Does God Play Dice in Game Theoretic Probability?

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Outline

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Matching Pennies

Game

No Wealth by Matching Pennies

Wealth by Matching Pennies

Question

Bimatrix presentation of 2-player games

- $n = 2$
- $A_1 = \{U, D\}$
- $A_2 = \{L, R\}$
- $u = \langle u_1, u_2 \rangle : A_1 \times A_2 \to \mathbb{R} \times \mathbb{R}$

	L	R
U	$u_2(U, L)$ $u_1(U, L)$	$u_2(U, R)$ $u_1(U, R)$
D	$u_2(D, L)$ $u_1(D, L)$	$u_2(D, R)$ $u_1(D, R)$

Matching Pennies

- players: A, B
- moves: $M_A = M_B = \{H, T\}$
- $u = \langle u_A, u_B \rangle : M_A \times M_B \to \mathbb{R} \times \mathbb{R}$

	H	T
H	1, -1	-1, 1
T	-1, 1	1, -1

Matching Pennies

Strategy: Randomize!

The only Nash equilibrium for Matching Pennies is the profile $\langle a, b \rangle$ where the players randomize

$$p(a = H) \; = \; \frac{1}{2} \; = \; p(b = H)$$

Matching Pennies

Randomness: Strategy!

The other way around, we can **define** that a sequence

$$H, T, T, H, T \dots$$

is random iff it is a strategy for Matching Pennies that does not lose against any opponent.

Matching Pennies

Randomness: Strategy!

The other way around, we can **define** that a sequence

$$H, T, T, H, T \dots$$

is random iff it is a strategy for Matching Pennies that does not lose against any opponent.

[**Reason**: If you can write a short program to predict the next move with probability $> \frac{1}{2}$, then you can win Matching Pennies.]

Matching Pennies

Suspicion

- Is this a bit like Game Theoretic Probability?

Matching Pennies

Suspicion

- Is this a bit like Game Theoretic Probability?
- Maybe not quite. . .

Matching Pennies against Nature

- players: A, B, N
- moves: $M_A = M_B = \{+, -\}$, $M_N = \{00, 01, 10, 11\}$
- $u = \langle u_{A,B}, u_N \rangle : M_{A,B} \times M_N \to \mathbb{R} \times \mathbb{R}$

	00,01,10	11
++, --	-1 1	1 -1
+-, -+	1 -1	-1 1

Matching Pennies against Nature

Game protocol

- N moves first with $xy \in \{0,1\}^2$
- A sees x (not y or b) and responds with $a \in \{+,-\}$
- B sees y (not x or a) and responds with $b \in \{+,-\}$

Matching Pennies against Nature

Game protocol

- N moves first with $xy \in \{0,1\}^2$
- A sees x (not y or b) and responds with $a \in \{+,-\}$
- B sees y (not x or a) and responds with $b \in \{+,-\}$

Remark

They play a game of imperfect information.

Coordinating pennies: Strategies

- N's moves xy are random and uniformly distributed.
- A and B should coordinate to specify
  - A's strategy: probability distribution $p(a \mid x)$
  - B's strategy: probability distribution $p(b \mid y)$

  to maximize their payoffs.

Coordinating pennies: Payoffs

$$\mathbb{U}_{AB} = \frac{1}{4}\Big(\mathbb{E}_{AB}(00) + \mathbb{E}_{AB}(01) + \mathbb{E}_{AB}(10) - \mathbb{E}_{AB}(11)\Big)$$

$$\mathbb{E}_{AB}(xy) = \sum_{a,b\in M_{AB}} a \cdot b \cdot p(ab|xy)$$

where we muliply $a, b \in \{+, -\}$ as if they are $+1$ and -1

Hidden Variable Theorem

Theorem

If the mutual dependency of x and y is expressed by a variable $\lambda \in \Lambda$ with density $q : \Lambda \to [0,1]$, so that

$$p(ab|xy) = \int_\Lambda p(a|x,\lambda) \cdot p(b|y,\lambda) \cdot q(\lambda) d\lambda \quad (1)$$

then

$$\mathbb{U}_{AB} \leq \frac{1}{2} \quad (2)$$

Hidden Variable Theorem

Proof

Write

$$
\begin{aligned}
\mathbb{E}_A(x,\lambda) &= \sum_{a\in M_A} a\cdot p(a|x,\lambda)\\
\mathbb{E}_B(y,\lambda) &= \sum_{b\in M_B} b\cdot p(b|y,\lambda)\\
\mathbb{E}_{AB}(xy,\lambda) &= \mathbb{E}_A(x,\lambda)\cdot\mathbb{E}_B(y,\lambda)
\end{aligned}
$$

Hidden Variable Theorem

Proof

Then

$$\mathbb{U}_{AB} = \int_{\Lambda} \mathbb{U}_{AB}(\lambda) \cdot q(\lambda) d\lambda$$

for

$$\mathbb{U}_{AB}(\lambda) = \frac{1}{4}\Big(\mathbb{E}_A(0,\lambda) \cdot \mathbb{E}_B(0,\lambda) + \mathbb{E}_A(0,\lambda) \cdot \mathbb{E}_B(1,\lambda) + \mathbb{E}_A(1,\lambda) \cdot \mathbb{E}_B(0,\lambda) - \mathbb{E}_A(1,\lambda) \cdot \mathbb{E}_B(1,\lambda)\Big)$$

Hidden Variable Theorem

Proof

Then

$$\mathbb{U}_{AB} = \int_{\Lambda} \mathbb{U}_{AB}(\lambda) \cdot q(\lambda) d\lambda$$

for

$$\begin{aligned}
\mathbb{U}_{AB}(\lambda) &= \frac{1}{4}\Big(\mathbb{E}_A(0,\lambda) \cdot \mathbb{E}_B(0,\lambda) + \mathbb{E}_A(0,\lambda) \cdot \mathbb{E}_B(1,\lambda) + \\
&\quad \mathbb{E}_A(1,\lambda) \cdot \mathbb{E}_B(0,\lambda) - \mathbb{E}_A(1,\lambda) \cdot \mathbb{E}_B(1,\lambda)\Big) \\
&= \frac{1}{4}\Big(\mathbb{E}_A(0,\lambda) \cdot \big(\mathbb{E}_B(0,\lambda) + \mathbb{E}_B(1,\lambda)\big) + \\
&\quad \mathbb{E}_A(1,\lambda) \cdot \big(\mathbb{E}_B(0,\lambda) - \mathbb{E}_B(1,\lambda)\big)\Big)
\end{aligned}$$

Hidden Variable Theorem

Proof

Since $-1 \leq \mathbb{E}_A(0,\lambda), \mathbb{E}_A(1,\lambda) \leq 1$

$$\mathbb{U}_{AB}(\lambda) \quad \leq \quad \frac{1}{4}\Big(\big|\mathbb{E}_B(0,\lambda) + \mathbb{E}_B(1,\lambda)\big| + \big|\mathbb{E}_B(0,\lambda) - \mathbb{E}_B(1,\lambda)\big|\Big)$$

Hidden Variable Theorem

Proof

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If $\mathbb{E}_B(0,\lambda) \geq \max\left\{0, \mathbb{E}_B(1,\lambda)\right\}$, then it follows that

$$\begin{aligned} \mathbb{U}_{AB}(\lambda) &\leq \frac{1}{4}\Big(\mathbb{E}_B(0,\lambda) + \mathbb{E}_B(1,\lambda) + \mathbb{E}_B(0,\lambda) - \mathbb{E}_B(1,\lambda)\Big) \\ &= \frac{1}{4}\Big(\mathbb{E}_B(0,\lambda) + \mathbb{E}_B(0,\lambda)\Big) \\ &\leq \frac{1}{2} \end{aligned}$$

since $0 \leq \mathbb{E}_B(0,\lambda) \leq 1$.

Hidden Variable Theorem

Proof

Since $-1 \le \mathbb{E}_A(0,\lambda), \mathbb{E}_A(1,\lambda) \le 1$

$$\mathbb{U}_{AB}(\lambda) \le \frac{1}{4}\Big(\big|\mathbb{E}_B(0,\lambda) + \mathbb{E}_B(1,\lambda)\big| + \big|\mathbb{E}_B(0,\lambda) - \mathbb{E}_B(1,\lambda)\big|\Big)$$

If $0 \ge \mathbb{E}_B(0,\lambda) \ge \mathbb{E}_B(1,\lambda)$, then it follows that

$$\begin{aligned}
\mathbb{U}_{AB}(\lambda) &\le \frac{1}{4}\Big(-\mathbb{E}_B(0,\lambda) - \mathbb{E}_B(1,\lambda) + \mathbb{E}_B(0,\lambda) - \mathbb{E}_B(1,\lambda)\Big) \\
&= \frac{1}{4}\Big(-\mathbb{E}_B(1,\lambda) - \mathbb{E}_B(1,\lambda)\Big) \\
&\le \frac{1}{2}
\end{aligned}$$

since $0 \ge \mathbb{E}_B(1,\lambda) \ge -1$.

Hidden Variable Theorem

Proof

Since $-1 \leq \mathbb{E}_A(0,\lambda), \mathbb{E}_A(1,\lambda) \leq 1$

$$\mathbb{U}_{AB}(\lambda) \quad \leq \quad \frac{1}{4}\Big(\big|\mathbb{E}_B(0,\lambda) + \mathbb{E}_B(1,\lambda)\big| + \big|\mathbb{E}_B(0,\lambda) - \mathbb{E}_B(1,\lambda)\big|\Big)$$

If $\mathbb{E}_B(0,\lambda) \leq \mathbb{E}_B(1,\lambda)$, then the two analogous cases again give

$$\mathbb{U}_{AB}(\lambda) \quad \leq \quad \frac{1}{2}$$

Hidden Variable Theorem

Proof

In all cases

$$\mathbb{U}_{AB} = \int_\Lambda \mathbb{U}_{AB}(\lambda)\cdot q(\lambda)d\lambda \leq \int_\Lambda \frac{1}{2}q(\lambda)d\lambda = \frac{1}{2}$$

□

Hidden Variable Theorem

Interpretation

Suppose that

- A, B and N repeat the game infinitely often, and
- A and B invest $\$\frac{1}{2}$ each for every bet.

Hidden Variable Theorem

Interpretation

Suppose that

- A, B and N repeat the game infinitely often, and
- A and B invest $\$\frac{1}{2}$ each for every bet.

Since N's moves are uniformly distributed, A and B's chances are

- $\frac{3}{4}$ to win \$1
- $\frac{1}{4}$ to lose \$1

i.e. the expected winnings for each of them are

$$\frac{3}{4}(\$1) + \frac{1}{4}(-\$1) \quad = \quad \$\frac{1}{2}$$

Hidden Variable Theorem

Interpretation

- So if A and B randomize their moves uniformly, in the long run their wealth remains unchanged.

Hidden Variable Theorem

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- So if *A* and *B* randomize their moves uniformly, in the long run their wealth remains unchanged.
  - This is the Nash equilibrium of Matching Pennies.

Hidden Variable Theorem

Interpretation

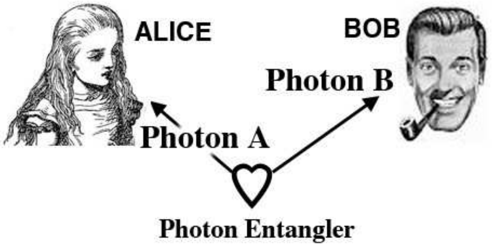
- So if *A* and *B* randomize their moves uniformly, in the long run their wealth remains unchanged.
  - This is the Nash equilibrium of Matching Pennies.
- The **question** is whether they can increase their wealth by coordinating.
  - The **answer** suggested by the Theorem is **NO**.

Hidden Variable Theorem

Another suspicion

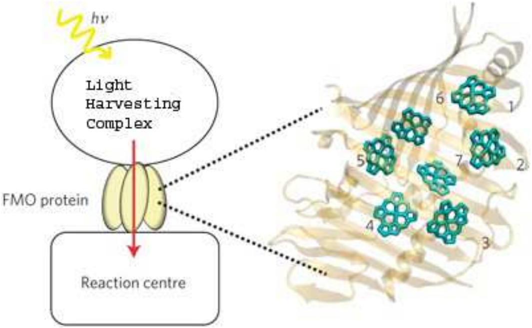
- Is averaging out the hidden variable λ really the only way in which A and B can coordinate?
- Maybe not?

Idea



E.g., they could also use entangled photons

Idea



Plants extract their strategic advantage similarly: photosynthesis is a quantum effect!

Disproving the Theorem

Claim

Using a physical device, A and B can disprove the Hidden Variable Theorem.

Disproving the Theorem

Claim

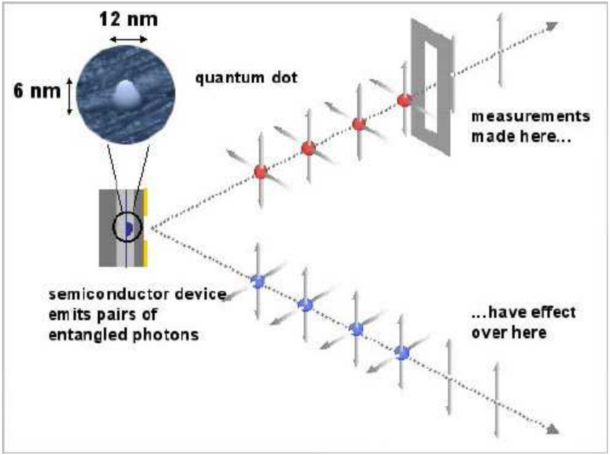
Using a physical device, *A* and *B* can disprove the Hidden Variable Theorem.

More precisely

Measuring entangled photons, *A* and *B* can coordinate their strategies to match pennies against *N* in such a way that their wealth will increase, infinitely in the long run.

Disproving the Theorem

A and *B*'s strategic device



Disproving the Theorem

A and *B*'s preparation

- The device emits $\vec{x}$ and $\vec{y}$ in the singlet state

$$\Psi = \frac{|{\downarrow}{\uparrow}\rangle - |{\uparrow}{\downarrow}\rangle}{\sqrt{2}} = \frac{|{\rightarrow}{\leftarrow}\rangle - |{\leftarrow}{\rightarrow}\rangle}{\sqrt{2}}$$

- *A* measures the spin of $\vec{x}$ in the basis

$$\left\{|{\downarrow}\rangle, \qquad |{\uparrow}\rangle\right\}$$

- *B* measures the spin of $\vec{y}$ in the basis

$$\left\{|{\rightarrow}\rangle = \frac{-(|{\downarrow}\rangle + |{\uparrow}\rangle)}{\sqrt{2}}, \qquad |{\leftarrow}\rangle = \frac{-(|{\downarrow}\rangle - |{\uparrow}\rangle)}{\sqrt{2}}\right\}$$

Disproving the Theorem

A and B's strategy

The response to N's move $xy \in \{00, 01, 10, 11\}$ is:

- A sees $x \in \{0, 1\}$
- if $x = 0$ measure $|\downarrow\rangle$
- if $x = 1$ measure $|\uparrow\rangle$
- if yes then play $a = +$
- otherwise play $a = -$

- B sees $y \in \{0, 1\}$
- if $y = 0$ measure $|\rightarrow\rangle$
- if $y = 1$ measure $|\leftarrow\rangle$
- if yes then play $b = +$
- otherwise play $b = -$

Disproving the Theorem

Expected payoff for A and B

Since $\mathbb{E}_{AB}(xy) = -\vec{x} \cdot \vec{y}$, it follows that

$$\begin{aligned} \mathbb{E}_{AB}(00) = \mathbb{E}_{AB}(01) = \mathbb{E}_{AB}(10) &= \frac{1}{\sqrt{2}} \\ \mathbb{E}_{AB}(11) &= -\frac{1}{\sqrt{2}} \end{aligned}$$

which gives

$$\begin{aligned} \mathbb{U}_{AB} &= \frac{1}{4}\Big(\mathbb{E}_{AB}(00) + \mathbb{E}_{AB}(01) + \mathbb{E}_{AB}(10) - \mathbb{E}_{AB}(11)\Big) \\ &= \frac{1}{\sqrt{2}} \end{aligned}$$

Disproving the Theorem

Expected payoff for *A* and *B*

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$$\begin{aligned} \mathbb{U}_{AB} &= \frac{1}{4}\Big(\mathbb{E}_{AB}(00) + \mathbb{E}_{AB}(01) + \mathbb{E}_{AB}(10) - \mathbb{E}_{AB}(11)\Big) \\ &= \frac{1}{\sqrt{2}} > \frac{1}{2} \end{aligned}$$

Empiric corollary

A and *B* can coordinate to win, but they coordined strategy is not realized through a hidden variable, i.e.

$$p(ab\,|\,xy) \neq \int_{\Lambda} p(a\,|\,x) \cdot p(b\,|\,y) \cdot q(\lambda) d\lambda$$

Outline

Matching Pennies

Question

Background

- *A* and *B*'s strategy is based on the Einstein-Podelsky-Rosen's setup (EPR) for *"spooky action at distance"*
  - Einstein's conclusion: since action at distance is impossible, there must be a hidden variable
- The Hidden Variable Theorem is based on John Bell's inequality.
  - Quantum theoretic prediction: Bell's Inequality can be violated

Background

- *A* and *B*'s strategy is based on the Einstein-Podelsky-Rosen's setup (EPR) for *"spooky action at distance"*
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- The Hidden Variable Theorem is based on John Bell's inequality.
  - Quantum theoretic prediction: Bell's Inequality can be violated $\leftsquigarrow$ experimentally confirmed

Game Theoretic Probability

What does this have to do with Game Theoretic Probability?

Philosophy of Probability

Question

Whence probability?

Answers

subjective: Because we average over hidden variables

- Bernoulli, Laplace, Einstein, 't Hooft

objective: Because God plays dice

- Darwin, Bachelier, Born, Zurek

Game Theoretic Probability

Strategy

Formulate a Probability Theory such that it

- arises from the strategies in a forecasting game
- provides a unified account of random processes
    - supports subjective and objective interpretation

Physics of Probability

But the interpretations can be tested experimentally!

Physics of Probability

Question

Does God play dice?

Answers

no: The world is deterministic

- Einstein, Bohm, superstrings...

yes: The world emerges from randomness

- Bell, Aspect, quantum darwinism...

Game Theoretic Probability

Question

Can we

- provide a unified account of random processes
- **that allows (thought) experimental testing**?