Derandomization in Game-Theoretic Probability

Kenshi Miyabe, Meiji University, Japan
(joint work with Akimichi Takemura)

Fifth Workshop on Game-Theoretic Probability
and Related Topics
13 Nov 2014

- This talk is mainly based on "Derandomization in game-theoretic probability" by K. Miyabe and A. Takemura, Stochastic Processes and their Applications, Vol.125, pp.39–59 (2015).

Abstract

- We give a general method for constructing a concrete deterministic strategy of Reality from a randomized strategy. The construction can be seen as derandomization.

Derandomization

- Randomized algorithm is everywhere.
- Numerical analysis: Monte Carlo Method etc.
- Complexity theory: BPP
- Statistics: mainly due to Fisher

- We sometimes want a deterministic strategy rather than a randomized strategy, because we can not construct a real random sequence.
- There are lots of work in complexity theory.
- Is Monte Carlo Method mathematically correct? See the work by Hiroshi Sugita at Osaka Univ.

Derandomization

- Can we always derandomize? Is there a general method?
- Yes, we can, in game-theoretic probability.
  Use the technique of algorithmic randomness.

SLLN in GTP

Unbounded Forecasting Game (UFG)

Players: Forecaster, Skeptic, Reality

Protocol:

$\mathcal{K}_0 := 1.$

For $n = 1, 2, \cdots :$

Forecaster announces $m_n \in \mathbb{R}$ and $v_n \geq 0$.

Skeptic announces $M_n \in \mathbb{R}$ and $V_n \geq 0$.

Reality announces $x_n \in \mathbb{R}$.

$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n(x_n - m_n) + V_n((x_n - m_n)^2 - v_n).$

Colateral Duties: Skeptic must keep $\mathcal{K}_n$ non-negative. Reality must keep $\mathcal{K}_n$ from tending to infinity.

Theorem (Proposition 4.1 in the book of Shafer and Vovk 2001)

In the unbounded forecasting game,

(i) Skeptic can force

$$\sum_{n=1}^{\infty} \frac{v_n}{n^2} < \infty \quad \Rightarrow \quad \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (x_i - m_i) = 0.$$

(ii) Reality can comply with

$$\sum_{n=1}^{\infty} \frac{v_n}{n^2} = \infty \Rightarrow \left(\frac{1}{n} \sum_{i=1}^{n} (x_i - m_i) \text{ does not conv. to } 0 \right).$$

Skeptic can force an event E in a game if there is a strategy of Skeptic by which Skeptic makes E happens or $\lim_n \mathcal{K}_n = \infty$. Then, we say that E happens almost surely.

Reality can comply with an event E in a game if there is a strategy of Reality by which Reality makes E happens and $\lim_n \mathcal{K}_n < \infty$.

In the book Shafer and Vovk used the terminology "can force" for both.

Skeptic's strategy

- How to construct?
- One way is that, find a measure-theoretic proof, "translate" into a proof via martingales (this part is sometimes non-trivial) and further "translate" into a game-theoretic proof.
- New ideas sometimes make proofs much more direct and simpler.

Reality's strategy

- How to construct?
- Not straightforward.

The following is Kolmogorov's strategy. Assume the sequence v_n such that $\sum_n \frac{v_n}{n^2} = \infty$ is given. Consider the measure such that, if $v_n < n^2$, then

$$x_n := \begin{pmatrix} n \\ -n \\ 0 \end{pmatrix} \text{ with probability } \begin{pmatrix} v_n/(2n^2) \\ v_n/(2n^2) \\ 1 - v_n/n^2 \end{pmatrix},$$

respectively, and if $v_n \geq n^2$ then

$$x_n := \begin{pmatrix} \sqrt{v_n} \\ -\sqrt{v_n} \end{pmatrix} \text{ with probability } \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}.$$

Then, by Borel-Cantelli lemma, $|x_n| \geq n$ infinitely often almost surely.

Theorem (Martin's theorem)
For a perfect-information game with two players, if the winning strategy is quasi-Borel, then the game is determined, that is, exact one of the two players has a winning strategy.

In the book of Shafer and Vovk, by combining Kolmogorov's strategy and Martin's theorem, they have proved Reality's compliance, but did not give a concrete strategy.

We can give a deterministic strategy of Reality. One such a strategy is given in the following note.

V. Vovk, Kolmogorov's strong law of large numbers in game-theoretic probability: reality's side, arXiv:1304.1074.

- Do we need to come up with a new strategy every time for another theorem?
- How related are Kolmogorov's strategy and Reality's strategy?
- If we have a general way to transform the strategy, we will have a strong method for derandomization.

$$x_n = \begin{cases} n & \text{if } v_n < n^2,\ V_n \le d_n, M_n < 0 \\ -n & \text{if } v_n < n^2,\ V_n \le d_n, M_n \ge 0 \\ 0 & \text{if } v_n < n^2,\ V_n > d_n, \\ \sqrt{v_n} & \text{if } v_n \ge n^2,\ M_n < 0, \\ -\sqrt{v_n} & \text{if } v_n \ge n^2,\ M_n \ge 0 \end{cases}$$

where

$$b_n = \#\{k < n \ :\ x_k \neq 0\}$$

$$c_n \in \mathbb{N} \text{ satisfying } c_n - 1 \le \sum_{k=1}^{n} \frac{v_k}{k^2} < c_n$$

$$d_n = C\frac{2^{-b_n-2} - 2^{-c_n-2}}{n^2} \text{ for some } C.$$

Unbounded Forecasting Game (UFG)

Players: Forecaster, Skeptic, Reality

Protocol:

$\mathcal{K}_0 := 1$.

For $n = 1, 2, \cdots$:

Forecaster announces $m_n \in \mathbb{R}$ and $v_n \geq 0$.

Skeptic announces $M_n \in \mathbb{R}$ and $V_n \geq 0$.

Reality announces $x_n \in \mathbb{R}$.

$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n(x_n - m_n) + V_n((x_n - m_n)^2 - v_n)$.

Colateral Duties: Skeptic must keep $\mathcal{K}_n$ non-negative. Reality must keep $\mathcal{K}_n$ from tending to infinity.

- The strategy (in particular the value d) is constructed by Kolmogorov’s randomized strategy. It is NOT by trial and error!!

Idea of construction

- (I) Take a randomized strategy
- (II) Construct a strategy of Skeptic that forces the random event.
- (III) Construct a strategy of Reality using it.

Step (I)

- (I) Take a randomized strategy
- This is Kolmogorov's strategy in this case. We also need its proof. The proof uses Borel-Cantelli's lemma.
- This reminds me of program extraction of intuitionistic logic

Step (II)

- (II) Construct a strategy of Skeptic that forces the random event.
- We know that an event happens almost surely when the probability is the given one. Then, we can construct Skeptic's strategy that forces the event. In this case we constructed a simple strategy that forces the Borel-Cantelli lemma.

Step (III)

- (III) Construct a strategy of Reality using it.
- We can do that because, if Skeptic can force an event, then Reality can force the event.

Let F be a strategy of Skeptic that forces an event E.

Reality fights against $(S+F)/2$ where S is the real strategy of Skeptic.

Here, "fights" means that Reality makes the capital $\mathcal{K}^{(S+F)/2}$ bounded.

Then $\mathcal{K}^{S}$ is bounded.

Since $\mathcal{K}^{F}$ is bounded, E must happen.

Hence, Reality complies with E via this strategy.

Stronger results

If an event has lower probability 1 then Reality usually can comply with the event with the condition $\mathcal{K}_n \leq \mathcal{K}_0$ for every n.

In that case, we say that Reality can strongly comply with it.

Furthermore, if an event has positive lower probability, then Reality can comply with the event.

Thus, the value $\sup_n \mathcal{K}_n$ has a strong relation to lower probability.

See the paper for more details.

Future work

- We can construct a sequence random enough for a specific purpose. For instance, if you give me a countable list of limit theorems and their game-theoretic proofs, I can construct a random sequence that satisfies all properties. Now we have a practical reason to give game-theoretic proofs!!

- We are looking for applications worth examining.

- Thank you for listening.