1)^2}\right) \end{array}\right\} \\
& + z_3^2\left\{\begin{array}{l} \dfrac{(\gamma_3-\gamma_1)^2}{p_2}\left(\dfrac{(\gamma_3-\gamma_1)^2}{p} + \dfrac{(\gamma_3-\gamma)^2}{p_1}\right) + \dfrac{(\gamma-\gamma_2)^2(\gamma_3-\gamma_1)^2}{pp_1} \\ - \dfrac{(\gamma_1-\gamma_2)^2(\gamma_3-\gamma)^2}{pp_1} - 2\dfrac{(\gamma_2-\gamma_1)(\gamma_2-\gamma)(\gamma_3-\gamma_1)(\gamma_3-\gamma)}{pp_1} \end{array}\right\}
\end{aligned}
$$

Whence

$$
\begin{aligned}
\frac{\mathrm{C}_2\,\mathrm{A}_2^2 - \mathrm{B}_2^2}{\mathrm{A}_1^2} = {} & \frac{(\mathrm{U}_3 + \gamma_1\,\mathrm{Z}_3)^2}{pp_2} + \frac{(\mathrm{U}_3 + \gamma\,\mathrm{Z}_3)^2}{p_1p_2} + \frac{(\mathrm{U}_3 + \gamma_2\,\mathrm{Z}_3)^2}{pp_1} \\
& - 2z_3\left\{\begin{array}{l} \dfrac{(\mathrm{U}_3 + \gamma_1\,\mathrm{Z}_3)(\gamma_3-\gamma_1)}{pp_2} + \dfrac{(\mathrm{U}_3 + \gamma\,\mathrm{Z}_3)(\gamma_3-\gamma)}{p_1p_2} \\ + \dfrac{(\mathrm{U}_3 + \gamma_1\,\mathrm{Z}_3)(\gamma_2-\gamma_3)(\gamma_2-\gamma)}{pp_1(\gamma-\gamma_1)} + \dfrac{(\mathrm{U}_3 + \gamma\,\mathrm{Z}_3)(\gamma_3-\gamma_1)(\gamma_3-\gamma_2)}{pp_1(\gamma-\gamma_1)} \end{array}\right\} \\
& + z_3^2\left\{\frac{(\gamma_3-\gamma_1)^2}{p_2p} + \frac{(\gamma_3-\gamma)^2}{p_1p_2} + \frac{[(\gamma-\gamma_2)(\gamma_3-\gamma_1) - (\gamma_1-\gamma_2)(\gamma_3-\gamma)]^2}{pp_1(\gamma-\gamma_1)^2}\right\}
\end{aligned}
$$

$$\frac{\mathrm{C}_2\,\mathrm{A}_2^2-\,\mathrm{B}_2^2}{\mathrm{A}_1^2}+\frac{z_3^2\,\mathrm{A}_2^2}{p_3}=\frac{(\,\mathrm{U}_3+\gamma_1\,\mathrm{Z}_3)^2}{pp_2}+\frac{(\,\mathrm{U}_3+\gamma\,\mathrm{Z}_3)^2}{p_1p_2}+\frac{(\,\mathrm{U}_3+\gamma_2\,\mathrm{Z}_3)^2}{pp_1}$$
$$-\,2z_3\left\{\frac{(\,\mathrm{U}_3+\gamma_1\,\mathrm{Z}_3)(\gamma_3-\gamma_1)}{pp_2}+\frac{(\,\mathrm{U}_3+\gamma\,\mathrm{Z}_3)(\gamma_3-\gamma)}{p_1p_2}+\frac{(\,\mathrm{U}_3+\gamma_2\,\mathrm{Z}_3)(\gamma_3-\gamma_2)}{pp_1}\right\}$$
$$+\,z_3^2\left\{\frac{(\gamma_3-\gamma_1)^2}{p_2p}+\frac{(\gamma-\gamma_2)^2}{p_1p_3}+\frac{(\gamma_1-\gamma_2)^2}{p_2p}+\frac{(\gamma_3-\gamma)^2}{p_1p_2}+\frac{(\gamma_3-\gamma_1)^2}{pp_2}+\frac{(\gamma_3-\gamma_2)^2}{pp_1}\right\}$$

If one puts at present

(59)
$$\frac{(\,\mathrm{U}_3+\gamma_1\,\mathrm{Z}_3)^2}{pp_2}+\frac{(\,\mathrm{U}_3+\gamma_2\,\mathrm{Z}_3)^2}{pp_1}+\frac{(\,\mathrm{U}_3+\gamma\,\mathrm{Z}_3)^2}{p_1p_2}=\,\mathrm{C}_3,$$
$$\frac{(\,\mathrm{U}_3+\gamma_1\,\mathrm{Z}_3)(\gamma_3-\gamma_1)}{pp_2}+\frac{(\,\mathrm{U}_3+\gamma\,\mathrm{Z}_3)(\gamma_3-\gamma)}{p_1p_2}+\frac{(\,\mathrm{U}_3+\gamma_2\,\mathrm{Z}_3)(\gamma_3-\gamma_2)}{pp_1}\;\mathrm{B}_3,$$
$$\frac{(\gamma-\gamma_1)^2}{p_3p_2}+\frac{(\gamma_1-\gamma_2)^2}{p_3p}+\frac{(\gamma_2-\gamma)^2}{p_3p_1}+\frac{(\gamma_3-\gamma)^2}{p_1p_2}+\frac{(\gamma_3-\gamma_1)^2}{pp_2}+\frac{(\gamma_3-\gamma_2)^2}{pp_1}=\,\mathrm{A}_3^2.$$

One will have

$$(60)\qquad -\frac{n^2}{2}\left(\frac{\mathrm{C}_2\,\mathrm{A}_2^2-\,\mathrm{B}_2^2}{\mathrm{A}_1^2\,\mathrm{A}_2^2}+\frac{z_3^2}{p_3}\right)=-\frac{n^2}{2\,\mathrm{A}_2^2}\left(\,\mathrm{A}_3^2z_3^2-2\,\mathrm{B}_3z_3+\,\mathrm{C}_3\right).$$

When one brings together this expression from the expression (55), one sees that the one is composed in z_3, $\mathrm{A}_2,\ \mathrm{A}_3,\ \mathrm{B}_3,\ \mathrm{C}_3$, in the same manner as the other in z_2, A_1, $\mathrm{A}_2,\ \mathrm{B}_2,\ \mathrm{C}_2$.

One is able therefore to write immediately the result of the transformation of z_3 into y_3.

$$(61)\qquad y_3=\frac{n}{\mathrm{A}_2\sqrt{2}}\left(\mathrm{A}_3z_3-\frac{\mathrm{B}_3}{\mathrm{A}_3}\right),\qquad dz_3=dy_3\frac{\mathrm{A}_2\sqrt{2}}{n\,\mathrm{A}_3},$$

in the integral (58), which, by suppressing beforehand the constant factor $\frac{\mathrm{A}_2\sqrt{2}}{n\,\mathrm{A}_3}$, and writing S_3 for the series S_2 transformed into y_3, is changed into

$$(62)\qquad {}_m\int_{b'}^{b}dy_2dy_3dz_4dz_5\ldots dz_mdu\,e^{\mathrm{K}_3+\,\mathrm{S}_3-y_2^2-y_3^2-\frac{n^2}{2\,\mathrm{A}_2^2}\left(\mathrm{C}_3-\frac{\mathrm{B}_3^2}{\mathrm{A}_3^2}\right)}$$

It is impossible to not recognize that the symmetric functions $\mathrm{A}_3,\ \mathrm{B}_3,\ \mathrm{C}_3$, (59) follow the same law as the functions $\mathrm{A}_1,\ \mathrm{A}_2,\ \mathrm{B}_2,\ \mathrm{C}_2$. It is therefore permitted to anticipate that the same transformation will be applied to the other variables z exactly in the same manner, and that one will find successively some symmetric functions of like nature. But one will be assured from it easily by the well-known method which consists in supposing that one is arrived to the form given after $(q-2)$ transformations of so many of the variables $z_2,\ z_3,\ldots\ z_{q-1}$ and to prove that under this hypothesis, the transformation of the following variable z_q, will give precisely a similar result.

One will admit that the expression

$$(63)\qquad K_{q-1}+\,\mathrm{S}_{q-1}-y_2^2-y_3^2\cdots-y_{q-1}^2-\frac{n^2}{2\,\mathrm{A}_{q-1}^2}\left(\mathrm{C}_{q-1}-\frac{\mathrm{B}_{q-1}^2}{\mathrm{A}_{q-1}^2}\right)=\,\mathrm{R}$$

is the exponent of e in the integral

$$_m\int_{b'}^{b} dy_2 dy_3 \dots dy_{q-1} dz_q dz_{q+1} \dots dz_m du\, e^{\,\mathrm{R}}$$

due to $(q-2)$ operations. It will be necessary to prove that the result of a new transformation of z_q into y_q will give for the exponent of e the similar expression

$$(64)\qquad K_q + \mathrm{S}_q - y_2^2 - y_3^2 \cdots - y_{q-1}^2 - y_q^2 - \frac{n^2}{2\,\mathrm{A}_{q-1}^2}\left(\mathrm{C}_q - \frac{\mathrm{B}_q^2}{\mathrm{A}_q^2}\right) = \mathrm{R}'$$

in the integral

$$_m\int_{b'}^{b} dy_2 dy_3 \dots dy_{q-1} dy_q dz_{q+1} \dots dz_m du\, e^{\,\mathrm{R}'}$$

the symmetric functions of like denominations conserving the values:

$$(65)\qquad \begin{aligned}
\mathrm{K}_{q-1} &= -\frac{n^2}{2}\left(\frac{z_q^2}{p_q} + \frac{z_{q+1}^2}{p_{q+1}} \cdots + \frac{z_m^2}{p_m}\right) = -\frac{n^2 z_q^2}{2p_q} + \mathrm{K}_q,\\
\mathrm{U}_{q-1} &= u - \gamma_q z_q - \gamma_{q+1} z_{q+1} \cdots - \gamma_m z_m = \mathrm{U}_q - \gamma_q z_q,\\
\mathrm{Z}_{q-1} &= z_q + z_{q+1} + \cdots + z_m = \mathrm{Z}_q + z_q,\\
\mathrm{A}_{q-1}^2 &= \frac{(\gamma-\gamma_1)^2}{p_2 p_3 \cdots p_{q-1}} + \frac{(\gamma-\gamma_2)^2}{p_1 p_3 \cdots p_{q-1}} + \cdots + \frac{(\gamma_{q-1}-\gamma_{q-3})^2}{p p_1 p_2 \cdots p_{q-3}}\\
&= \frac{1}{p_{q-1}}\,\mathrm{A}_{q-2}^2 + \frac{(\gamma_{q-1}-\gamma)^2}{p_1 p_2 \cdots p_{q-2}} + \frac{(\gamma_{q-1}-\gamma_1)^2}{p_1 p_2 \cdots p_{q-2}} + \cdots + \frac{(\gamma_{q-1}-\gamma_{q-2})^2}{p p_1 \cdots p_{q-3}}\\
\mathrm{B}_{q-1} &= \frac{(\mathrm{U}_{q-1} + \gamma\,\mathrm{Z}_{q-1})(\gamma_{q-1}-\gamma)}{p_1 p_2 \cdots p_{q-1}} + \frac{(\mathrm{U}_{q-1} + \gamma_1\,\mathrm{Z}_{q-1})(\gamma_{q-1}-\gamma_1)}{p_1 p_2 \cdots p_{q-2}}\\
&\quad + \cdots + \frac{(\mathrm{U}_{q-1} + \gamma_{q-2}\,\mathrm{Z}_{q-1})(\gamma_{q-1}-\gamma_{q-2})}{p_1 p_2 \cdots p_{q-3}},\\
\mathrm{C}_{q-1} &= \frac{(\mathrm{U}_{q-1} + \gamma\,\mathrm{Z}_{q-1})^2}{p_1 p_2 \cdots p_{q-2}} + \frac{(\mathrm{U}_{q-1} + \gamma_1\,\mathrm{Z}_{q-1})^2}{p_1 p_2 \cdots p_{q-3}} + \frac{(\mathrm{U}_{q-1} + \gamma_{q-2}\,\mathrm{Z}_{q-1})^2}{p_1 p_2 \cdots p_{q-3}}
\end{aligned}$$

In order to arrive to this demonstration, one will dispose first separately in the expression (63), the terms of the exponent of e which contain z_q and z_q^2, this which will give

$$(66)\quad K_q + \mathrm{S}_{q-1} - y_2^2 - y_3^2 \cdots - y_{q-1}^2 - \frac{n^2}{2\,\mathrm{A}_{q-1}^2}\left(\frac{z_q^2\,\mathrm{A}_{q-1}}{p_q} - \frac{\mathrm{C}_{q-1}\,\mathrm{A}_{q-1}^2 - \mathrm{B}_{q-1}^2}{\mathrm{A}_{q-2}^2}\right).$$

The question will no longer be but to develop the part multiplied by $\frac{n^2}{2\,\mathrm{A}_{q-1}^2}$ to the mean of the values of the symmetric functions (65). Here is the calculation of it.

Decomposing each of the squares contained in C_{q-1}, such that

$$(\mathrm{U}_{q-1} + \gamma\,\mathrm{Z}_{q-1})^2 = (\mathrm{U}_q + \gamma\,\mathrm{Z}_q)^2 - 2z_q(\mathrm{U}_q + \gamma\,\mathrm{Z}_q)(\gamma_q - \gamma) + z_q^2(\gamma_q - \gamma)^2$$

hence

$$\mathrm{C}_{q-1} = \frac{(\mathrm{U}_q + \gamma\,\mathrm{Z}_q)^2}{p_1p_2\cdots p_{q-2}} + \frac{(\mathrm{U}_q + \gamma_1\,\mathrm{Z}_q)^2}{p_1p_2\cdots p_{q-3}} + \frac{(\mathrm{U}_q + \gamma_{q-2}\,\mathrm{Z}_q)^2}{p_1p_2\cdots p_{q-3}}$$
$$- 2z_q\left\{\frac{(\mathrm{U}_q + \gamma\,\mathrm{Z}_q)(\gamma_q-\gamma)}{p_1p_2\cdots p_{q-2}} + \frac{(\mathrm{U}_q + \gamma_1\,\mathrm{Z}_q)(\gamma_q-\gamma_1)}{pp_2\cdots p_{q-2}} + \cdots + \frac{(\mathrm{U}_q + \gamma_{q-2}\,\mathrm{Z}_q)(\gamma_q-\gamma_{q-2})}{pp_1\cdots p_{q-3}}\right\},$$
$$+ z_q^2\left\{\frac{(\gamma_q-\gamma)^2}{p_1p_2\cdots p_{q-2}} + \frac{(\gamma_q-\gamma_1)^2}{p_1p_2\cdots p_{q-2}} + \cdots + \frac{(\gamma_q-\gamma_{q-2})^2}{pp_1\cdots p_{q-3}}\right\}^2.$$

One sees immediately that

$$\mathrm{C}_{q-1} = p_{q-1}\left(\mathrm{C}_q - \frac{(\mathrm{U}_q + \gamma_{q-1}\,\mathrm{Z}_q)^2}{p_1p_2\cdots p_{q-2}}\right) - 2z_qp_{q-1}\left(\mathrm{B}_q - \frac{(\mathrm{U}_q + \gamma_{q-1}\,\mathrm{Z}_q)(\gamma_q-\gamma_{q-1})}{pp_1\cdots p_{q-2}}\right)$$
$$+ z_q^2\left(\mathrm{A}_q^2 - \frac{\mathrm{A}_{q-1}^2}{p_q} - \frac{(\gamma_q-\gamma_{q-1})^2}{pp_1\cdots p_{q-2}}\right)p_{q-1},$$

$$\mathrm{C}_{q-1} = p_{q-1}\left(\mathrm{A}_q^2 z_q - 2\,\mathrm{B}_q z_q^2 + \mathrm{C}_q\right) - z_q^2 p_{q-1}\frac{\mathrm{A}_{q-1}^2}{p_q}$$
$$- \frac{[\mathrm{U}_q + \gamma_{q-1}\,\mathrm{Z}_q - z_q(\gamma_q-\gamma_{q-1})]^2}{pp_{1_{p_2}}\cdots p_{q-2}}p_{q-1},$$

$$\mathrm{C}_{q-1}\,\mathrm{A}_{q-1}^2 = \mathrm{A}_{q-2}^2\left(\mathrm{A}_q^2 z_q^2 - 2\,\mathrm{B}_q z_q + \mathrm{C}_q\right)$$
$$- z_q^2\frac{\mathrm{A}_{q-1}^2}{p_q}\,\mathrm{A}_{q-2}^2 - \frac{(\mathrm{U}_{q-1} + \gamma_{q-1}\,\mathrm{Z}_{q-1})^2}{pp_1\cdots p_{q-2}}\,\mathrm{A}_{q-2}^2$$
$$+ \mathrm{C}_{q-1}\left\{\frac{(\gamma_{q-1}-\gamma)^2}{p_1p_2\cdots p_{q-2}} + \frac{(\gamma_{q-1}-\gamma_1)^2}{pp_2\cdots p_{q-2}} + \cdots + \frac{(\gamma_{q-1}-\gamma_{q-2})^2}{pp_1\cdots p_{q-3}}\right\}$$

$$\frac{\mathrm{C}_{q-1}\,\mathrm{A}_{q-1}^2 - \mathrm{B}_{q-1}^2}{\mathrm{A}_{q-2}^2} + \frac{z_q^2\,\mathrm{A}_{q-1}^2}{p_q} = \mathrm{A}_q^2 z_q^2 - 2\,\mathrm{B}_q z_q + \mathrm{C}_q \tag{67}$$
$$- \left\{\begin{array}{l} \dfrac{\mathrm{B}_{q-1}^2}{\mathrm{A}_{q-1}^2} + \dfrac{(\mathrm{U}_{q-1} + \gamma_{q-1}\,\mathrm{Z}_{q-1})^2}{pp_1\cdots p_{q-2}} \\ - \dfrac{\mathrm{C}_{q-1}}{\mathrm{A}_{q-1}^2}\left(\dfrac{(\gamma_{q-1}-\gamma)^2}{p_1p_2\cdots p_{q-2}} + \dfrac{(\gamma_{q-1}-\gamma_1)^2}{pp_2\cdots p_{q-2}} + \cdots + \dfrac{(\gamma_{q-1}-\gamma_{q-2})^2}{pp_1\cdots p_{q-3}}\right)\end{array}\right\}$$

There remains to prove that the sum of the terms contained between parentheses, in this last equation, is identically null. For this, decomposing B_{q-1}^2, it becomes

$$\mathrm{B}_{q-1}^2 = \frac{(\mathrm{U}_{q-1} + \gamma\,\mathrm{Z}_{q-1})^2(\gamma_{q-1}-\gamma)^2}{(p_1p_2\cdots p_{q-2})^2} + \frac{(\mathrm{U}_{q-1} + \gamma_1\,\mathrm{Z}_{q-1})^2(\gamma_{q-1}-\gamma_1)^2}{(pp_2\cdots p_{q-2})^2}$$
$$+ \cdots + \frac{(\mathrm{U}_{q-1} + \gamma_{q-2}\,\mathrm{Z}_{q-1})^2(\gamma_{q-1}-\gamma_{q-2})^2}{(pp_1\cdots p_{q-3})^2}$$
$$+ 2\frac{(\mathrm{U}_{q-1} + \gamma\,\mathrm{Z}_{q-1})(\mathrm{U}_{q-1} + \gamma_1\,\mathrm{Z}_{q-1})(\gamma_{q-1}-\gamma)(\gamma_{q-1}-\gamma_1)}{p_1p_2\cdots p_{q-2}\times pp_2\cdots p_{q-2}}$$
$$+ \cdots + 2\frac{(\mathrm{U}_{q-1} + \gamma_{q-3}\,\mathrm{Z}_{q-1})(\mathrm{U}_{q-1} + \gamma_{q-2}\,\mathrm{Z}_{q-1})(\gamma_{q-1}-\gamma_{q-3})(\gamma_{q-1}-\gamma_{q-2})}{pp_1\cdots p_{q-1}p_{q-2}\times pp_1\cdots p_{q-4}p_{q-3}}$$

In the same way one has already seen, the numerator of any of the double products which enter in B^2_{q-1}, is able to be considered as result of the negative square of a binomial; for example,

$$\begin{aligned}
&2(\,U_{q-1}+\gamma\,Z_{q-1})(\,U_{q-1}+\gamma_1\,Z_{q-1})(\gamma_{q-1}-\gamma)(\gamma_{q-1}-\gamma_1)\\
&=-\,\{(\,U_{q-1}+\gamma\,Z_{q-1})(\gamma_{q-1}-\gamma_1)-(\,U_{q-1}+\gamma_1\,Z_{q-1})(\gamma_{q-1}-\gamma)\}\\
&\quad+(\,U_{q-1}+\gamma\,Z_{q-1})^2(\gamma_{q-1}-\gamma_1)^2+(\,U_{q-1}+\gamma_1\,Z_{q-1})^2(\gamma_{q-1}-\gamma)^2\\
&=-\,(\,U_{q-1}+\gamma_{q-1}\,Z_{q-1})^2(\gamma-\gamma_1)^2+(\,U_{q-1}+\gamma\,Z_{q-1})^2(\gamma_{q-1}-\gamma_1)^2\\
&\quad+(\,U_{q-1}+\gamma_1\,Z_{q-1})^2(\gamma_{q-1}-\gamma)^2
\end{aligned}$$

Taking account of all the similar reductions in the value of B^2_{q-1},

$$\begin{aligned}
B^2_{q-1}=&\ \frac{(\,U_{q-1}+\gamma\,Z_{q-1})^2(\gamma_{q-1}-\gamma_1)^2}{(p_1p_2\cdots p_{q-2})^2}+\frac{(\,U_{q-1}+\gamma_1\,Z_{q-1})^2(\gamma_{q-1}-\gamma_1)^2}{(pp_2\cdots p_{q-2})^2}\\
&+\cdots+\frac{(\,U_{q-1}+\gamma_{q-2}\,Z_{q-1})^2(\gamma_{q-1}-\gamma_{q-2})^2}{(pp_1\cdots p_{q-3})^2}+\frac{(\,U_{q-1}+\gamma\,Z_{q-1})^2(\gamma_{q-1}-\gamma_1)^2}{p_1p_2\cdots p_{q-2}\times pp_2\cdots p_{q-2}}\\
&+\frac{(\,U_{q-1}+\gamma_1\,Z_{q-1})^2(\gamma_{q-1}-\gamma)^2}{pp_2\cdots p_{q-2}\times p_1p_2\cdots p_{q-2}}+\frac{(\,U_{q-1}+\gamma\,Z_{q-1})^2(\gamma_{q-1}-\gamma_1)^2}{p_1p_2\cdots p_{q-2}\times pp_1p_3\cdots p_{q-2}}\\
&+\frac{(\,U_{q-1}+\gamma_2\,Z_{q-1})^2(\gamma_{q-1}-\gamma)^2}{pp_1p_3\cdots p_{q-2}\times p_1p_2\cdots p_{q-2}}+\cdots+\frac{(\,U_{q-1}+\gamma_{q-3}\,Z_{q-1})^2(\gamma_{q-1}-\gamma_{q-2})^2}{pp_1\cdots p_{q-1}p_{q-2}\times pp_1p_3\cdots p_{q-4}p_{q-3}}\\
&+\frac{(\,U_{q-1}+\gamma_{q-2}\,Z_{q-1})^2(\gamma_{q-1}-\gamma_{q-3})^2}{pp_1\cdots p_{q-4}p_{q-3}\times pp_1\cdots p_{q-3}p_{q-2}}-\frac{(\,U_{q-1}+\gamma_{q-1}\,Z_{q-1})^2(\gamma-\gamma_1)^2}{p_1p_2\cdots p_{q-2}\times pp_2\cdots p_{q-2}}\\
&-\frac{(\,U_{q-1}+\gamma_{q-1}\,Z_{q-1})^2(\gamma-\gamma_2)^2}{pp_1\cdots p_{q-2}\times p_1p_2\cdots p_{q-2}}-\cdots-\frac{(\,U_{q-1}+\gamma_{q-1}\,Z_{q-1})^2(\gamma_{q-2}-\gamma_{q-3})^2}{pp_1\cdots p_{q-3}\times pp_1\cdots p_{q-4}p_{q-2}}
\end{aligned}$$

Comparing the positive terms of this development to the squares furnished by the product

$$\begin{aligned}
&C_{q-1}\left(\frac{(\gamma_{q-1}-\gamma)^2}{p_1p_2\cdots p_{q-2}}+\frac{(\gamma_{q-1}-\gamma_1)^2}{pp_2\cdots p_{q-2}}+\cdots+\frac{(\gamma_{q-1}-\gamma_{q-2})^2}{pp_1\cdots p_{q-3}}\right)\\
&\quad=\left\{\frac{(\,U_{q-1}+\gamma\,Z_{q-1})^2}{p_1p_2\cdots p_{q-2}}+\frac{(\,U_{q-1}+\gamma_1\,Z_{q-1})^2}{pp_2\cdots p_{q-3}}+\cdots+\frac{(\,U_{q-1}+\gamma_{q-2}\,Z_{q-1})^2}{pp_1\cdots p_{q-3}}\right\}\\
&\times\left\{\frac{(\gamma_{q-1}-\gamma)^2}{p_1p_2\cdots p_{q-2}}+\frac{(\gamma_{q-1}-\gamma_1)^2}{pp_2\cdots p_{q-2}}+\cdots+\frac{(\gamma_{q-1}-\gamma_{q-2})^2}{pp_1\cdots p_{q-3}}\right\}
\end{aligned}$$

one will recognize that both are identical.

If, moreover, one pays attention that the negative terms which terminate B^2_{q-1}, are able to be united under the form

$$\begin{aligned}
&-\frac{(\,U_{q-1}+\gamma_{q-1}\,Z_{q-1})^2}{pp_1\cdots p_{q-2}}\left\{\frac{(\gamma-\gamma)^2}{p_2p_3\cdots p_{q-2}}+\frac{(\gamma-\gamma_2)^2}{p_1p_3\cdots p_{q-2}}+\cdots+\frac{(\gamma_{q-3}-\gamma_{q-2})^2}{pp_1\cdots p_{q-4}}\right\}\\
&=\frac{(\,U_{q-1}+\gamma_{q-1}\,Z_{q-1})^2}{pp_1p_2\cdots p_{q-2}}\,A^2_{q-2},
\end{aligned}$$

one will conclude that

$$\mathrm{B}_{q-1}^2 = \mathrm{C}_{q-1}\left\{\frac{(\gamma_{q-1}-\gamma)^2}{p_1p_2\cdots p_{q-2}} + \frac{(\gamma_{q-1}-\gamma_1)^2}{pp_2\cdots p_{q-2}} + \cdots + \frac{(\gamma_{q-1}-\gamma_{q-2})^2}{pp_1\cdots p_{q-3}}\right\}$$
$$- \frac{(\mathrm{U}_{q-1}+\gamma_{q-1}\mathrm{Z}_{q-1})^2}{pp_1p_2\cdots p_{q-2}}\mathrm{A}_{q-2}^2$$

Whence

$$\frac{(\mathrm{U}_{q-1}+\gamma_{q-1}\mathrm{Z}_{q-1})^2}{pp_1p_2\cdots p_{q-2}} - \left\{\frac{(\gamma_{q-1}-\gamma)^2}{p_1p_2\cdots p_{q-2}} + \frac{(\gamma_{q-1}-\gamma_1)^2}{pp_2\cdots p_{q-2}} + \cdots + \frac{(\gamma_{q-1}-\gamma_{q-2})^2}{pp_1\cdots p_{q-3}}\right\}\frac{\mathrm{C}_{q-1}}{\mathrm{A}_{q-2}^2} + \frac{\mathrm{B}_{q-1}^2}{\mathrm{A}_{q-2}^2} = 0.$$

This is also the sum of the terms of the expression (67), of which it was necessary to prove the identical nullity. This equation (67) is reduced therefore to

$$\frac{\mathrm{C}_{q-1}^2\,\mathrm{A}_{q-1} - \mathrm{B}_{q-1}^2}{\mathrm{A}_{q-2}^2} + \frac{z_q^2\,\mathrm{A}_{q-1}^2}{p_q} = z_q^2\,\mathrm{A}_q^2 - 2z_q\,\mathrm{B}_q + \mathrm{C}_q,$$

and the exponent (66) becomes

$$\mathrm{K}_q + \mathrm{S}_{q-1} - y_2^2 - y_3^2 - \cdots - y_{q-1}^2 - \frac{n^2}{2\,\mathrm{A}_{q-1}^2}(z_q^2\,\mathrm{A}_q^2 - 2z_q\,\mathrm{B}_q + \mathrm{C}_q)$$

The form of the quantity between parentheses which is reproduced here is too known in order to not put immediately

$$z_q = y_q\frac{\sqrt{2}\,\mathrm{A}_{q-1}}{n\,\mathrm{A}_q} - \frac{\mathrm{B}_q}{\mathrm{A}_q^2},$$

and hence the result of the substitution which restores the exponent (63) or (66), and the integral on which it depends, to

$$\text{(68)}\qquad \mathrm{K}_q + \mathrm{S}_{q-1} - y_2^2 - y_3^2 - \cdots - y_{q-1}^2 - y_q^2 - \frac{n^2}{2\,\mathrm{A}_{q-1}^2}\left(\mathrm{C}_q - \frac{\mathrm{B}_q^2}{\mathrm{A}_q^2}\right) = \mathrm{R}'$$

$$_m\int_{b'}^{b} dy_2dy_3\cdots dy_{q-1}dy_qdz_{q+1}dz_{q+2}\cdots dz_mdue^{\mathrm{R}'}$$

by suppressing in advance the common factor $\frac{\sqrt{2}\,\mathrm{A}_{q-1}}{n\,\mathrm{A}_q}$.

The expression (68) is precisely the same as the exponent (64). It is therefore established generally that the transformation of any variable gives a result of the same form as the results of the preceding transformations, if these last have led to the form (63), (64) or (68).

This special form having been established for one and two variables, it is demonstrated that it will be reproduced for the third, for the fourth, and to complete exhaustion of the variables z.

It is agreeable at present to examine that which will arrive under this last hypothesis, and how the operation will be achieved with respect to u.

By making $q = m$ in the formula (68), it gives the integral

$$(69) \qquad {}_m\int_{b'}^{b} dy_2 dy_3 \cdots dy_m du e^{\mathrm{S}_m - y_2^2 - y_3^2 - \cdots - y_m^2 - \frac{n^2}{2\,\mathrm{A}_{m-1}^2}\left(\mathrm{C}_m - \frac{\mathrm{B}_m^2}{\mathrm{A}_m^2}\right)}$$

Changing also $(q-1)$ into m in formulas (65), the quantities designated by K_m and Z_m will be null, since one will have employed all the variables z, of which they had for unique object to represent some sums.

The quantity U_m will be equal to u.

One will have next

$$\begin{aligned}
\mathrm{A}_m^2 &= \frac{(\gamma-\gamma_1)^2}{p_2p_3\cdots p_m} + \frac{(\gamma-\gamma_2)^2}{p_1p_3\cdots p_m} + \cdots + \frac{(\gamma_m-\gamma_{m-1})^2}{pp_1p_2\cdots p_{m-2}} \\
&= \frac{1}{p_m}\,\mathrm{A}_{m-1}^2 + \frac{(\gamma_m-\gamma)^2}{p_1p_2\cdots p_{m-1}} + \frac{(\gamma_m-\gamma_1)^2}{pp_2\cdots p_{m-1}} + \cdots + \frac{(\gamma_m-\gamma_{m-1})^2}{pp_1\cdots p_{m-2}}, \\
\mathrm{B}_m &= u\left\{\frac{(\gamma_m-\gamma)^2}{p_1p_2\cdots p_{m-1}} + \frac{(\gamma_m-\gamma_1)^2}{pp_2\cdots p_{m-1}} + \cdots + \frac{(\gamma_m-\gamma_{m-1})^2}{pp_1\cdots p_{m-2}}\right\}, \\
\mathrm{C}_m &= u^2\left\{\frac{1}{p_1p_2\cdots p_{m-1}} + \frac{1}{pp_2\cdots p_{m-1}} + \cdots + \frac{1}{pp_1\cdots p_{m-2}}\right\} = u^2\frac{n-p_m}{pp_1\cdots p_{m-1}},
\end{aligned}$$

Hence

$$\begin{aligned}
\mathrm{C}_m\,\mathrm{A}_m^2 &= \frac{u^2(n-p_m)}{pp_1\cdots p_{m-1}}\left[\mathrm{A}_{m-1}^2 + p_m\left(\frac{(\gamma_m-\gamma)^2}{p_1p_2\cdots p_{m-1}} + \frac{(\gamma_m-\gamma_1)^2}{pp_2\cdots p_{m-1}} + \cdots + \frac{(\gamma_m-\gamma_{m-1})^2}{pp_1\cdots p_{m-2}}\right)\right] \\
\mathrm{C}_m\,\mathrm{A}_m^2 - \mathrm{B}_m^2 &= \frac{u^2 n\,\mathrm{A}_{m-1}^2}{pp_1p_2\cdots p_m} \\
&- u^2\left[\frac{\mathrm{A}_{m-1}^2}{pp_1\cdots p_{m-1}} - \frac{n-p_m}{pp_1\cdots p_{m-1}}\left(\frac{(\gamma_m-\gamma)^2}{p_1p_2\cdots p_{m-1}} + \frac{(\gamma_m-\gamma_1)^2}{pp_2\cdots p_{m-1}} + \cdots + \frac{(\gamma_m-\gamma_{m-1})^2}{pp_1\cdots p_{m-2}}\right) + \frac{\mathrm{B}_m^2}{u^2}\right].
\end{aligned}$$

But

$$\mathrm{B}_m^2 = u^2\left\{\begin{array}{l}\dfrac{(\gamma_m-\gamma)^2}{(p_1p_2\cdots p_{m-1})^2} + \dfrac{(\gamma_m-\gamma_1)^2}{(pp_2\cdots p_{m-1})^2} + \cdots + \dfrac{(\gamma_m-\gamma_{m-1})^2}{(pp_1\cdots p_{m-2})^2} \\ + 2\dfrac{(\gamma_m-\gamma)(\gamma_m-\gamma_1)}{p_1p_2\cdots p_{m-1}\times pp_2\cdots p_{m-1}} + \cdots + 2\dfrac{(\gamma_m-\gamma_{m-2})(\gamma_m-\gamma_{m-1})}{pp_1\cdots p_{m-3}p_{m-1}\times pp_2\cdots p_{m-2}}\end{array}\right\}$$

Executing on the double products the change already employed, one has for example

$$2(\gamma_m-\gamma)(\gamma_m-\gamma_1) = -(\gamma-\gamma_1)^2 + (\gamma_m-\gamma)^2 + (\gamma_m-\gamma_1)^2,$$

this which permits to write

$$\begin{aligned}\frac{\mathrm{B}_m^2}{u^2} =&\frac{1}{pp_1p_2\cdots p_{m-1}}\left\{\frac{(\gamma_m-\gamma)^2p}{p_1p_2\cdots p_{m-1}}+\frac{(\gamma_m-\gamma_1)^2p_1}{pp_2\cdots p_{m-1}}+\cdots+\frac{(\gamma_m-\gamma_{m-1})^2p_{m-1}}{pp_1\cdots p_{m-2}}\right.\\ &+\frac{(\gamma_m-\gamma)^2p_1}{p_1p_2\cdots p_{m-1}}+\frac{(\gamma_m-\gamma_1)^2p}{pp_2\cdots p_{m-1}}+\frac{(\gamma_m-\gamma)^2p_2}{p_1p_2\cdots p_{m-1}}+\frac{(\gamma_m-\gamma_2)^2p}{pp_2\cdots p_{m-1}}+\cdots\\ &+\frac{(\gamma_m-\gamma_{m-2})^2p_{m-1}}{pp_1\cdots p_{m-1}}+\frac{(\gamma_m-\gamma_{m-1})^2p_{m-2}}{pp_1\cdots p_{m-2}}-\frac{(\gamma-\gamma_1)^2}{p_2p_3\cdots p_{m-1}}-\frac{(\gamma-\gamma_2)^2}{p_1p_3\cdots p_{m-1}}\\ &\left.-\cdots-\frac{(\gamma_{m-1}-\gamma_{m-2})^2}{pp_1\cdots p_{m-3}}\right\}\end{aligned}$$

Reusing the similar positive terms as for the functions γ, and observing that they are all multiplied by $p+p_1+p_2+\cdots+p_{m-1}$ is that is to say by $n-p_m$; that besides the negative terms are precisely the terms of A_{m-1}^2, one is able to affirm that

$$\begin{aligned}\frac{\mathrm{B}_m^2}{u^2} = \frac{n-p_m}{pp_1p_2\cdots p_{m-1}}\left\{\frac{(\gamma_m-\gamma)^2}{p_1p_2\cdots p_{m-1}}+\frac{(\gamma_m-\gamma_1)^2}{pp_2\cdots p_{m-1}}+\cdots+\frac{(\gamma_m-\gamma_{m-1})^2}{pp_1\cdots p_{m-2}}\right\}\\ -\frac{\mathrm{A}_{m-1}^2}{pp_1\cdots p_{m-1}}\end{aligned}$$

The coefficient of u^2 between parentheses in the development of $\mathrm{C}_m\ \mathrm{A}_m^2-\ \mathrm{B}_m^2$, is therefore identically null; and there remains

$$\frac{\mathrm{C}_m\ \mathrm{A}_m^2-\ \mathrm{B}_m^2}{\mathrm{A}_{m-1}^2}=\frac{u^2n}{pp_1p_2\cdots p_m}.$$

The part of the exponent of e which, in the integral (69), contained further A_{m-1}, B_m, C_m, is reduced consequently to

$$-\frac{n^3}{2pp_1p_2\cdots p_m\ \mathrm{A}_m^2}u^2,$$

and if one makes

$$u=t\sqrt{\frac{2pp_1p_2\cdots p_m\ \mathrm{A}_m^2}{n^3}},$$

one will have for the integral, by continuing to suppress in advance the constant factors outside the $\int$ sign, calling c' and c, the limits of t, and S_{m+1} the series of superior powers transformed into l,

(70) $$_m\int_{c'}^{c} dy_2dy_3\cdots dy_mdte^{\,\mathrm{S}_{m+1}-y^2-y_3^2-\cdots-y_m-t^2}$$

In order to achieve the integration, it will suffice to make the series S_{m+1} exit from the exponent: this which gives

$$_m\int_{c'}^{c} dy_2dy_3\cdots dy_mdte^{-y^2-y_3^2-\cdots-y_m-t^2}\left(1+\ \mathrm{S}_{m+1}+\frac{1}{2}\,\mathrm{S}_{m+1}^2+\ \text{etc.}\right)$$

One should remark then that by integrating any term with respect to y_2, the magnitude of the limits of this variable which is of the order $\sqrt{n}$, will reduce to zero, if the exponent of y_2 is odd, and to a constant factor multiplied by $\sqrt{\pi}$, if the exponent is even and if the very great limits are of contrary signs.

One will conclude from it easily that after having integrated relatively to all the variables y, it will remain only some terms in t, those which were found multiplied by some even powers of each of these variables y.

The integration, with respect to t, will lead next to two series: the one of terms multiplied by $\frac{1}{2}e^{-c^2}$, $\frac{1}{2}e^{-c'^2}$, of which the coefficients, being of the order $\frac{1}{\sqrt{n}}$ and its powers, will be quite small when one will take for n a large number, and $c > 2$, $c' > 2$.

One is able therefore to neglect this series, of which, moreover, the preceding analysis is capable to take account.

The second series will be composed of constant terms affected of $\int_{c'}^{c} e^{-t^2}dt$. By designating it by T, the sought integral, becomes finally

$$((\sqrt{\pi})^{m-1} + \mathrm{T})\int_{c'}^{c} e^{-t^2}dt. \tag{71}$$

Taking $c' = c$, one will have the numerator of the probability P. As for the denominator, it is necessary to integrate the expression (71), for all the possible values of t. The limits which contain all the possible values are γ and γ_m, the smallest and the greatest of the arbitrary γ: as

$$u = v - \frac{p\gamma + p_1\gamma_1 + p_2\gamma_2 + \cdots + p_m\gamma_m}{n},$$

the extreme limits of u will be

$$+\frac{p(\gamma_m-\gamma) + p_1(\gamma_m-\gamma_1) + \cdots + p_{m-1}(\gamma_m-\gamma_{m-1})}{n}$$
$$-\frac{p_1(\gamma_1-\gamma) + p_2(\gamma_2-\gamma) + \cdots + p_m(\gamma_m-\gamma)}{n},$$

and one will have for the limits of t

$$l = \{p(\gamma_m-\gamma) + p_1(\gamma_m-\gamma_1) + \cdots + p_{m-1}(\gamma_m-\gamma_{m-1})\}\sqrt{\frac{n}{2pp_1p_2\cdots p_m\,\mathrm{A}_m^2}}$$
$$l' = \{p(\gamma_m-\gamma) + p_1(\gamma_2-\gamma) + \cdots + p_m(\gamma_m-\gamma)\}\sqrt{\frac{n}{2pp_1p_2\cdots p_m\,\mathrm{A}_m^2}}$$

quantities of the very great order $\sqrt{n}$. One is therefore by right to suppose in the denominator of the probability $t = \pm\infty$: this which gives for the value of this term

$$((\sqrt{\pi})^{m-1} + \mathrm{T})\sqrt{\pi},$$

and one arrives, as in the preceding cases, to

$$P = \frac{2}{\sqrt{\pi}}\int_0^c e^{-t^2}dt. \tag{72}$$

It is the probability that the value of v is contained between the limits

$$\frac{p\gamma + p_1\gamma_1 + p_2\gamma_2 + \cdots + p_m\gamma_m}{n} \pm c\sqrt{\frac{2pp_1p_2\cdots p_m\ \mathrm{A}_m^2}{n^3}},$$

or else that the difference between the true value of v and the mean deduced from observations fall between

$$\pm c\sqrt{\frac{2pp_1p_2\cdots p_m\ \mathrm{A}_m^2}{n^3}},$$

an expression which, by bringing together for A_m^2 the always symmetric function that this letter represents, takes the form already established:

(73)

$$\pm c\sqrt{\frac{2}{n^3}\left\{pp_1(\gamma-\gamma_1)^2 + pp_2(\gamma-\gamma_2)^2 + p_1p_2(\gamma_1-\gamma_2)^2 + \cdots + p_{m-1}p_m(\gamma_m - \gamma_{m-1^2})\right\}}$$

If one develops the squares between the parentheses, one obtains for a term such as $pp_1(\gamma-\gamma_1)^2$,

$$pp_1\gamma^2 + pp_1\gamma_1^2 - 2pp_2\gamma\gamma_1.$$

It becomes thus manifest that the radical containing the squares of the differences between γ and the m other arbitraries, each multiplied by the numbers pp_1, etc., correlatives; the product of a square γ^2 by the number p, will be multiplied by the sum of all the other numbers $p_1 + p_2 + \cdots + p_m = n - p$.

The terms of the parenthesis under the radical (73) will be able therefore to be written

$$p(n-p)\gamma^2 + p_1(n-p_1)\gamma_1^2 + p_2(n-p_2)\gamma_2^2 + \cdots - 2pp_1\gamma\gamma_1 - 2pp_2\gamma\gamma_2$$
$$- 2p_1p_2\gamma_1\gamma_2 - \cdots - 2p_{m-1}p_m\gamma_{m-1}\gamma_m$$

or else

$$n(p\gamma^2 + p_1\gamma_1^2 + p_2\gamma_2^2 + \cdots + p_m\gamma_m^2) - (p^2\gamma^2 + p_1^2\gamma_1^2 + p_2^2\gamma_2$$
$$+ \cdots + p_m^2\gamma_m^2 - 2pp_1\gamma\gamma_1 - 2pp_2\gamma\gamma_2 - \cdots - 2p_mp_{m-1}\gamma_m\gamma_{m-1}$$

which is evidently equal to

$$n^2\frac{p\gamma^2 + p_1\gamma_1^2 + p_2\gamma_2^2 + \cdots + p_m^2\gamma_m^2}{n} - n^2\left(\frac{p\gamma + p_1\gamma_1 + p_2\gamma_2 + \cdots + p_m\gamma_m}{n}\right)^2$$

whence one obtains for the limits (73) of the deviation

(74)

$$\pm c\sqrt{\frac{2}{n}\left\{\frac{p\gamma^2 + p_1\gamma_1^2 + p_2\gamma_2^2 + \cdots + p_m\gamma_m^2}{n} - \left(\frac{p\gamma + p_1\gamma_1 + p_2\gamma_2 + \cdots + p_m\gamma_m^2}{n}\right)^2\right\}}$$

Here figures only the mean of the squares of the n observations, less the square of the mean: it is the form that Laplace has given in order to measure the deviations of the calculated mean life according to the ages γ, γ_1, γ_2, etc., of a large number of individuals.

In calling μ the mean, one is able to remark that

$$\frac{p\gamma^2 + p_1\gamma_1^2 + p_2\gamma_2^2 + \cdots + p_m\gamma_m}{n} - \mu^2 = \frac{p\gamma^2 + p_1\gamma_1^2 + p_2\gamma_2^2 + \cdots + p_m\gamma_m - n\mu^2}{n}$$

$$= \frac{\left\{\begin{array}{l} p(\gamma^2 + \mu^2 - 2\gamma\mu) + p_1(\gamma_1^2 + \mu^2 - 2\gamma_1\mu) + \cdots + p_m(\gamma_m^2 + \mu^2 - 2\gamma_m\mu) \\ -\, 2n\mu^2 + 2(p\gamma + p_1\gamma_1 + \cdots p_m\gamma_m)\mu \end{array}\right\}}{n}$$

$$= \frac{p(\gamma - \mu)^2 + p_1(\gamma_1 - \mu)^2 + \cdots + p_m(\gamma_m - \mu)^2 - 2n\mu^2 + 2n\mu^2}{n}$$

The limits of the deviation is able therefore to be written equally

$$\pm\frac{c}{n}\sqrt{2\left\{p(\gamma-\mu)^2 + p_1(\gamma_1-\mu)^2 + p_2(\gamma_2-\mu)^2 + \cdots + p_m(\gamma_m-\mu)^2\right\}} \tag{75}$$

There is no more under the radical than the squares of the differences between each value γ resulting from the observation and the mean of all.

Laplace has discovered under this last form the constant which measures the errors of the mean results from astronomical observations.

The analysis which has just been exposed no longer permits to raise the least doubt on the use of this constant so remarkable. It shows that this quantity is certainly the complete coefficient of the first term of the series which expresses the probability of the deviation of the mean of any number of observations, likewise that the radical $\sqrt{\frac{2pp_1}{n^3}}$, on which there has never been raised doubt, is the coefficient of the first term in the probability of the deviation of the real ratio of possibility to the mean ratio of the repetitions of two events which exclude both.

If the direct demonstration of this truth was able to obtain the suffrage of the Academy, perhaps the observers would make use more often of the rule of Laplace, of which the rigorous exactitude appears to have been until here well judged only by a small number of scholars. This Memoir would then be completely fulfilled, since at present, the intention which has made it written, although the default of the times has not at all permitted to develop the consequences of them.