Intoduction to the Fifth Workshop
Game-Theoretic Probability and Related Topics

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1. Basics of game-theoretic probability
2. Probability-free finance
3. Prediction

Contrast

Conventional wisdom

Maybe you have the wrong model.

Rare event more likely than you think. (Taleb)

Good prediction means getting the model right.

Game-theoretic alternative

Often no correct model. Only a game.

Many events have no probability at all. (Kolmogorov)

Prediction is a game that can be played well.

Basics of game-theoretic probability

Pascal instead of Fermat

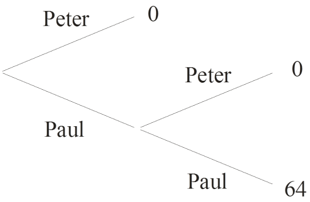
- Rules for betting instead of stochastic model.
- Expectation = cost of replication

Game-theoretic testing

- Proof that E happens with high probability
  = strategy for getting very rich if E does not happen
- Proof that E happens for sure
  = strategy for getting infinitely rich if E does not happen

Pascal saw a game.

Pascal's question



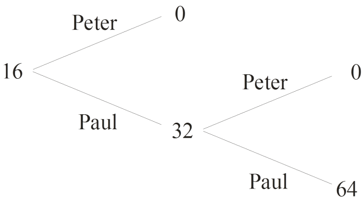
Rules of the game

Even odds.
Paul gets 64 if he wins twice.

If the game ends now, how much should Paul get?

Pascal's answer

Paul can replicate his payoff starting with 16.



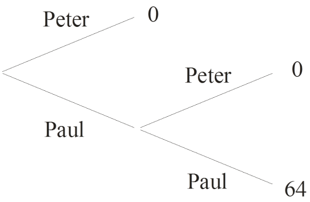
Expectation = cost of replication

Expectation = 16
Probability of winning $= \frac{16}{64} = \frac{1}{4}$

Fermat saw a stochastic model.

Fermat's model

Suppose they play two rounds!



Four equally possible outcomes

1. Peter wins, Peter wins.
2. Peter wins, Paul wins.
3. Paul wins, Peter wins.
4. Paul wins, Paul wins.

Fermat's answer

Paul gets 64 only in outcome 4.
So Paul should get 16 .

Pierre Fermat, 1601-1665

Basics of game-theoretic probability

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Game-theoretic testing

- Proof that E happens with high probability
  = strategy for getting very rich if E fails
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Example of a probability game

Three players: Forecaster, Skeptic, Reality

On each round...

Forecaster announces the price m for a payoff x.

Skeptic buys M units of x.

Reality announces the value of x.

Skeptic receives the net gain $M(x - m)$.

Perfect information: Players see and remember each other's moves.

Roles

Forecaster is the model.

Skeptic buys M units of x.

Skeptic tests the prices offered by Forecaster.

Bounded Forecasting Protocol

$\mathcal{K}_n$ = Skeptic's capital

$\mathcal{K}_0 := 1$.
FOR $n = 1, 2, \ldots, N$:
  Forecaster announces $m_n \in [-1, 1]$.
  Skeptic announces $M_n \in \mathbb{R}$.
  Reality announces $x_n \in [-1, 1]$.
  $\mathcal{K}_n := \mathcal{K}_{n-1} + M_n(x_n - m_n)$.

An event with high probability

$\mathcal{K}_0 := 1$.
FOR $n = 1, 2, \ldots, N$:
  Forecaster announces $m_n \in [-1, 1]$.
  Skeptic announces $M_n \in \mathbb{R}$.
  Reality announces $x_n \in [-1, 1]$.
  $\mathcal{K}_n := \mathcal{K}_{n-1} + M_n(x_n - m_n)$.

Game-theoretic testing

Proof that E happens with high probability
  = strategy for getting very rich if E fails

Example where E is the event $\left|\frac{1}{N}\sum_{n=1}^{N}(x_n - m_n)\right| < \epsilon$

Proposition

Skeptic has a strategy that turns his initial capital of 1 into $\frac{1}{2}\exp \epsilon^2 N/2$ if the event $\left|\frac{1}{N}\sum_{n=1}^{N}(x_n - m_n)\right| < \epsilon$ fails.

GET VERY RICH means GREATLY MULTIPLY THE CAPITAL YOU RISK

Game-theoretic testing

Proof that E happens with high probability
= strategy for greatly multiplying the capital risked if E fails

Proposition

Skeptic has a strategy that does not risk bankruptcy and turns his initial capital of 1 *into* $\frac{1}{2}\exp \epsilon^2 N/2$ *if* $\left|\frac{1}{N}\sum_{n=1}^{N}(x_n - m_n)\right| < \epsilon$ *if fails.*

A strategy that risks bankruptcy does not qualify as a proof.

An event that happens for sure

$\mathcal{K}_0 := 1$.
FOR $n = 1, 2, \ldots$:
Forecaster announces $m_n \in [-1, 1]$.
Skeptic announces $M_n \in \mathbb{R}$.
Reality announces $x_n \in [-1, 1]$.
$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n(x_n - m_n)$.

Game-theoretic testing

Proof that E happens for sure
= strategy for getting infinitely rich if E fails

Proposition

Skeptic has a strategy that does not risk bankruptcy and turns his initial capital of 1 into ∞ if $\frac{1}{N}\sum_{n=1}^{N}(x_n - m_n) \to \infty$ does not happen.

Ways of using game-theoretic probability

We just learned...

Pure probability: Prove theorems about what happens with high probability or for sure by constructing strategies for Skeptic.

Statistical testing: Forecaster is the model. Use Skeptic's strategies to test the model.

Now let's talk about...

Probability-free finance: The hypothesis that Skeptic will not become rich without risking bankruptcy becomes a form of the efficient-market hypothesis.

Prediction: Construct strategies for Forecaster that will pass the most important tests.

Probability-free finance

GAME-THEORETIC EFFICIENT-MARKET HYPOTHESIS:
A speculator will not greatly multiply the capital he risks.

Some consequences

1. Volatility of prices proportional to $\sqrt{dt}$
2. Ito calculus in the limit with more and more frequent trading
3. CAPM
4. Equity premium close to squared volatility of index

Prediction

One way of achieving good prediction without a stochastic model

Construct strategy for Forecaster that passes the most important tests.

- Formulate each test with as a strategy for Skeptic.
- Average the strategies for Skeptic.
- Forecaster pays against the average.